The Logic of Conditional Beliefs: Neighbourhood Semantics and Sequent Calculus

Marianna Girlando, Sara Negri, Nicola Olivetti, Vincent Risch

Aix Marseille Université, Laboratoire des Sciences de l'Information et des Systèmes;
University of Helsinki, Department of Philosophy

Advances in Modal Logics
Budapest, August 30 - September 2, 2016

Outline

(1) The logic $\mathbb{CDL}$

(2) Semantics

(3) Labelled Sequent Calculus

(4) Main results: Soundness, Termination and Completeness

(5) Conclusions

Outline

(1) The logic $\mathbb{CDL}$

(2) Semantics

(3) Labelled Sequent Calculus

(4) Main results: Soundness, Termination and Completeness

(5) Conclusions

The Logic of Conditional Beliefs ($\mathbb{CDL}$)

The Logic of Conditional Beliefs

Multi-agent modal epistemic logic, featuring the *conditional belief* operator:

$Bel_i(B|A)$, *"agent i believes B having learnt A"*

Three-wise-men puzzle

- *Agent a believes that she is wearing a white hat:* $Bel_a W_a$
- *Agent a learns that agent b knows the colour of the hat that b herself is wearing, and changes her beliefs: she is now convinced that she is wearing a black hat:* $Bel_a(B_a|K_b W_b \vee K_b B_b)$

References

Baltag and Smets (2006); Baltag and Smets (2008); Board (2004); Pacuit (2013).

The Logic of Conditional Beliefs ($\mathbb{CDL}$)

Language of $\mathbb{CDL}$

$$A \ := \ P \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A \mid Bel_i(A|A)$$

Epistemic operators

- Conditional belief (primitive): $Bel_i(C|B)$, "*agent i believes C, given B*"
- Unconditional belief (defined): $Bel_i B =_{df} Bel_i(B|\top)$, "*agent i believes B*"
- Knowledge (defined): $K_i B =_{df} Bel_i(\bot|\neg B)$, "*agent i knows B*"

Axiomatic presentation of $\mathbb{CDL}$ [Board, 2004]

Inference rules

(1) If $\vdash B$, then $\vdash Bel_i(B|A)$ (epistemization rule)

(2) If $\vdash A \supset\subset B$, then $\vdash Bel_i(C|A) \supset\subset Bel_i(C|B)$ (rule of logical equivalence)

Axioms

Any axiomatization of the classical propositional calculus, plus:

(3) $(Bel_i(B|A) \wedge Bel_i(B \supset C|A)) \supset Bel_i(C|A)$ (distribution axiom)

(4) $Bel_i(A|A)$ (success axiom)

(5) $Bel_i(B|A) \supset\subset (Bel_i(C|A \wedge B) \supset Bel_i(C|A))$ (minimal change principle 1)

(6) $\neg Bel_i(\neg B|A) \supset (Bel_i(C|A \wedge B) \supset\subset Bel_i(B \supset C|A))$ (minimal change principle 2)

(7) $Bel_i(B|A) \supset Bel_i(Bel_i(B|A)|C)$ (positive introspection)

(8) $\neg Bel_i(B|A) \supset Bel_i(\neg Bel_i(B|A)|C)$ (negative introspection)

(9) $A \supset \neg Bel_i(\bot|A)$ (consistency axiom)

The axiomatization is related to the AGM postulates of belief revision.

Outline

(1) The logic $\mathbb{CDL}$

(2) Semantics

(3) Labelled Sequent Calculus

(4) Main results: Soundness, Termination and Completeness

(5) Conclusions

Epistemic Plausibility Models for $\mathbb{CDL}$

Epistemic plausibility models [Board, 2004; Baltag and Smets, 2008; Pacuit, 2013]

Let $\mathcal{A}$ be a set of agents; an *epistemic plausibility model* (EPM) has the form

$$\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\leq_i\}_{i\in\mathcal{A}}, [\![\ ]\!]\rangle$$

where

- W is a non-empty set of elements called “worlds”;
- for each $i \in \mathcal{A}$, $\sim_i$ is an equivalence relation over W;
- for each $i \in \mathcal{A}$, $\leq_i$ is a well-founded pre-order over W;
- $[\![\ ]\!]$: *Atm* $\rightarrow \mathcal{P}(W)$ is the evaluation for atomic formulas.

