Unification in modal logic Alt_1

Philippe Balbiani[1] and Tinko Tinchev[2]

[1]Institut de recherche en informatique de Toulouse
CNRS — Université de Toulouse
[2]Department of Mathematical Logic and Applications
Sofia University

Introduction

Unification problem in a logical system L

- Given a formula $\psi(x_1, \ldots, x_n)$
- Determine whether there exists formulas φ_1, ..., φ_n such that $\psi(\varphi_1, \ldots, \varphi_n)$ is in L

Admissibility problem in a logical system L

- Given a rule of inference $\frac{\varphi_1(x_1,\ldots,x_n),\ldots,\varphi_m(x_1,\ldots,x_n)}{\psi(x_1,\ldots,x_n)}$
- Determine whether for all formulas χ_1, ..., χ_n, if $\varphi_1(\chi_1, \ldots, \chi_n)$, ..., $\varphi_m(\chi_1, \ldots, \chi_n)$ are in L then $\psi(\chi_1, \ldots, \chi_n)$ is in L

Introduction

Rybakov (1984)

- The admissibility problem in *IPL* and *S*4 is decidable

Chagrov (1992)

- There exists a decidable normal modal logic with an undecidable admissibility problem

Ghilardi (1999, 2000)

- *IPL*, *K*4, etc have a finitary unification type

Wolter and Zakharyaschev (2008)

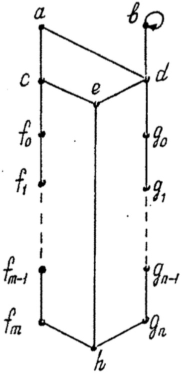
- The unification problem for any normal modal logic between K_U and $K4_U$ is undecidable

Introduction

Chagrov (1992)

- There exists a decidable normal modal logic with an undecidable admissibility problem

Proof: For all integers m, n, let $\mathcal{F}(m, n)$ be the frame



Introduction

Chagrov (1992)

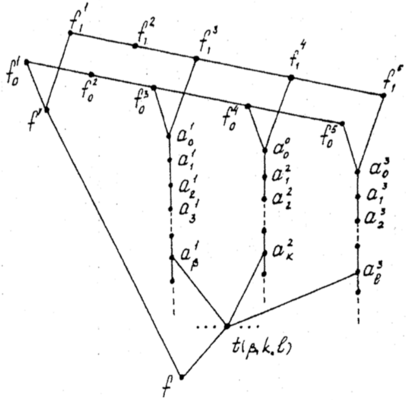
- There exists a decidable normal modal logic with an undecidable admissibility problem

Proof:

- For all integers m, n, let $\mathcal{F}(m, n)$ be the frame...
- For all sets S of pairs of integers, let $L(S) = Log\{\mathcal{F}(m, n) : (\frac{m-1}{2}, \frac{n-1}{2}) \notin S\}$
- If S is recursive then $L(S)$-membership is decidable
- If $Pr_2 S$ is nonrecursive then $L(S)$-admissibility is undecidable

Introduction

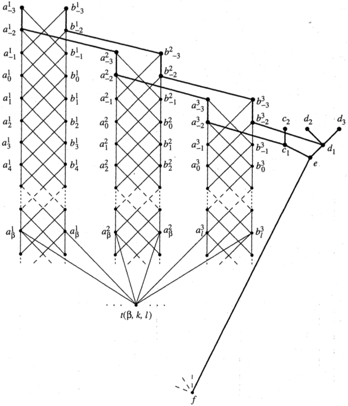
Other frames $\mathcal{F}(\mathbf{P}, \mathfrak{a})$ associated to a Minsky program **P** and a configuration $\mathfrak{a}$



Chagrov, A. *Undecidable properties of extensions of the logic of provability.* Algebra i Logika **29** (1990) 350–367.

