Computability of definability in the class of all KD45 frames

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Introduction

We consider a first-order language FOL with a single binary predicate symbol r, the basic modal language $\mathrm{ML}(\Box)$ and the basic modal language with the added universal modality $\mathrm{ML}(\Box, [U])$.

A *Kripke frame* is an ordered pair of the kind $\langle W, R \rangle$, where W is a non-empty set and $R \subseteq W \times W$ is a binary relation over W.

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The Correspondence Problems

First-Order Definability:
Given a modal formula A, decide if there is a first-order formula ψ such that for every Kripke frame F: $\mathrm{F} \Vdash A$ iff $\mathrm{F} \vDash \psi$.

Modal Definability:
Given first-order formula ψ, decide if there is a $\mathrm{ML}(\Box)$ formula A such that for every Kripke frame F: $\mathrm{F} \vDash \psi$ iff $\mathrm{F} \Vdash A$.

These problems ware answered in Lidia Chagrova's theorem:

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Let C_{S5} be the class of all $S5$-frames (all frames with an equivalence relation).

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Let C_{KD45} be the class of all KD45-frames (all frames whose relation is serial, transitive and Euclidean).

Are the correspondence problems over C_{KD45} computable?

We show that:

- Every $\mathrm{ML}(\Box)$ formula is first-order definable over C_{KD45}.
- Modal definability of FOL formulas in the language $\mathrm{ML}(\Box)$ over C_{KD45} is PSPACE-complete.
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We say that a frame $\mathrm{F} = \langle W, R \rangle$ is a *daisy* iff $W = P(\mathrm{F}) \cup S(\mathrm{F})$, where $P(\mathrm{F}) \cap S(\mathrm{F}) = \emptyset$, $S(\mathrm{F}) \neq \emptyset$, $P(\mathrm{F})$ is the set of *petals*, $S(\mathrm{F})$ is the set of *stamens*, and the following hold:

(Daisy 1). $\forall x \in P(\mathrm{F}) \neg \exists y \in W(\langle y, x \rangle \in R)$

(Daisy 2). $\forall x \in P(\mathrm{F}) \forall y \in S(\mathrm{F})(\langle x, y \rangle \in R)$

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Any KD45-frame F is a disjoint union of daisies.

Let C_0 be the class of finite daisies without petals (equivalence classes).
Let C_1 be the class of finite daisies with a single petal.
Denote by D_i the finite daisy without petals and i stamens.
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Lemma

Let A be a $\mathrm{ML}(\Box)$-formula.
Exactly one of the following three holds: either $C_{S5} \Vdash A$, $D_1 \nVdash A$, or there is a number $n > 1$, such that for all i: $D_i \Vdash A \Leftrightarrow i < n$.
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Denote $\psi_n(x) =_{def} \forall y_1 \ldots \forall y_n(\bigwedge\{(x\ r\ y_k) \mid 1 \leq k \leq n\} \rightarrow \bigvee\{(y_k = y_\ell) \mid 1 \leq k < \ell \leq n\})$ for $n \geq 1$.

Theorem

Let A be a $\mathrm{ML}(\Box)$-formula. Then there is a first-order formula ψ, such that A and ψ are globally correspondent over the class of frames C_{KD45}. Also, ψ can be effectively computed.

Sketch of proof. The definition of A over C_0, ψ_{C_0}, is either $\top$ ($C_{\mathrm{S5}} \Vdash A$), $\bot$ ($D_1 \nVdash A$), or $\psi_n(x)$ for some $n > 1$. The definition of A over C_1, ψ_{C_1}, is either $\top$ ($C_{\mathrm{KD45}} \Vdash A$), $\bot$ ($D_1' \nVdash A$), or $\psi_{n'}(x)$ for some $n' > 1$. It can be shown that a definition of A over C_{KD45} is:
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Definitions

Let $\mathrm{F} \in C_{\mathrm{KD45}}$ and D be the unique set of daisies, up to isomorphism, such that F is the disjoint union of D, i.e. $\mathrm{F} = \biguplus D$.
Let $s(\mathrm{F}) =_{def} sup(\{Card(S(x)) \mid x \in D\})$.
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For $n \geq 1$, denote:
$A_n =_{def} \bigwedge\{\Diamond p_k \mid 1 \leq k \leq n\} \to \bigvee\{\Diamond(p_i \wedge p_j) \mid 1 \leq i < j \leq n\}$.
Denote $A_0 =_{def} \bot$ and $A_\omega =_{def} \top$.