The relations $\sim_i$ and $\leq_i$ satisfy the following properties:

- *Plausibility implies possibility*: If $w \leq_i v$ then $w \sim_i v$
- *Local connectedness*: If $w \sim_i v$ then $w \leq_i v$ or $v \leq_i w$

Epistemic Plausibility Models for $\mathbb{CDL}$

Truth conditions for formulas in EPM

- $[\![\neg A]\!] \equiv W - [\![A]\!]$
- $[\![A \wedge B]\!] \equiv [\![A]\!] \cap [\![B]\!]$
- $[\![A \vee B]\!] \equiv [\![A]\!] \cup [\![B]\!]$
- $[\![A \supset B]\!] \equiv (W - [\![A]\!]) \cup [\![B]\!]$
- $[\![Bel_i(B|A)]\!] \equiv \{x \in W \,|\, Min_{\leq_i}([x]_{\sim_i} \cap [\![A]\!]) \subseteq [\![B]\!]\}$
  where $[x]_{\sim_i} = \{w \mid w \sim_i x\}$
  and $Min_{\leq_i}(S) = \{u \in S \mid \forall z \in S \; (u \leq_i z)\}$

Theorem: Completeness of the axiomatization [Board, 2004]

A formula A is a theorem of $\mathbb{CDL}$ if and only if it is valid in the class of epistemic plausibility models.

Neighbourhood Models for $\mathbb{CDL}$

Neighbourhood models

- These models associate to each world a set of sets of worlds, used to interpret modalities; they were originally proposed to give an interpretation of non-normal modal logics: Scott (1970), Montague (1970), Chellas (1980)...
- Semantics of counterfactuals: Sphere models, Lewis (1973);
- Semantics of belief revision: Grove (1988);
- Studied recently also by Pacuit (2007); Marti and Pinosio (2013); Negri and Olivetti (2015); Negri (2016).

Neighbourhood Models for $\mathbb{CDL}$

Multi-agent neighbourhood models

Let $\mathcal{A}$ be a set of agents; a *multi-agent neighbourhood model* (NM) has the form

$$\mathcal{M} = \langle W, \{I\}_{i \in \mathcal{A}}, [\![\ ]\!] \rangle$$

where

- W is a non empty set of elements called "worlds" ;
- for each $i \in \mathcal{A}$, $I_i : W \to \mathcal{P}(\mathcal{P}(W))$ is the neighbourhood function, satisfying the following properties:
  - *Non-emptiness*: $\forall \alpha \in I_i(x), \alpha \neq \emptyset$
  - *Nesting*: $\forall \alpha, \beta \in I_i(x), \alpha \subseteq \beta$ or $\beta \subseteq \alpha$
  - *Total reflexivity*: $\exists \alpha \in I_i(x)$ such that $x \in \alpha$
  - *Local absoluteness*: If $\alpha \in I_i(x)$ and $y \in \alpha$ then $I_i(x) = I_i(y)$
  - *Closure under intersection*: If $S \subseteq I_i(x)$ and $S \neq \emptyset$ then $\bigcap S \in S$ (always holds in finite models)
- $[\![\ ]\!]$: *Atm* $\to \mathcal{P}(W)$ is the evaluation for atomic formulas.