Introduction

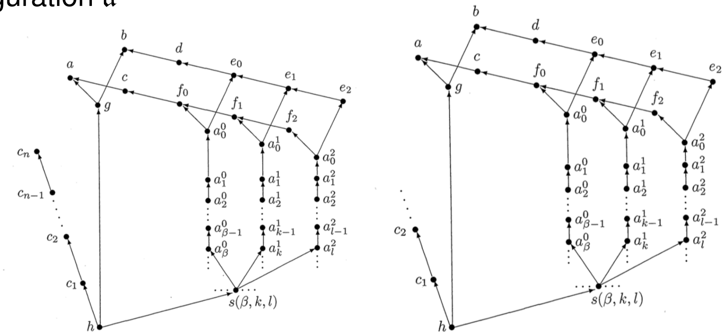
Other frames $\mathcal{F}(\mathbf{P}, \mathfrak{a})$ associated to a Minsky program **P** and a configuration $\mathfrak{a}$



Chagrov, A., Zakharyaschev, M. *The undecidability of the disjunction property of propositional logics and other related problems.* The Journal of Symbolic Logic **58** (1993) 967–1002.

Introduction

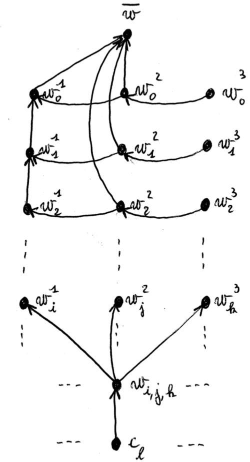
Other frames $\mathcal{F}(\mathbf{P}, \mathfrak{a})$ associated to a Minsky program **P** and a configuration $\mathfrak{a}$



Chagrov, A., Chagrova, L. The truth about algorithmic problems in correspondence theory. In: Advances in Modal Logic. Vol. 6. College Publications (2006) 121–138.

Introduction

Other frames $\mathcal{F}(\mathbf{P})$ associated to a Minsky program **P**



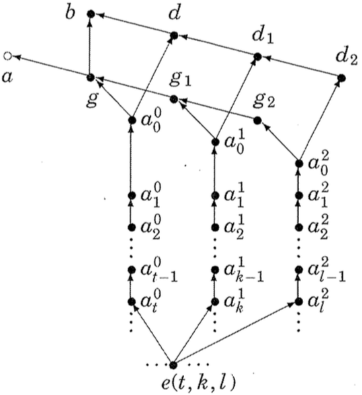
Isard, S. *A finitely axiomatizable undecidable extension of K.* Theoria **43** (1977) 195–202.

Introduction

Wolter and Zakharyaschev (2008)

- The unification problem for any normal modal logic between K_U and $K4_U$ is undecidable

Proof: Let **P** be a Minsky program, $\mathfrak{a} = (s, m, n)$ be a configuration and $\mathcal{F}(\mathbf{P}, \mathfrak{a})$ be the frame



Introduction

Wolter and Zakharyaschev (2008)

- The unification problem for any normal modal logic between K_U and $K4_U$ is undecidable

Proof:

- Let **P** be a Minsky program, $\mathfrak{a} = (s, m, n)$ be a configuration and $\mathcal{F}(\mathbf{P}, \mathfrak{a})$ be the frame...
- Let α, β, etc be formulas characterizing the points in $\mathcal{F}(\mathbf{P}, \mathfrak{a})$
- With each configuration $\mathfrak{b}$, associate a modal formula $\psi(\mathfrak{b})$
- If $K_U \subseteq L \subseteq K4_U$ then $\mathbf{P} : \ \mathfrak{a} \rightarrow \mathfrak{b}$ iff $\psi(\mathfrak{b})$ is unifiable in L

Introduction

Unification problem in a logical system L

- Given a formula $\psi(x_1, \ldots, x_n)$
- Determine whether there exists formulas φ_1, ..., φ_n such that $\psi(\varphi_1, \ldots, \varphi_n)$ is in L

Example: $\Box x \lor \Box\neg x$ is unifiable in all normal logics

- K (class of all frames)
- KD (class of all serial frames)
- $K4$ (class of all transitive frames)
- $S4$ (class of all reflexive transitive frames)
- $S5$ (class of all partitions)

Introduction

Computability and type of unification in L

L	Computability	Type
K	?	Nullary
KD	NP-complete	?
$K4$	Decidable	Finitary
$KD4$	NP-complete	Finitary
$K45$	NP-complete	Unitary
$KD45$	NP-complete	Unitary
$S4$	NP-complete	Finitary
$S5$	NP-complete	Unitary
$S4.3$	NP-complete	Unitary

