Post Completeness in Congruential Modal Logics

Peter Fritz

University of Oslo
peter.fritz@ifikk.uio.no

AiML
September 2, 2016

Post completeness

Let L be a set (the formulas);
let $C \subseteq \mathcal{P}(L)$ such that $L \in C$ (the logics).

Post completeness

Let L be a set (the formulas);
let $C \subseteq \mathcal{P}(L)$ such that $L \in C$ (the logics).

$\Lambda \in C$ is *Post complete in* C iff
$\Lambda \neq L$ and there is no $\Lambda' \in C$ such that $\Lambda \subset \Lambda' \subset L$.

Post completeness

Let L be a set (the formulas);
let $C \subseteq \mathcal{P}(L)$ such that $L \in C$ (the logics).

$\Lambda \in C$ is *Post complete in* C iff
$\Lambda \neq L$ and there is no $\Lambda' \in C$ such that $\Lambda \subset \Lambda' \subset L$.

In short: being Post complete is being a co-atom.

Post completeness

Let L be a set (the formulas);
let $C \subseteq \mathcal{P}(L)$ such that $L \in C$ (the logics).

$\Lambda \in C$ is *Post complete in* C iff
$\Lambda \neq L$ and there is no $\Lambda' \in C$ such that $\Lambda \subset \Lambda' \subset L$.

In short: being Post complete is being a co-atom.

Theorem (Makinson 1971):
There are two logics Post complete in normal modal logics,

$\mathbf{Triv} = \mathbf{K}\Box p \leftrightarrow p$ and

$\mathbf{Ver} = \mathbf{K}\Box p$

Post completeness

Let L be a set (the formulas);
let $C \subseteq \mathcal{P}(L)$ such that $L \in C$ (the logics).

$\Lambda \in C$ is *Post complete in* C iff
$\Lambda \neq L$ and there is no $\Lambda' \in C$ such that $\Lambda \subset \Lambda' \subset L$.

In short: being Post complete is being a co-atom.

Theorem (Makinson 1971):
There are two logics Post complete in normal modal logics,

$\mathbf{Triv} = \mathbf{K}\Box p \leftrightarrow p$ and

$\mathbf{Ver} = \mathbf{K}\Box p$

What about other lattices of modal logics?

Congruential modal logics

$\mathcal{L}$: propositional language with operators $\top$, $\neg$, $\wedge$ and $\Box$.

Congruential modal logics

$\mathcal{L}$: propositional language with operators $\top$, $\neg$, $\wedge$ and $\Box$.

Modal logic:
$\Lambda \subseteq \mathcal{L}$ containing all tautologies and closed under MP and US.

Congruential modal logics

$\mathcal{L}$: propositional language with operators $\top$, $\neg$, $\wedge$ and $\Box$.

Modal logic:
$\Lambda \subseteq \mathcal{L}$ containing all tautologies and closed under MP and US.

Congruential modal logic (CML):
Modal logic Λ closed under $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$.

Congruential modal logics

$\mathcal{L}$: propositional language with operators $\top$, $\neg$, $\wedge$ and $\Box$.

Modal logic:
$\Lambda \subseteq \mathcal{L}$ containing all tautologies and closed under MP and US.

Congruential modal logic (CML):
Modal logic Λ closed under $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$.

Modal algebra:
$\mathfrak{A} = \langle A, 1, -, \sqcap, * \rangle$ such that $\langle A, 1, -, \sqcap \rangle$ is a Boolean algebra and $* : A \to A$.

Congruential modal logics

$\mathcal{L}$: propositional language with operators $\top$, $\neg$, $\wedge$ and $\Box$.

Modal logic:
$\Lambda \subseteq \mathcal{L}$ containing all tautologies and closed under MP and US.

Congruential modal logic (CML):
Modal logic Λ closed under $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$.

Modal algebra:
$\mathfrak{A} = \langle A, 1, -, \sqcap, * \rangle$ such that $\langle A, 1, -, \sqcap \rangle$ is a Boolean algebra and $* : A \to A$.