Neighbourhood Models for $\mathbb{CDL}$

Forcing relation [Negri, 2016]

- variables for worlds: $x, y, z \dots$
- variables for neighbourhoods: $\alpha, \beta, \gamma \dots$
- "*x forces A* ", for A formula: $x \Vdash A$ iff $x \in [\![A]\!]$
- "α *universally forces A* ": $\alpha \Vdash^{\forall} A$ iff $\forall y \in \alpha\ (y \Vdash A)$
- "α *existentially forces A* ": $\alpha \Vdash^{\exists} A$ iff $\exists y \in \alpha\ (y \Vdash A)$

Truth conditions for formulas in NM

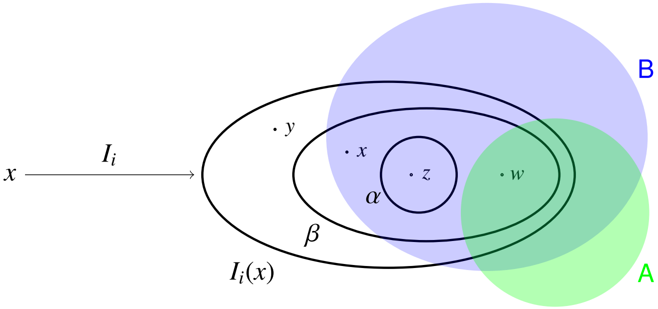
- Truth conditions for propositional formulas are the ones defined for EPM

Conditional Belief

Truth condition

$$x \Vdash Bel_i(B|A) \quad \text{iff} \quad \forall \alpha \in I_i(x)(\alpha \cap [\![A]\!] = \emptyset) \text{ or } \exists \beta \in I_i(x)(\beta \cap [\![A]\!] \neq \emptyset \text{ and } \beta \cap [\![A]\!] \subseteq [\![B]\!])$$

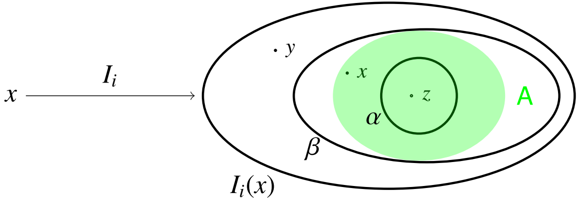
$$\text{iff} \quad \forall \alpha \in I_i(x)(\alpha \Vdash^{\forall} \neg A) \quad \text{or} \quad \exists \beta \in I_i(x)(\beta \Vdash^{\exists} A \text{ and } \beta \Vdash^{\forall} A \supset B)$$



Belief

Truth condition

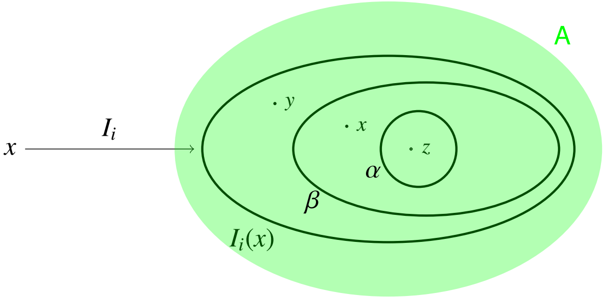
$$x \Vdash Bel_i A \quad \text{iff} \quad \exists \beta \in I_i(x)\ (\beta \subseteq [\![A]\!])$$
$$\quad \text{iff} \quad \exists \beta \in I_i(x)\ (\beta \Vdash^{\forall} A)$$



Knowledge

Truth condition

$$x \Vdash K_i A \quad \textit{iff} \quad \forall \beta \in I_i(x)\ (\beta \subseteq [\![A]\!])$$
$$\quad \textit{iff} \quad \forall \beta \in I_i(x)\ (\beta \Vdash^{\forall} A)$$



Equivalence Between Plausibility Models and Neighbourhood Models

Theorem: Equivalence between models

A formula A is valid in the class of epistemic plausibility models if and only if it is valid in the class of multi-agent neighbourhood models.

Proof.

Generalization of the canonical “topological construction” considered by Pacuit (2013) and Marti and Pinosio (2013), and going back to Alexandroff (1937). □

Corollary: Completeness of the axiomatization

A formula A is a theorem of $\mathbb{CDL}$ if and only if it is valid in the class of neighbourhood models.