Introduction

Our results

- The unification problem in Alt_1 is decidable (*PSPACE*)
- Alt_1 has a nullary unification type

Normal logics: syntax and semantics

Syntax

- $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$

Semantics

- $\mathcal{M} = (W, R, V)$

where

- $W \neq \emptyset$
- $R \subseteq W \times W$
- for all variables x, $V(x) \subseteq W$

Truth-conditions

- $\mathcal{M}, s \models x$ iff $s \in V(x)$
- $\mathcal{M}, s \models \Box\varphi$ iff for all $t \in W$, if sRt then $\mathcal{M}, t \models \varphi$

Normal logics: unification in L

Substitutions

- σ: variable $x \mapsto$ formula $\sigma(x)$

Composition of substitutions

- $\sigma \circ \tau$: variable $x \mapsto$ formula $\tau(\sigma(x))$

Equivalence relation between substitutions

- $\sigma \simeq_L \tau$ iff for all variables x, $\sigma(x) \leftrightarrow \tau(x) \in L$

Partial order between substitutions

- $\sigma \preceq_L \tau$ iff there exists a substitution μ such that $\sigma \circ \mu \simeq_L \tau$

Normal logics: unification in L

Unifiers

- A substitution σ is a unifier of a formula φ iff $\sigma(\varphi) \in L$

Complete sets of unifiers

- A set Σ of unifiers of a formula φ is complete iff
  - For all unifiers τ of φ, there exists a unifier σ of φ in Σ such that $\sigma \preceq_L \tau$

Important questions

- Given a formula, has it a unifier?
- If so, has it a minimal complete set of unifiers?
- If so, how large is this set?

Why unification is NP-complete when $KD \subseteq L$

Computability and type of unification in L

L	Computability	Type
K	?	Nullary
KD	NP-complete	?
$K4$	Decidable	Finitary
$KD4$	NP-complete	Finitary
$K45$	NP-complete	Unitary
$KD45$	NP-complete	Unitary
$S4$	NP-complete	Finitary
$S5$	NP-complete	Unitary
$S4.3$	NP-complete	Unitary

Why unification is *NP*-complete when $KD \subseteq L$

Proposition: If $KD \subseteq L$, unification in L is *NP*-complete
Proof:

- A substitution σ is **ground** if it replaces each variable by a variable-free formula
- If a formula has a unifier then it has a ground unifier
- Since $\Diamond\top \in L$, therefore there are **only two** non-equivalent variable-free formulas: $\bot$ and $\top$
- Thus, to decide whether a formula has a unifier, it suffices to check whether any of the ground substitutions makes it equivalent to $\top$ (which can be done in polynomial time)

Why unification is nullary in K

Computability and type of unification in L

L	Computability	Type
K	?	Nullary
KD	NP-complete	?
$K4$	Decidable	Finitary
$KD4$	NP-complete	Finitary
$K45$	NP-complete	Unitary
$KD45$	NP-complete	Unitary
$S4$	NP-complete	Finitary
$S5$	NP-complete	Unitary
$S4.3$	NP-complete	Unitary

Why unification is nullary in K

Proposition: The formula $\varphi = x \to \Box x$ has no minimal complete set of unifiers

Proof:

- The following substitutions are unifiers of φ
  - $\sigma_\top(x) = \top$
  - $\sigma_i(x) = \Box^{<i} x \wedge \Box^i \bot$
- If $i \leq j$ then $\sigma_j \preceq_K \sigma_i$
- If $i < j$ then $\sigma_i \not\preceq_K \sigma_j$
- If τ is a unifier of φ then either $\sigma_\top \preceq_K \tau$, or $\sigma_i \preceq_K \tau$ when $\deg(\tau(x)) \leq i$

Jeřábek, E. Blending margins: the modal logic K has nullary unification type. Journal of Logic and Computation **25** (2015) 1231–1240.