$\Lambda(\mathfrak{A})$, the *logic of* $\mathfrak{A}$:
$\{\varphi \in \mathcal{L} : \varphi$ mapped to 1 by all interpretations in $\mathfrak{A}\}$

Congruential modal logics

$\mathcal{L}$: propositional language with operators $\top$, $\neg$, $\wedge$ and $\Box$.

Modal logic:
$\Lambda \subseteq \mathcal{L}$ containing all tautologies and closed under MP and US.

Congruential modal logic (CML):
Modal logic Λ closed under $\varphi \leftrightarrow \psi / \Box\varphi \leftrightarrow \Box\psi$.

Modal algebra:
$\mathfrak{A} = \langle A, 1, -, \sqcap, * \rangle$ such that $\langle A, 1, -, \sqcap \rangle$ is a Boolean algebra and $* : A \to A$.

$\Lambda(\mathfrak{A})$, the *logic of* $\mathfrak{A}$:
$\{\varphi \in \mathcal{L} : \varphi$ mapped to 1 by all interpretations in $\mathfrak{A}\}$

Theorem (Hansson & Gärdenfors 1973):
$\Lambda \subseteq \mathcal{L}$ is a CML iff Λ is the logic of some modal algebra.

A continuum of Post complete logics

C-Post complete:
Post complete in (the lattice of) congruential modal logics.

A continuum of Post complete logics

C-Post complete:
Post complete in (the lattice of) congruential modal logics.

Theorem:
The number of C-Post complete modal logics is $\beth_1$.

A continuum of Post complete logics

C-Post complete:
Post complete in (the lattice of) congruential modal logics.

Theorem:
The number of C-Post complete modal logics is $\beth_1$.

Proof:
By Lindenbaum's Lemma, every consistent CML can be extended to a C-Post complete one.

A continuum of Post complete logics

C-Post complete:
Post complete in (the lattice of) congruential modal logics.

Theorem:
The number of C-Post complete modal logics is $\beth_1$.

Proof:
By Lindenbaum's Lemma, every consistent CML can be extended to a C-Post complete one.

So it suffices to construct $\beth_1$ CMLs such that any two of them have an inconsistent join.

A continuum of Post complete logics

C-Post complete:
Post complete in (the lattice of) congruential modal logics.

Theorem:
The number of C-Post complete modal logics is $\beth_1$.

Proof:
By Lindenbaum's Lemma, every consistent CML can be extended to a C-Post complete one.

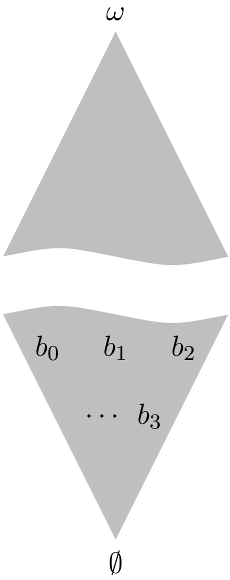
So it suffices to construct $\beth_1$ CMLs such that any two of them have an inconsistent join.

We construct one for every set of natural numbers $S \subseteq \omega$.

A continuum of Post complete logics

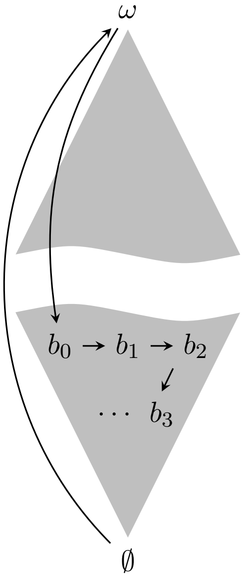
$\mathfrak{A}_S$ based on algebra of finite/cofinite subsets of ω.