Outline

A Labelled Sequent Calculus for $\mathbb{CDL}$

Sequent calculus $G3CDL$

$G3CDL$ is a labelled sequent calculus which internalizes the neighbourhood semantics of $\mathbb{CDL}$.

- labels for worlds: $x, y, z \dots$
- labels for neighbourhoods: $a, b, c \dots$

- $a \Vdash^{\exists} A \;\equiv\; \exists x\,(x \in a \textit{ and } x \Vdash A)$
- $a \Vdash^{\forall} A \;\equiv\; \forall x\,(x \in a \textit{ implies } x \Vdash A)$
- $x \Vdash_i B|A \;\equiv\; \exists c\,(c \in I_i(x) \textit{ and } c \Vdash^{\exists} A \textit{ and } c \Vdash^{\forall} A \supset B)$
- $x \Vdash Bel_i(B|A) \;\equiv\; \forall a \in I_i(x)(a \Vdash^{\forall} \neg A) \textit{ or } x \Vdash_i B|A$

A Labelled Sequent Calculus for $\mathbb{CDL}$

G3CDL Rules (1)

Initial sequents $\qquad x : P, \Gamma \Rightarrow \Delta, x : P$

Rules for local forcing

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A}{\Gamma \Rightarrow \Delta, a \Vdash^{\forall} A} \; R{\Vdash^{\forall}} \; (x \textit{ fresh}) \qquad \frac{x : A, x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{x \in a, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta} \; L{\Vdash^{\forall}}$$

$$\frac{x \in a, \Gamma \Rightarrow \Delta, x : A, a \Vdash^{\exists} A}{x \in a, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A} \; R{\Vdash^{\exists}} \qquad \frac{x \in a, x : A, \Gamma \Rightarrow \Delta}{a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta} \; L{\Vdash^{\exists}} \; (x \textit{ fresh})$$

Propositional rules: rules of **G3K** [Negri 2005]

A Labelled Sequent Calculus for $\mathbb{CDL}$

G3CDL Rules (2)

Rules for conditional belief

$$\frac{a \in I_i(x), a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_i B|A}{\Gamma \Rightarrow \Delta, x : Bel_i(B|A)} \; RB \; (a \; fresh)$$

$$\frac{a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad x \Vdash_i B|A, a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta}{a \in I_i(x), x : Bel_i(B|A), \Gamma \Rightarrow \Delta} \; LB$$

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\exists} A \quad a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A, a \Vdash^{\forall} A \supset B}{a \in I_i(x), \Gamma \Rightarrow \Delta, x \Vdash_i B|A} \; RC$$

$$\frac{a \in I_i(x), a \Vdash^{\exists} A, a \Vdash^{\forall} A \supset B, \Gamma \Rightarrow \Delta}{x \Vdash_i B|A, \Gamma \Rightarrow \Delta} \; LC (a \; fresh)$$

A Labelled Sequent Calculus for $\mathbb{CDL}$

G3CDL Rules (3)

Rules for inclusion

$$\frac{a \subseteq a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathit{Ref} \qquad \frac{c \subseteq a, c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}{c \subseteq b, b \subseteq a, \Gamma \Rightarrow \Delta}\ \mathit{Tr}$$

$$\frac{x \in a, a \subseteq b, x \in b, \Gamma \Rightarrow \Delta}{x \in a, a \subseteq b, \Gamma \Rightarrow \Delta}\ L\subseteq$$

Rules for semantic conditions

$$\frac{a \subseteq b, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta \quad b \subseteq a, a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta}{a \in I_i(x), b \in I_i(x), \Gamma \Rightarrow \Delta}\ S$$

$$\frac{x \in a, a \in I_i(x), \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ T\ (a\ \mathit{fresh})$$

$$\frac{a \in I_i(x), y \in a, b \in I_i(x), b \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, b \in I_i(x), \Gamma \Rightarrow \Delta}\ A_1 \qquad \frac{a \in I_i(x), y \in a, a \in I_i(y), \Gamma \Rightarrow \Delta}{a \in I_i(x), y \in a, \Gamma \Rightarrow \Delta}\ A_2$$

