Classical and Empirical Negation in Subintuitionistic Logic

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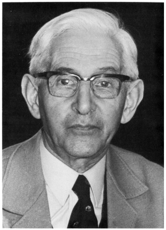
Outline

Motivation for empirical negation



- Intuitionism: a verificationism restricted to mathematical discourse.
- An attempt to generalize intuitionism to empirical discourse presents various challenges.
- Question: will $\wedge$, $\vee$, $\rightarrow$ and $\bot$ suffice as propositional connectives for an empirical language?
- Answer: no, especially concerning negation. E.g., Dummett says:

Motivation for empirical negation (Cont'd)

Dummet in "The seas of language" (1996), p.473.

Negation . . . is highly problematic. In mathematics, given the meaning of "if . . . then", it is trivial to explain "Not A" as meaning "If A, then 0 = 1"; by contrast, a satisfactory explanation of "not", as applied to empirical statements for which bivalence is not, in general, taken as holding, is very difficult to arrive at. Given that the sentential operators cannot be thought of as explained by means of the two-valued truth-tables, the possibility that the laws of classical logic will fail is evidently open: but it is far from evident that the correct logical laws will always be the intuitionistic ones. More generally, it is by no means easy to determine what should serve as the analogue, for empirical statements, of the notion of proof as it figures in intuitionist semantics for mathematical statements.

Motivation for empirical negation (Cont'd)

Problem

The "arrow-falsum" definition of negation is often too strong to serve as the negation for empirical statements.

Example

- Attempt: express in our generalized intuitionistic language the fact that Goldbach's conjecture is not decided.
- What we get: any warrant for 'Goldbach's conjecture is decided' can be transformed into warrant for an absurdity (say '0=1'). This implies that Goldbach's conjecture is undecidable!
- No! In the future, someone may prove or refute Goldbach's conjecture. So the fact that Goldbach's conjecture is not decided does not imply that it is undecid*able*, as our translation gives us.
- What we want: there is no sufficient evidence at present for the truth of the conjecture.

Semantics for **IPC**$^{\sim}$

Definition (Semantics for **IPC**$^{\sim}$)

IPC$^{\sim}$ model M is $\langle W, \leq, @, v \rangle$ where

- W: non-empty set partially ordered by $\leq$ with the least element @.
- $V : W \times \text{Prop} \rightarrow \{0, 1\}$ where if $v(w_1, p) = 1$ and $w_1 \leq w_2$ then $v(w_2, p) = 1$ for all $p \in \text{Prop}$ and $w_1, w_2 \in W$.

Valuations V are then extended to interpretations I as follows:

- $I(w, p) = V(w, p)$
- $I(w, \sim A) = 1$ iff $I(@, A) = 0$
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$
- $I(w, A \rightarrow B) = 1$ iff for all $x \in W$: if $w \leq x$ & $I(x, A) = 1$ then $I(x, B) = 1$.

Classical negation

$$I(w, \neg A) = 1 \text{ iff } I(w, A) \neq 1$$

Subintuitionistic logic (or Logics with strict implication)

Some earlier works

- Giovanna Corsi: "Weak logics with strict implication" (1987)
- Kosta Došen: "Modal translations in **K** and **D**" (1993)
- Greg Restall: "Subintuitionistic Logic" (1994)

We follow Restall's presentation which accommodates our approach.

Subintuitionistic logic (due to Restall): semantics

Definition (Semantics for **SJ**)

SJ model M is $\langle W, @, R, v \rangle$ where

- W: non-empty set with $@ \in W$.
- R: binary relation on W with $@Rw$ for all $w \in W$.
- $V : W \times \text{Prop} \to \{0, 1\}$.

Valuations V are then extended to interpretations I as follows:

- $I(w, p) = V(w, p)$
- $I(w, A \wedge B) = 1$ iff $I(w, A) = 1$ and $I(w, B) = 1$
- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$
- $I(w, A \to B) = 1$ iff for all $x \in W$: if wRx & $I(x, A) = 1$ then $I(x, B) = 1$.