Why unification is decidable and finitary in $K4$

Computability and type of unification in L

L	**Computability**	**Type**
K	?	Nullary
KD	NP-complete	?
$K4$	Decidable	Finitary
$KD4$	NP-complete	Finitary
$K45$	NP-complete	Unitary
$KD45$	NP-complete	Unitary
$S4$	NP-complete	Finitary
$S5$	NP-complete	Unitary
$S4.3$	NP-complete	Unitary

Why unification is decidable and finitary in $K4$

A formula φ is **projective** if it has a unifier σ such that

- $\varphi \wedge \Box\varphi \rightarrow (\sigma(x) \leftrightarrow x) \in K4$

Remark

- Such unifier is a most general unifier of φ

Proposition: The projectivity problem in $K4$ is decidable
Proposition If the substitution σ is a unifier of the formula φ then there exists a projective formula ψ, $depth(\psi) \leq depth(\varphi)$, such that

- σ is a unifier of ψ
- $\psi \wedge \Box\psi \rightarrow \varphi \in K4$

Ghilardi, S. Best solving modal equations. Annals of Pure and Applied Logic **102** (2000) 183–198.

Why unification is unitary in $S5$

Computability and type of unification in L

L	Computability	Type
K	?	Nullary
KD	NP-complete	?
$K4$	Decidable	Finitary
$KD4$	NP-complete	Finitary
$K45$	NP-complete	Unitary
$KD45$	NP-complete	Unitary
$S4$	NP-complete	Finitary
$S5$	NP-complete	Unitary
$S4.3$	NP-complete	Unitary

Why unification is unitary in *S*5

Proposition: If a formula has a unifier then it has a most general unifier

Proof:

- Let σ be a unifier of φ
- Let τ be the following "Löwenheim" substitution
  - $\tau(x) = (\Box\varphi \wedge x) \vee (\Diamond\neg\varphi \wedge \sigma(x))$
- $\Box\varphi \to (\tau(\psi) \leftrightarrow \psi) \in S5$
- $\Diamond\neg\varphi \to (\tau(\psi) \leftrightarrow \sigma(\psi)) \in S5$
- τ is a unifier of φ
- If μ is a unifier of φ then $\tau \preceq_{S5} \mu$
- Thus, τ is a most general unifier of φ

Baader, F., Ghilardi, S. Unification in modal and description logics. Logic Journal of the IGPL **19** (2011) 705–730.

Normal logic Alt_1: syntax and semantics

Syntax

- $\varphi ::= x \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$

Semantics

- Class of all deterministic frames

Axiomatization

- $K + \Diamond x \to \Box x$

Computability

- co*NP*-complete

Why unification is nullary in Alt_1

Proposition: The formula $\varphi = x \to \Box x$ has no minimal complete set of unifiers

Proof: Following the line of reasoning suggested by

- **Jeřábek, E.** Blending margins: the modal logic *K* has nullary unification type. Journal of Logic and Computation **25** (2015) 1231–1240.

Why unification is decidable (*PSPACE*) in Alt_1

Let $\varphi(x)$ be a formula and k be an integer

Proposition: The following conditions are equivalent

1. $\varphi(x)$ has a unifier
2. There exists a variable-free formula ψ such that $\varphi(\psi) \in Alt_1$
3. There exists a variable-free formula ψ such that $\Box^k\bot \to \varphi(\psi) \in Alt_1$ and $\Diamond^k\top \to \varphi(\psi) \in Alt_1$

Why unification is decidable (*PSPACE*) in Alt_1

Let ψ be a variable-free formula

If n is an integer, define

- $\models_n \psi$ iff $(0, \ldots, n), 0 \models \psi$

If i, k, n are integers such that $i \leq k \leq n$, define the bit

- $V_k(\psi, n, i) =$ "if $\models_{n-k+i} \psi$ then 1 else 0"

If k, n are integers such that $k \leq n$, define the $(k+1)$-tuples

- $V_k(\psi, n) = (V_k(\psi, n, 0), \ldots, V_k(\psi, n, k))$
- $a_k(\psi, n) = V_k(\psi, n \cdot (k+1) + k)$

If k is an integer, define the nonempty set of pairs

- $g_k(\psi) = \{(a_k(\psi, n), a_k(\psi, n+1)) : \ n \geq 0\}$

Why unification is decidable (*PSPACE*) in Alt_1

Let $\varphi(x)$ be a formula and k be an integer

Proposition: For all variable-free formulas ψ, χ such that $g_k(\psi) = g_k(\chi)$, the following conditions are equivalent