Derivation Example: Axiom (6) $\neg Bel_i(\neg B|A) \supset (Bel_i(B \supset C|A) \supset Bel_i(C|A \wedge B))$

$$\mathcal{D}\quad\left\{\;\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{y : A \cdots \Rightarrow \ldots y : A \qquad y : B \cdots \Rightarrow \ldots y : B}{y : A, y : B, y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots y : A \wedge B}\,(R\wedge)}{y : A, y : B, y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B}\,(R \Vdash^{\exists})}{y \in b, c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, y : A \supset \neg B}\,(R\supset,\ R\neg)}{c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, b \Vdash^{\forall} A \supset \neg B}\,(R \Vdash^{\forall})}{c \in I_i(x), c \Vdash^{\exists} A, b \in I_i(x) \cdots \Rightarrow \ldots b \Vdash^{\exists} A \wedge B, x \Vdash_i \neg B|A}\,(RC)}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots x : Bel_i(\neg B|A), b \Vdash^{\exists} A \wedge B}\,(RB)\right.$$

$$\mathcal{E}\quad\left\{\;\dfrac{\dfrac{\dfrac{\dfrac{z : A \cdots \Rightarrow \ldots z : A \qquad z : c \cdots \Rightarrow \ldots z : C}{z : A \supset C, z : A, z : B, z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B, \cdots \Rightarrow \ldots z : C}\,(L\supset)}{z : A, z : B, z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots z : C}\,(L \Vdash^{\forall})}{z \in b, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots z : (A \wedge B) \supset C}\,(R\supset,\ L\wedge)}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \Vdash^{\exists} A \wedge B \cdots \Rightarrow \ldots b \Vdash^{\forall} (A \wedge B) \supset C}\,(R \Vdash^{\forall})\right.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D} \qquad\qquad \mathcal{E}}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset C, a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\,(RC)}{x \Vdash_i C|A, a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\,(LC)}{a \in I_i(x), a \Vdash^{\exists} A \wedge B, x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x \Vdash_i C|A \wedge B}\,(LB)}{x : Bel_i(C|A) \Rightarrow x : Bel_i(\neg B|A), x : Bel_i(C|A \wedge B)}\,(RB)}{x : \neg(Bel_i(\neg B|A)), x : Bel_i(C|A) \Rightarrow x : Bel_i(C|A \wedge B)}\,(L\neg)$$

Structural Properties of $G3CDL$

Admissibility of Weakening

The rules of left and right weakening are height-preserving admissible in $G3CDL$.

Invertibility

All the rules of $G3CDL$ are height-preserving invertible.

Admissibility of Contraction

The rules of left and right contraction are height-preserving admissible in $G3CDL$.

Admissibility of Cut

Rule of Cut is admissible in $G3CDL$.

Adding Knowlege and Belief

G3CDL Rules (4)

Rules for knowledge and belief

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : K_i A} \; LK \; (a \; new) \qquad \frac{a \in I_i(x), x : K_i A, a \Vdash^{\forall} A, \Gamma \Rightarrow \Delta}{a \in I_i(x), x : K_i A, \Gamma \Rightarrow \Delta} \; RK$$

$$\frac{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i A, a \Vdash^{\forall} A}{a \in I_i(x), \Gamma \Rightarrow \Delta, x : Bel_i A} \; LSB \qquad \frac{a \in I_i(x), a \Vdash^{\forall} A \Rightarrow \Delta}{x : Bel_i A, \Gamma \Rightarrow \Delta} \; RSB \; (a \; new)$$

Admissibility of the rules

The rules for knowledge and belief are admissible in *G3CDL*.

Outline

(1) The logic $\mathbb{CDL}$

(2) Semantics

(3) Labelled Sequent Calculus

(4) Main results: Soundness, Termination and Completeness

(5) Conclusions

Main Results

Soundness

If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $G3CDL$, then it is valid in the class of multi-agent neighbourhood models.