A continuum of Post complete logics



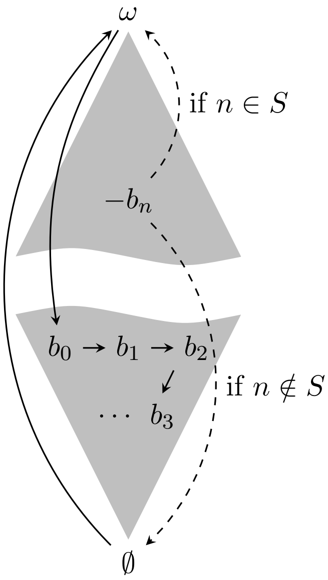
$\mathfrak{A}_S$ based on algebra of finite/cofinite subsets of ω.

A continuum of Post complete logics



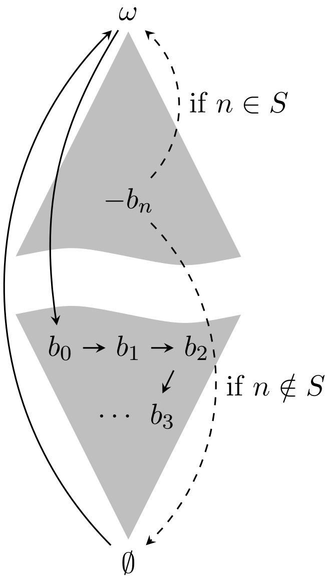
$\mathfrak{A}_S$ based on algebra of finite/cofinite subsets of ω.

A continuum of Post complete logics



$\mathfrak{A}_S$ based on algebra of finite/cofinite subsets of ω.

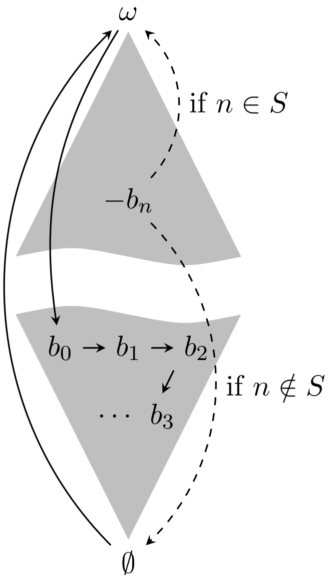
A continuum of Post complete logics



$\mathfrak{A}_S$ based on algebra of finite/cofinite subsets of ω.

Consider $\varphi_n = \Box\neg\Box^n\Box\top$

A continuum of Post complete logics

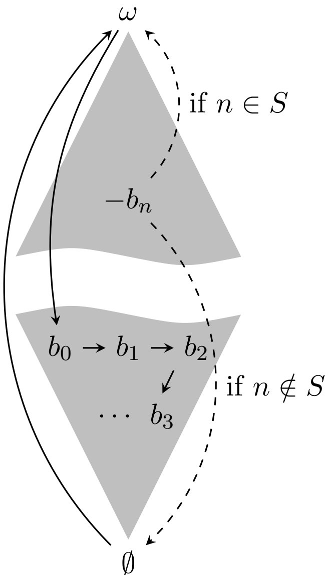


$\mathfrak{A}_S$ based on algebra of finite/cofinite subsets of ω.

Consider $\varphi_n = \Box\neg\Box^n\Box\top$

$\varphi_n \in \Lambda(\mathfrak{A}_S)$ iff $n \in S$
$\neg\varphi_n \in \Lambda(\mathfrak{A}_S)$ iff $n \notin S$

A continuum of Post complete logics



$\mathfrak{A}_S$ based on algebra of finite/cofinite subsets of ω.

Consider $\varphi_n = \Box\neg\Box^n\Box\top$

$\varphi_n \in \Lambda(\mathfrak{A}_S)$ iff $n \in S$
$\neg\varphi_n \in \Lambda(\mathfrak{A}_S)$ iff $n \notin S$

$\Box$

Neighborhood Semantics

Neighborhood frame:
Pair $\langle W, N\rangle$ such that W is a set and $N : \mathcal{P}(W) \to \mathcal{P}(W)$.

Neighborhood Semantics

Neighborhood frame:
Pair $\langle W, N\rangle$ such that W is a set and $N : \mathcal{P}(W) \to \mathcal{P}(W)$.