Semantic consequence is defined in terms of truth preservation at $@$:
$\Sigma \models A$ iff for all models $\langle W, @, R, I \rangle$, $I(@, A) = 1$ if $I(@, B) = 1$ for all $B \in \Sigma$.

Subintuitionistic logic (due to Restall): proof theory

Definition (Axioms and rules of inference for SJ)

$$A{\to}A \qquad ((C{\to}A) \wedge (C{\to}B)){\to}(C{\to}(A \wedge B)))$$

$$A{\to}(B{\to}B) \qquad A{\to}(A \vee B)$$

$$((A{\to}B) \wedge (B{\to}C)){\to}(A{\to}C) \qquad B{\to}(A \vee B)$$

$$(A \wedge B){\to}A \qquad ((A{\to}C) \wedge (B{\to}C)){\to}((A \vee B){\to}C))$$

$$(A \wedge B){\to}B \qquad (A \wedge (B \vee C)){\to}((A \wedge B) \vee (A \wedge C))$$

$$\frac{A \quad A{\to}B}{B} \qquad \frac{A \quad B}{A \wedge B}$$

$$\frac{A \vee C \quad (A{\to}B) \vee C}{B \vee C} \qquad \frac{(A{\to}B) \vee E \quad (C{\to}D) \vee E}{((B{\to}C){\to}(A{\to}D)) \vee E}$$

$\Gamma \vdash A$ iff there is a sequence of formulae $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of **SJ**; (iii) is obtained by one of the rules from formulae preceding it in sequence.

Aim

Previous results

- De: a discussion of empirical negation
- De & O.: an axiomatization of $\mathbf{IPC}^{\sim}$
- Farinas & Herzig: combines classical and intuitionistic negations
- Restall: subintuitionistic logics

Aim

- explore classical and empirical negations in subintuitionistic logic
- observe two more results related to empirical negation in relevant and superintuitionistic logics

Classical negation

Theorem (inspired by Routley and Meyer)

Subintuitionistic logic with classical negation is axiomatized by **SJ** plus:

$$\neg\neg A \to A \qquad \frac{((A \wedge B) \to \neg C) \vee D}{((A \wedge C) \to \neg B) \vee D}$$

Some basic results

Deduction theorem

$$\Gamma, A \vdash_c B \text{ iff } \Gamma \vdash_c \neg A \vee B$$

Validities

- $(A \wedge \neg A) \to B$
- $B \to (A \vee \neg A)$
- $\neg\neg A \leftrightarrow A$
- $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$
- $\dfrac{A \to B}{\neg B \to \neg A}$
- $\dfrac{A \to \neg B}{B \to \neg A}$

Invalidities

- $A \to (\neg A \to B)$
- $(A \to B) \to (\neg B \to \neg A)$
- $(\neg B \to \neg A) \to (A \to B)$

Some basic results

"Combining classical and intuitionistic logic" (1996)

- An attempt to combine classical and intuitionistic logics.
- Axiom system is formulated along the line of conditional logic.

Theorem

SJ with classical negation extended by axioms for reflexivity, transitivity and restricted heredity is sound and complete with respect to C+J-models.

Empirical negation

Theorem

Subintuitionistic logic with empirical negation is axiomatized by **SJ** plus:

$$A \vee {\sim} A \qquad ({\sim} A \wedge {\sim} B) \to {\sim}(A \vee B)$$

$${\sim}{\sim} A \to ({\sim} A \to B) \qquad \frac{(A \vee B) \vee C}{({\sim} A \to {\sim}{\sim} B) \vee C}$$

$${\sim} A \to (B \to {\sim} A)$$

Remark

IPC$^{\sim}$ is axiomatized by **IPC**$^{+}$ plus:

$$A \vee {\sim} A \qquad {\sim}{\sim} A \to ({\sim} A \to B) \qquad \frac{A \vee B}{{\sim} A \to B}$$

Remark

Addition of empirical negation implies the loss of disjunctive property.