Completeness

If a formula A is valid in the class of multi-agent neighbourhood models, then it is derivable in $G3CDL$.

Termination

Adopting a suitable strategy, proof search for any sequent of the form $\Rightarrow x_0 : A$ always comes to an end after a finite number of steps.

Finite model property

If a formula A is satisfiable in the class of neighbourhood models, then it is satisfiable in the class of *finite* neighbourhood models.

Proof sketch

- Definition of *saturated sequent*
- Definition of a suitable *proof search strategy*

Claim: Terminating derivation tree

Each sequent that occurs as a leaf of a derivation tree built in accordance with the search strategy is either an *initial sequent* or a *saturated sequent*.

Claim: Existence of a *finite* countermodel

Let $\Gamma_i \Rightarrow \Delta_i$ be a saturated sequent occurring as a leaf of a derivation branch. Then there exists a finite countermodel $\mathcal{M}$ to $\Gamma_i \Rightarrow \Delta_i$ that *satisfies* all formulas in $\downarrow \Gamma_i$ and *falsifies* all formulas in $\downarrow \Delta_i$ (where $\downarrow \Gamma_i = \bigcup_{j \leq i} \Gamma_j$ and $\downarrow \Delta_i = \bigcup_{j \leq i} \Delta_j$).

Outline

(1) The logic $\mathbb{CDL}$

(2) Semantics

(3) Labelled Sequent Calculus

(4) Main results: Soundness, Termination and Completeness

(5) Conclusions

Conclusions

Results

- A new simple neighbourhood semantics for $\mathbb{CDL}$
- A labelled sequent calculus based on it with good properties:
  - analyticity, cut-freeness
  - terminating proof-search
- Constructive proof of the finite model property of $\mathbb{CDL}$

Future Research

- Provide an interpretation in NM of other epistemic operators defined in the literature: *safe belief*, *strong belief* [Baltag and Smets, 2008];
- Provide a direct proof of completeness of the axiomatization with respect to the semantics defined in terms of neighbourhood models;
- Long term goal: to obtain modular and uniform calculi covering all logics at least as strong as $\mathbb{CDL}$, including the family of Lewis' logic of counterfactuals.

Thank you !

References

Alexandroff, P. (1937). Diskrete Räume. *Mat.Sb. (NS)*, 2(3):501–519.

Baltag, A. and Smets, S. (2006). Conditional doxastic models: A qualitative approach to dynamic belief revision. *Electronic Notes in Theoretical Computer Science*, 165:5–21.

Baltag, A. and Smets, S. (2008). A qualitative theory of dynamic interactive belief revision. *Logic and the foundations of game and decision theory (LOFT 7)*, 3:9–58.

Board, O. (2004). Dynamic interactive epistemology. *Games and Economic Behavior*, 49(1):49–80.

Chellas, B. F. (1980). *Modal logic: an introduction*, volume 316. Cambridge University Press.

Grove, A. (1988). Two modellings for theory change. *Journal of philosophical logic*, 17(2):157–170.

Lewis, D. K. (1973). *Counterfactuals*. Blackwell.

Marti, J. and Pinosio, R. (2013). Topological semantics for conditionals. *The Logica Yearbook*.

Montague, R. (1970). Pragmatics and intensional logic. *Synthese*, 22(1-2):68–94.

Negri, S. (2016). Proof theory for non-normal modal logics: The neighbourhood formalism and basic results. *to appear,*. http://www.helsinki.fi/~negri/negri_ifcolog.pdf.

Negri, S. and Olivetti, N. (2015). A sequent calculus for preferential conditional logic based on neighbourhood semantics. In *Automated Reasoning with Analytic Tableaux and Related Methods*, pages 115–134. Springer.

Pacuit, E. (2007). Neighborhood semantics for modal logic. *Notes of a course on neighborhood structures for modal logic: http://ai.stanford.edu/ epacuit/classes/esslli/nbhdesslli.pdf*.