$\langle W, N, V\rangle, w \vDash \Box\varphi$ iff $w \in N(\{v \in W : \langle W, N, V\rangle, v \vDash \varphi\})$

Neighborhood Semantics

Neighborhood frame:
Pair $\langle W, N \rangle$ such that W is a set and $N : \mathcal{P}(W) \to \mathcal{P}(W)$.

$\langle W, N, V \rangle, w \models \Box\varphi$ iff $w \in N(\{v \in W : \langle W, N, V \rangle, v \models \varphi\})$

Neighborhood frames are (effectively) modal algebras based on powerset algebras.

Neighborhood Semantics

Neighborhood frame:
Pair $\langle W, N\rangle$ such that W is a set and $N : \mathcal{P}(W) \to \mathcal{P}(W)$.

$\langle W, N, V\rangle, w \vDash \Box\varphi$ iff $w \in N(\{v \in W : \langle W, N, V\rangle, v \vDash \varphi\})$

Neighborhood frames are (effectively) modal algebras based on powerset algebras.

Theorem:
There are at least $\aleph_0$ C-Post complete modal logics each of which is the logic of a class of neighborhood frames.

Neighborhood Semantics

Neighborhood frame:
Pair $\langle W, N\rangle$ such that W is a set and $N : \mathcal{P}(W) \to \mathcal{P}(W)$.

$\langle W, N, V\rangle, w \vDash \Box\varphi$ iff $w \in N(\{v \in W : \langle W, N, V\rangle, v \vDash \varphi\})$

Neighborhood frames are (effectively) modal algebras based on powerset algebras.

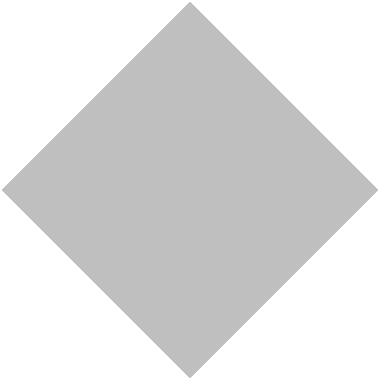
Theorem:
There are at least $\aleph_0$ C-Post complete modal logics each of which is the logic of a class of neighborhood frames.

Proof:
We construct one as $\Lambda(\mathfrak{A}_n)$ for each $n < \omega$.

Neighborhood Semantics

$\mathfrak{A}_n$ based on $\mathcal{P}(n)$; $\Lambda_n = \Lambda(\mathfrak{A}_n)$

$n = \{0, \ldots, n-1\}$

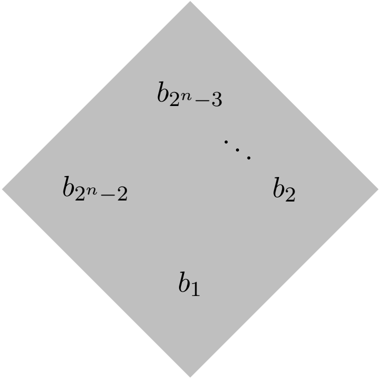


$\emptyset = b_0$

Neighborhood Semantics

$\mathfrak{A}_n$ based on $\mathcal{P}(n)$; $\Lambda_n = \Lambda(\mathfrak{A}_n)$