Lemma

For all $n \geq 1$, $\psi_n(x)$ is locally correspondent to A_n with respect to C_{KD45}.

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For all $1 \leq i \leq j$, $\forall x(((x\, r\, x) \wedge \psi_j) \vee (\neg(x\, r\, x) \wedge \psi_i))$ is globally correspondent to $((q \to \Diamond q) \wedge A_j) \vee A_i$ with respect to C_{KD45}.

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Let ψ be a FOL sentence. Then ψ is modally definable iff there are ordinals σ_0, σ_1 such that $0 \leq \sigma_1 \leq \sigma_0 \leq \omega$ and for every $\mathrm{F} \in C^b$: $\mathrm{F} \vDash \psi$ iff $s_0(\mathrm{F}) \leq \sigma_0$ and $s_1(\mathrm{F}) \leq \sigma_1$.

Sketch of proof. The right-to-left direction is easier, using the properties of generated subframes and p-morphisms (bounded morphisms). For the other direction, we show by examining nine cases (using the properties of Ehrenfeucht-Fraïssé games, generated subframes and p-morphic images) that the following ML(□) formula is a definition of ψ:
$A =_{def} ((q \to \Diamond q) \wedge A_{\alpha_0}) \vee A_{\alpha_1}$, where for $i \in \{0, 1\}$, α_i is either ω if σ_i is ω, or $\sigma_i + 1$, otherwise. □

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Using the previous lemma, it is not hard to show that:

Theorem

The problem of modal definability of a FOL sentence ψ over the class C_{KD45} is in PSPACE.

Sketch of proof. Let $\psi_0 =_{def} \bot, \psi_\omega =_{def} \top$. Let m be the quantifier rank of ψ. Let $Q =_{def} \{\psi_0, \psi_1, \ldots, \psi_m, \psi_{m+1}, \psi_\omega\}$, which is a finite set. The proof works by showing that ψ is modally definable iff there are ordinals α_0, α_1 such that $\psi_{\alpha_0}, \psi_{\alpha_1} \in Q$, $0 \le \alpha_1 \le \alpha_0 \le \omega$ and $C_{\mathrm{KD45}} \vDash \psi \leftrightarrow \forall x(((x\, r\, x) \wedge \psi_{\alpha_0}) \vee (\neg(x\, r\, x) \wedge \psi_{\alpha_1}))$. □

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Let C be a class of frames. We say that C is *stable* with respect to a modal language L iff there is a FOL formula $\psi_1(\bar{x}, x)$ and a FOL sentence ψ_2, such that:

(a) for all frames F in C, for all lists $\bar{w}$ of worlds in F, and for all frames F', if F' is the relativized reduct of F with respect to $\psi_1(\bar{x}, x)$ and $\bar{w}$, then F' is in C,

(b) for all frames F_0 in C, there are frames F, F' in C and there is a list $\bar{w}$ of worlds in F, such that F_0 is the relativized reduct of F with respect to $\psi_1(\bar{x}, x)$ and $\bar{w}$, $\mathrm{F} \models \psi_2$, $\mathrm{F}' \not\models \psi_2$, and for all modal L-formulas ϕ: if $\mathrm{F} \Vdash \phi$, then $\mathrm{F}' \Vdash \phi$.

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Theorem (P. Balbiani, T. Tinchev)

If C is a stable class of frames with respect to the modal language L, then the problem of deciding the validity of FOL *sentences in C is reducible to the problem of deciding the modal definability of* FOL *sentences in the language L with respect to C.*

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C_{KD45} *is a stable class with respect to* ML(□).

Theorem

The problem of modal definability of FOL *formulas over* C_{KD45} *is PSPACE-hard.*

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The problem of modal definability of FOL formulas over C_{KD45} is PSPACE-hard.

Definitions

Denote by C_b the class of all finite KD45-frames.

Let $\mathrm{G} \in C_b$, m be the maximal number of petals in a daisy in G, and n be the maximal number of stamens in a daisy in G. The *pattern of* G is the matrix $[x_{ij}]_{\substack{0 \le i \le m, \\ 1 \le j \le n}}$ where x_{ij} is the number of daisies in G with i petals and j stamens.