Pacuit, E. (2013). Dynamic epistemic logic I: Modeling knowledge and belief. *Philosophy Compass*, 8(9):798–814.

Scott, D. (1970). Advice on modal logic. In *Philosophical problems in logic*, pages 143–173. Springer.

Equivalence of Plausibility Models and Neighbourhood Models (1)

Theorem 1

if a formula A is valid in the class of multi-agent Neighbourhood Models then it is valid in the class of Epistemic Plausibility Models

Proof.

Let $\mathcal{M}_P = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{\leq_i\}_{i \in \mathcal{A}}, [\ \ ] \rangle$ be an *P*-model. Let $u \in W$ define its downward closed set:

$$\downarrow^{\leq_i} u = \{v \in W \mid v \leq_i u\}$$

We define the *N*-modelmodel $\mathcal{M}_N = \langle W, \{I\}_{i \in \mathcal{A}}, [\ \ ] \rangle$, where for $x \in W$

$$I_i(x) = \{\downarrow^{\leq_i} u \mid u \sim_i x\}$$

□

Equivalence of Plausibility Models and Neighbourhood Models (2)

Theorem 2

if a formula A is valid in the class of Epistemic Plausibility Models then it is valid in the class of multi-agent Neighbourhood Models

Proof.

Let $\mathcal{M}_N = \langle W, \{I\}_{i \in \mathcal{A}}, [\ \ ] \rangle$ be a multi-agent *N*-model. We construct an *P*-model $\mathcal{M}_P = \langle W, \{\sim_i\}_{i \in \mathcal{A}}, \{\preceq_i\}_{i \in \mathcal{A}}, [\ \ ] \rangle$, by stipulating:

- $x \sim_i y$ iff $\exists \alpha \in I_i(x), y \in \alpha$
- $x \preceq_i y$ iff $\forall \alpha \in I_i(y)$, if $y \in \alpha$ then $x \in \alpha$.

□

Corollary

A formula A is a theorem of $\mathbb{CDL}$ if and only if it is valid in the class of neighbourhood models.

Adding Knowlege and Belief

Admissibility of LK in *G3CDL*

- $K_i A =_{df} Bel_i(\bot|\neg A)$

$$\dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{\Gamma \Rightarrow \Delta, x : K_i A}\ LK\ (a\ new)$$

$$\dfrac{\dfrac{a \in I_i(x), a \Vdash^{\exists} \neg A, \Gamma \Rightarrow \Delta, x \Vdash_i \bot|\neg A, a \Vdash^{\exists} \neg A \quad \dfrac{a \in I_i(x), \Gamma \Rightarrow \Delta, a \Vdash^{\forall} A}{a \in I_i(x), a \Vdash^{\exists} \neg A, \Gamma \Rightarrow \Delta, x \Vdash_i \bot|\neg A, a \Vdash^{\forall} A}\ Wk}{a \in I_i(x), a \Vdash^{\exists} \neg A, \Gamma \Rightarrow \Delta, x \Vdash_i \bot|\neg A}\ RC}{\Gamma \Rightarrow \Delta, x : Bel_i(\bot|\neg A)}\ RB$$

Derivation Example: Axiom (9) $A \supset \neg Bel_i(\bot|A)$

$$\mathcal{D}: \quad \dfrac{x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow a \Vdash^{\exists} A, x : A}{x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow a \Vdash^{\exists} A} \; R\Vdash^{\exists}$$

$$\dfrac{\begin{array}{c}\vdots \\ \mathcal{D}\end{array} \quad \dfrac{\dfrac{\dfrac{\dfrac{y \in b, y : A, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset \bot, x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow y : A; \quad y : \bot, y \in b, y : A, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset \bot, x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow}{y : A \supset \bot, y \in b, y : A, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset \bot, x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow} L\supset}{y \in b, y : A, b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset \bot, x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow} L\Vdash^{\forall}}{b \in I_i(x), b \Vdash^{\exists} A, b \Vdash^{\forall} A \supset \bot, x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow} L\Vdash^{\exists}}{x \Vdash_i \bot|A, x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow} LC}{\dfrac{\dfrac{x \in a, a \in I_i(x), x : A, x : Bel_i(\bot|A) \Rightarrow}{x : A, x : Bel_i(\bot|A) \Rightarrow} T}{x : A \Rightarrow x : \neg Bel_i(\bot|A)} R\neg} LB$$