$n = \{0, \ldots, n-1\}$

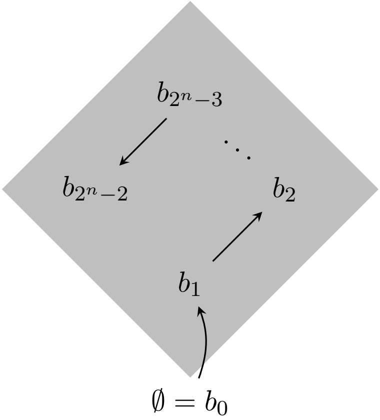


$\emptyset = b_0$

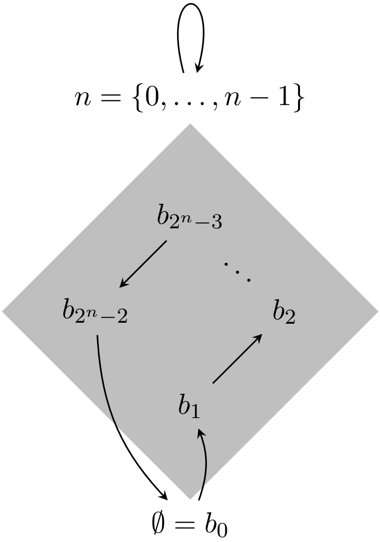
Neighborhood Semantics

$\mathfrak{A}_n$ based on $\mathcal{P}(n)$; $\Lambda_n = \Lambda(\mathfrak{A}_n)$

$n = \{0, \ldots, n-1\}$

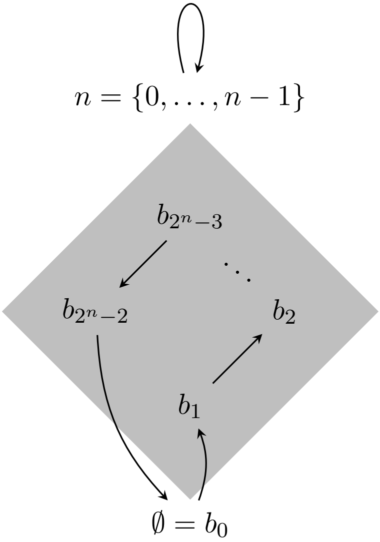


Neighborhood Semantics



$\mathfrak{A}_n$ based on $\mathcal{P}(n)$; $\Lambda_n = \Lambda(\mathfrak{A}_n)$

Neighborhood Semantics



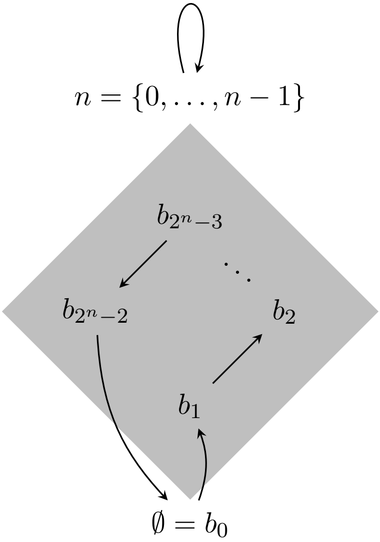
$\mathfrak{A}_n$ based on $\mathcal{P}(n)$; $\Lambda_n = \Lambda(\mathfrak{A}_n)$

If $n < n'$, then

$\neg\Box^{2^n-1}\bot \in \Lambda_n$

$\neg\Box^{2^n-1}\bot \notin \Lambda_{n'}$

Neighborhood Semantics



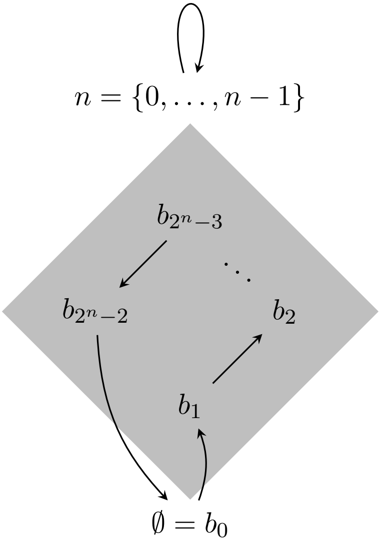
$\mathfrak{A}_n$ based on $\mathcal{P}(n)$; $\Lambda_n = \Lambda(\mathfrak{A}_n)$

If $n < n'$, then

$$\neg\Box^{2^n-1}\bot \in \Lambda_n$$
$$\neg\Box^{2^n-1}\bot \notin \Lambda_{n'}$$

Let $\Lambda \supset \Lambda_n$ and $\varphi \in \Lambda \backslash \Lambda_n$. Mapped to non-top element by some interpretation; replace proposition letters by "definitions" accordingly: φ'.