1. $\Diamond^k \top \to \varphi(\psi) \in Alt_1$
2. $\Diamond^k \top \to \varphi(\chi) \in Alt_1$

Define the equivalence relation $\simeq_k$ between variable-free formulas

- $\psi \simeq_k \chi$ iff $g_k(\psi) = g_k(\chi)$

Proposition: The equivalence relation $\simeq_k$ has finitely many equivalence classes

Why unification is decidable (*PSPACE*) in Alt_1

Let k be an integer

A nonempty set B of pairs of $(k+1)$-tuples of bits is **modally definable** iff

- There exists a variable-free formula ψ such that $B = g_k(\psi)$

Define the **domino relation** $\triangleright_B$ on a nonempty set B of pairs of $(k+1)$-tuples of bits

- $(b'_1, b''_1) \triangleright_B (b'_2, b''_2)$ iff $b''_1 = b'_2$

A path in the directed graph $(B, \triangleright_B)$ is **weakly Hamiltonian** iff

- It visits each vertex at least once

Proposition: For all nonempty sets B of pairs of $(k+1)$-tuples of bits, the following conditions are equivalent

1. B is modally definable
2. The directed graph $(B, \triangleright_B)$ contains a weakly Hamiltonian path either ending with $(\vec{1}_{k+1}, \vec{1}_{k+1})$, or ending with $(\vec{0}_{k+1}, \vec{0}_{k+1})$

Unification in Alt_1: a 1st sub-Boolean fragment

Syntax

- $\varphi ::= x \mid \top \mid (\varphi \wedge \psi) \mid \Box\varphi$

Unifiers

- A substitution σ is a unifier of a finite set $\{(\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n)\}$ of pairs of formulas iff $\sigma(\varphi_i) \leftrightarrow \sigma(\psi_i) \in Alt_1$, ..., $\sigma(\varphi_n) \leftrightarrow \sigma(\psi_n) \in Alt_1$

Proposition: The unification problem in Alt_1 is trivially decidable for this 1st fragment

Proof:

- Very easy

Unification in Alt_1: a 1st sub-Boolean fragment

Syntax

- $\varphi ::= x \mid \top \mid (\varphi \wedge \psi) \mid \Box\varphi$

Unifiers

- A substitution σ is a unifier of a finite set $\{(\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n)\}$ of pairs of formulas iff $\sigma(\varphi_i) \leftrightarrow \sigma(\psi_i) \in Alt_1$, ..., $\sigma(\varphi_n) \leftrightarrow \sigma(\psi_n) \in Alt_1$

Proposition: $\{(\Box x \wedge \Box y, y \wedge \Box\Box z)\}$ has no minimal complete set of unifiers
Proof: Following the line of reasoning suggested by

- **Baader, F.** Unification in commutative theories. Journal of Symbolic Computation **8** (1989) 479–497.

Unification in Alt_1: a 2nd sub-Boolean fragment

Syntax

- $\varphi ::= x \mid \top \mid (\varphi \wedge \psi) \mid \Diamond\varphi$

Unifiers

- A substitution σ is a unifier of a finite set $\{(\varphi_1, \psi_1), \ldots, (\varphi_n, \psi_n)\}$ of pairs of formulas iff $\sigma(\varphi_i) \leftrightarrow \sigma(\psi_i) \in Alt_1, \ldots, \sigma(\varphi_n) \leftrightarrow \sigma(\psi_n) \in Alt_1$

Proposition: The unification problem in Alt_1 is decidable (*PSPACE*) for this 2nd fragment

Proof:

- By means of a normal form property

Open question: The unification type of Alt_1 for this 2nd fragment

Open problems

Admissibility problem in Alt_1

Unification problem in Alt_1 with parameters

- given a formula $\psi(p_1, \ldots, p_m, x_1, \ldots, x_n)$
- determine whether there exists formulas φ_1, ..., φ_n such that $\psi(p_1, \ldots, p_m, \varphi_1, \ldots, \varphi_n)$ is in Alt_1

Admissibility problem in Alt_1 with parameters

Case when the ordinary modal language is extended by the difference modality or the universal modality