Main results

Saturated sequent

Consider a derivation branch of the form $\Gamma_0 \Rightarrow \Delta_0, ..., \Gamma_k \Rightarrow \Delta_k, \Gamma_{k+1} \Rightarrow \Delta_{k+1}, ...$ where $\Gamma_0 \Rightarrow \Delta_0$ is the sequent $\Rightarrow x_0 : A$, and $\downarrow\Gamma_i = \bigcup_{j\leq i} \Gamma_j$ and $\downarrow\Delta_i = \bigcup_{j\leq i} \Delta_j$. For each rule (R), we say that a sequent $\Gamma \Rightarrow \Delta$ *satisfies the saturation condition associated to* (R) if the following hold:

- $(R \Vdash^{\forall})$ *If* $a \Vdash^{\forall} A$ *is in* $\downarrow\Delta$*, then for some* x *there is* $x \in a$ *in* Γ *and* $x : A$ *in* $\downarrow\Delta$*;*
- $(L \Vdash^{\forall})$ *If* $x \in a$ *and* $a \Vdash^{\forall} A$ *are in* Γ*, then* $x : A$ *is in* Γ*;*
- (RB) *If* $x : Bel_i(B|A)$ *is in* $\downarrow\Delta$*, then for some* $i \in \mathcal{A}$ *and for some* a*,* $a \in I_i(x)$ *is in* Γ*,* $a \Vdash^{\exists} A$ *is in* $\downarrow\Gamma$ *and* $x \Vdash_i B|A$ *is in* $\downarrow\Delta$*;*
- (LB) *If* $a \in I_i(x)$ *and* $x : Bel_i(B|A)$ *are in* Γ*, then either* $a \Vdash^{\exists} A$ *is in* $\downarrow\Delta$ *or* $x \Vdash_i B|A$ *is in* $\downarrow\Gamma$*;*
- (T) *For all* x *occurring in* $\downarrow\Gamma\cup\downarrow\Delta$*, for all* $i \in \mathcal{A}$ *there is an* a *such that* $a \in I_i(x)$ *and* $x \in a$ *are in* Γ*;*
- (S) *If* $a \in I_i(x)$ *and* $b \in I_i(x)$ *are in* Γ*, then* $a \subseteq b$ *or* $b \subseteq a$ *are in* Γ*;*
- *...*

A sequent $\Gamma \Rightarrow \Delta$ is *saturated* if (Init) There is no $x : P$ in $\Gamma \cap \Delta$; $(L\bot)$ There is no $x : \bot$ in Γ; $\Gamma \Rightarrow \Delta$ satisfies *all* saturation conditions listed above.

Main results

Proof search strategy

When constructing root-first a derivation tree for a sequent $\Rightarrow x_0 : A$, apply the following strategy:

(1) No rule can be applied to an initial sequent;

(2) If $k(x) < k(y)$ all rules applicable to x are applied before any rule applicable to y.

(3) Rule (T) is applied as the first one to each world label x.

(4) Rules which do not introduce a new label (static rules) are applied *before* the rules which do introduce new labels (dynamic rules), with the exception of (T), as in (iii);

(5) Rule (RB) is applied *before* rule (LC);

(6) A rule (R) cannot be applied to a sequent $\Gamma_i \Rightarrow \Delta_i$ if $\downarrow \Gamma_i$ and / or $\downarrow \Delta_i$ satisfy the saturation condition associated to (R).