Neighborhood Semantics



$\mathfrak{A}_n$ based on $\mathcal{P}(n)$; $\Lambda_n = \Lambda(\mathfrak{A}_n)$

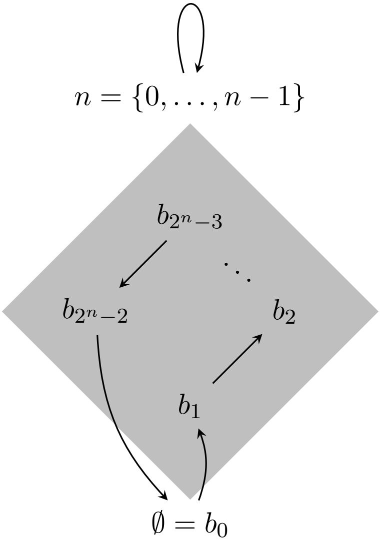
If $n < n'$, then

$$\neg\Box^{2^n-1}\bot \in \Lambda_n$$
$$\neg\Box^{2^n-1}\bot \notin \Lambda_{n'}$$

Let $\Lambda \supset \Lambda_n$ and $\varphi \in \Lambda \backslash \Lambda_n$. Mapped to non-top element by some interpretation; replace proposition letters by "definitions" accordingly: φ'.

$\neg\Box^k\varphi' \in \Lambda_n$ for some k. $\Box^k\top \leftrightarrow \Box^k\varphi' \in \Lambda$. But $\Box^k\top \in \Lambda_n \subseteq \Lambda$, so $\Box^k\varphi' \in \Lambda$. So $\Lambda = \mathcal{L}$.

Neighborhood Semantics



$\mathfrak{A}_n$ based on $\mathcal{P}(n)$; $\Lambda_n = \Lambda(\mathfrak{A}_n)$

If $n < n'$, then

$$\neg\Box^{2^n-1}\bot \in \Lambda_n$$
$$\neg\Box^{2^n-1}\bot \notin \Lambda_{n'}$$

Let $\Lambda \supset \Lambda_n$ and $\varphi \in \Lambda \backslash \Lambda_n$. Mapped to non-top element by some interpretation; replace proposition letters by "definitions" accordingly: φ'.

$\neg\Box^k\varphi' \in \Lambda_n$ for some k. $\Box^k\top \leftrightarrow \Box^k\varphi' \in \Lambda$. But $\Box^k\top \in \Lambda_n \subseteq \Lambda$, so $\Box^k\varphi' \in \Lambda$. So $\Lambda = \mathcal{L}$.

$\Box$

Neighborhood Semantics

Is every C-Post complete modal logic the logic of a class of neighborhood frames?

Neighborhood Semantics

Is every C-Post complete modal logic the logic of a class of neighborhood frames?

Left open in the paper. (In work in progress: *No* – some consistent CML is not valid on any neighborhood frame.)

Truth-functionality

Truth-functional:
The logic of a two-element modal algebra.

Truth-functionality

Truth-functional:
The logic of a two-element modal algebra.

$\emptyset$-*Post complete*:
Post complete in (the lattice of) all modal logics.

Truth-functionality

Truth-functional:
The logic of a two-element modal algebra.

$\emptyset$-*Post complete*:
Post complete in (the lattice of) all modal logics.

Theorem (Makinson 1971/Segerberg 1972):
The following are equivalent for an NML:

- $\emptyset$-Post completeness
- Truth-functionality
- Post completeness in normal/congruential modal logics

Truth-functionality

Truth-functional:
The logic of a two-element modal algebra.

$\emptyset$-*Post complete*:
Post complete in (the lattice of) all modal logics.