Let $\mathrm{G} \in C_b$. We define a $\mathrm{ML}(\Box, [U])$ formula A_{G}, *the Jankov-Fine formula of* G, with the following properties:

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Let $\mathrm{F} \in C_{\mathrm{KD45}}$, $\mathrm{G} \in C_b$. *Let* A_{G} *be the Jankov-Fine formula of* G. *Then* $\mathrm{F} \Vdash \neg A_{\mathrm{G}}$ *iff* G *is not a p-morphic image of* F.

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Definitions

Let $k > 0$. We denote by C_b^k the class of finite KD45-frames with at most k daisies, each of them with at most k petals and k stamens. When discussing patterns of frames from C_b^k, we only consider $(k+1) \times k$ matrices.

Pattern Transformation 1

This is our first pattern transformation:

Lemma

Let ψ be a sentence with quantifier rank k and modally definable by an $\mathrm{ML}(\Box, [U])$ formula A over C_{KD45}. If there is some $\mathrm{F} \in C_b$ with pattern $\mathcal{P}_1$, where some $x_{0j} = k$ and $\mathrm{F} \vDash \psi$, then there is a frame $\mathrm{F}' \in C_b$ with a pattern $\mathcal{P}_2$, which is equal to $\mathcal{P}_1$, except that all $x_{01} = \cdots = x_{0j} = k$, and $\mathrm{F}' \vDash \psi$.

Sketch of proof. Using the properties of p-morphisms and Ehrenfeucht-Fraïssé games.

Pattern Transformation 2

This is our second pattern transformation:

Lemma

Let ψ be a sentence with quantifier rank k and modally definable by a formula A from $\mathrm{ML}(\Box, [U])$ over C_{KD45}. If there is some $\mathrm{F} \in C_b$ with pattern $\mathcal{P}_1$, where there is some $x_{ij} = k$ with $i > 0$, and F is such that $\mathrm{F} \models \psi$, then there is a frame $\mathrm{F}' \in C_b$ with a pattern $\mathcal{P}_2$ such that $\mathrm{F}' \models \psi$ and $\mathcal{P}_2$ is equal to $\mathcal{P}_1$, except that $x_{01} = \cdots = x_{0j} = \cdots = x_{m0} = \cdots = x_{mj} = k$.

Sketch of proof. Using the properties of p-morphisms and Ehrenfeucht-Fraïssé games.

Let ψ be a sentence with quantifier rank k. Denote by $C_b^k(\psi)$ the class off all $\mathrm{F} \in C_b^k$ such that $\mathrm{F} \vDash \psi$.

Theorem

Let ψ be a sentence with quantifier rank k, let $C_{\mathrm{KD45}} \nvDash \psi$ and $C_{\mathrm{KD45}} \nvDash \neg\psi$. Then ψ is modally definable over C_{KD45} with a formula of $\mathrm{ML}(\Box, [U])$ iff $C_b^k(\psi)$ satisfies the following conditions:
(1) $\emptyset \neq C_b^k(\psi) \neq C_b^k$; and
(2) $C_b^k(\psi)$ is closed under p-morphisms and the two pattern transformations.

This guarantees that the problem of modal definability of FOL formulas in the language $\mathrm{ML}(\Box, [U])$ over C_{KD45} is in PSPACE.

Lemma

Let ψ be a FOL sentence. The problem of deciding $C_{\mathrm{KD45}} \models \psi$ is PSPACE-complete.

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Let ψ be a sentence with quantifier depth k. Let τ_k be a FOL sentence which says 'there are at least k^4 daisies, each with at least $k+1$ petals and $k+1$ stamens'. Then $\psi \vee \tau_k$ is modally definable in $\mathrm{ML}(\Box, [U])$ over C_{KD45} iff $C_{\mathrm{KD45}} \models \psi$.

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Conclusion

We have shown that:

- Every $\mathrm{ML}(\Box)$ formula is first-order definable over C_{KD45}.
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Future work

- First-order definability of $\mathrm{ML}(\Box, [U])$ formulas over C_{KD45} must be examined.
- Definability in C_{K45} and C_{K5} may also be explored.
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