Some results

Deduction theorem

$$\Gamma, A \vdash_e B \text{ iff } \Gamma \vdash_e {\sim} A \vee B$$

Corollary

$$\Gamma, A \vdash_e B \text{ iff } \Gamma \vdash_e {\sim}{\sim} A \rightarrow {\sim}{\sim} B$$
$$\text{iff } \Gamma \vdash_e {\sim} B \rightarrow {\sim} A$$

Validities and invalidities

- $\nvdash_e A \rightarrow ({\sim} A \rightarrow B)$ but $A, {\sim} A \vdash_e B$
- $\nvdash_e {\sim}{\sim} A \rightarrow A$ but ${\sim}{\sim} A \vdash_e A$
- $\nvdash_e A \rightarrow {\sim}{\sim} A$ but $A \vdash_e {\sim}{\sim} A$
- ${\sim}(A \wedge B) \leftrightarrow ({\sim} A \vee {\sim} B)$
- ${\sim}(A \vee B) \leftrightarrow ({\sim} A \wedge {\sim} B)$

Below subintuitionistic logic

Theorem (Routley and Priest)

SJ minus

- $B \rightarrow (A \rightarrow A)$
- $((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$

is sound and complete wrt simplified semantics.

Question

Can we add empirical negation to weak relevant logics?

Below subintuitionistic logic

Theorem

$\mathbf{B}^+$ with empirical negation can be axiomatized by adding the following:

$$A \vee {\sim} A \qquad ({\sim} A \wedge {\sim} B) \to {\sim}(A \vee B)$$

$$\sim\sim A \to ({\sim} A \to B) \qquad \frac{(A \vee B) \vee C}{({\sim} A \to {\sim}{\sim} B) \vee C}$$

$$\sim A \to (B \to {\sim} A)$$

Remark

- Even if we lose $B \to (A \to A)$, we still have ${\sim}(A \to A) \to B$.
- Classical negation can be added too!

Beyond intuitionistic logic

Definition (Semantics for $\mathbf{G}^{\sim}$)

$\mathbf{G}^{\sim}$ model M is $\langle W, \leq, @, v \rangle$ where

- W: non-empty set linearly ordered by $\leq$ with the least element $@$.
- $V : W \times \text{Prop} \rightarrow \{0, 1\}$ where if $v(w_1, p) = 1$ and $w_1 \leq w_2$ then $v(w_2, p) = 1$ for all $p \in \text{Prop}$ and $w_1, w_2 \in W$.

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- $I(w, A \vee B) = 1$ iff $I(w, A) = 1$ or $I(w, B) = 1$
- $I(w, A {\rightarrow} B) = 1$ iff for all $x{\in}W$: if $w{\leq}x$ & $I(x, A){=}1$ then $I(x, B){=}1$.

Semantic consequence is defined in terms of truth preservation at $@$.

Beyond intuitionistic logic

Definition

In Gödel logics: $\Delta A = 1$ iff $A = 1$, otherwise $\Delta A = 0$.

Theorem

$\sim\sim$ is Baaz' delta! ($\because I(w, \sim\sim A) = 1$ iff $I(@, A) = 1$.)

Remark

Another example of negative modality being powerful!

Conclusion

Summary

- Axiomatized **SJ** expanded by classical and empirical negation.
- We can go even weaker to work with $\mathbf{B}^{+}$.
- Observed a connection between Baaz' delta and empirical negation.

In sum: empirical negation is quite flexible!

(This implies the ubiquity of classical negation?)

Future directions

- Add quantifiers!
- Relate $\mathbf{SJ}^{\sim}$ and its extensions with other systems of modal logic!

KÖSZÖNÖM!!!