Theorem (Makinson 1971/Segerberg 1972):
The following are equivalent for an NML:

- $\emptyset$-Post completeness
- Truth-functionality
- Post completeness in normal/congruential modal logics

Theorem:
The following are equivalent for a CML:

- $\emptyset$-Post completeness
- Truth-functionality

Truth-functionality

Truth-functional:
The logic of a two-element modal algebra.

$\emptyset$-*Post complete*:
Post complete in (the lattice of) all modal logics.

Theorem (Makinson 1971/Segerberg 1972):
The following are equivalent for an NML:

- $\emptyset$-Post completeness
- Truth-functionality
- Post completeness in normal/congruential modal logics

Theorem:
The following are equivalent for a CML:

- $\emptyset$-Post completeness
- Truth-functionality

(The proof is a variant of the proof for NMLs.)

Characterizing intersections

Theorem (Humberstone 2016):
$\bigcap$ NMLs $\emptyset$-Post complete = NML axiomatized by $p \to \Box p$
$\bigcap$ CMLs $\emptyset$-Post complete = CML ax. by $(p \leftrightarrow q) \to (\Box p \leftrightarrow \Box q)$

Characterizing intersections

Theorem (Humberstone 2016):
$\bigcap$ NMLs $\emptyset$-Post complete = NML axiomatized by $p \to \Box p$
$\bigcap$ CMLs $\emptyset$-Post complete = CML ax. by $(p \leftrightarrow q) \to (\Box p \leftrightarrow \Box q)$

Question: Can we characterize $\bigcap$ $\emptyset$-Post complete modal logics closed under some rules using the corresponding conditionals?

Characterizing intersections

Theorem (Humberstone 2016):
$\bigcap$ NMLs $\emptyset$-Post complete = NML axiomatized by $p \to \Box p$
$\bigcap$ CMLs $\emptyset$-Post complete = CML ax. by $(p \leftrightarrow q) \to (\Box p \leftrightarrow \Box q)$

Question: Can we characterize $\bigcap$ $\emptyset$-Post complete modal logics closed under some rules using the corresponding conditionals?

Theorem:

$$\bigcap(\emptyset\text{-Post} \cap L(R)) = \varepsilon_0(\Lambda_\emptyset(\overrightarrow{R})).$$

(See paper for details and proof.)

Characterizing intersections (details)

(Substitution-invariant) rule:
Set of finite non-empty sequences of formulas closed under US.

$\Gamma \subseteq \mathcal{L}$ *closed under* a rule R:
If $\langle \rho_0, \ldots, \rho_n \rangle \in R$ and $\rho_i \in \Gamma$ for all $i < n$, then $\rho_n \in \Gamma$.

- $\emptyset$-Post(Γ): set of $\emptyset$-Post complete modal logics extending Γ
- $L(R)$: set of modal logics closed under R
- $\overrightarrow{R} = \{\bigwedge_{i<n} \rho_i \to \rho_n : \langle \rho_0, \ldots, \rho_n \rangle \in R\}$
- $\Lambda_\emptyset(\Gamma)$: modal logic axiomatized by Γ
- $\varepsilon_0(\Gamma) = \{\varphi \in \mathcal{L} :$ all substitution instances of φ without proposition letters are in $\Gamma\}$

Theorem: $\bigcap(\emptyset\text{-Post}(\Gamma) \cap L(R)) = \varepsilon_0(\Lambda_\emptyset(\Gamma \cup \overrightarrow{R}))$.

Open Question: How can we characterize $\bigcap R$-Post(Γ)?

A CML without neighborhood frames

The CML axiomatized by

$$(A1)\quad (\Box\top \wedge p) \leftrightarrow \Box(\Box\top \to (p \wedge \Box(\Box\top \wedge p)))$$
$$(A2)\quad (\Box\top \wedge p) \leftrightarrow \Box(\Box\top \to (p \wedge \neg\Box(\Box\top \wedge p)))$$
$$(A3)\quad \Box\Box\Box\top$$
$$(A4)\quad \neg\Box\bot$$

is not valid on any modal algebra based on an atomic Boolean algebra.