Big and Small Steps for Fast and Slow Provability

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September 1, 2016

Gödel-Löb provability logic GL

- K together with Löb's axiom: (L) $\Box(\Box A \to A) \to \Box A$
- Complete w.r.t. *transitive converse well-founded* trees

Theorem (Solovay)

GL *is the provability logic of any reasonable theory* T.

The bimodal system GLT

Contains GL for both $\triangle$ and $\Box$, together with:

(T1) $\triangle A \to \Box A$

(T2) $\Box A \to \triangle\Box A$

(T3) $\Box A \to \Box\triangle A$

(T4) $\Box\triangle A \to \Box A$

(T1) $\Diamond A \to \triangledown A$

(T2) $\triangledown\Diamond A \to \Diamond A$

(T3) $\Diamond\triangledown A \to \Diamond A$

(T4) $\Diamond A \to \Diamond\triangledown A$

Lindström-frame: $\langle W, \prec, \prec_\infty\rangle$, with $\langle W, \prec\rangle$ a GL-frame, and $x \prec_\infty y :\Leftrightarrow |\{z \mid x \prec z \prec y\}| = \infty$.

$$x \Vdash \triangledown A :\Leftrightarrow y \Vdash A \text{ for some } y \text{ with } x \prec y.$$

$$x \Vdash \Diamond A :\Leftrightarrow y \Vdash A \text{ for some } y \text{ with } x \prec_\infty y.$$

Theorem (Lindström)

GLT *is sound and complete w.r.t. Lindström-frames.*

Fast provability

PA^* is Peano Arithmetic (PA) together with Parikh's rule:

$$\text{if } \Box_{\mathsf{PA}}\varphi, \text{ then } \varphi.$$

Theorem (Parikh)

PA^* *has speed-up over* PA.

Theorem (Lindström)

GLT *is the joint provability logic of* $\Box_{\mathsf{PA}}$ *and* $\triangle_{\mathtt{p}}$.

Lemma (Lindström)

$\mathsf{PA} \vdash \triangledown_{\mathtt{p}}\varphi \leftrightarrow \Diamond^{\omega}_{\mathsf{PA}}\varphi$

Slow provability

Friedman, Rathjen, and Weiermann:

$$\mathsf{PA}{\restriction_{\mathtt{F}}} := \bigcup_{n\in\omega}\{\mathrm{I}\Sigma_n \mid \mathtt{F}(n){\downarrow}\},$$

where $\mathtt{F}$ is a certain recursive function with $\mathsf{PA} \nvdash \mathtt{F}{\downarrow}$.
$\triangle_{\mathtt{s}}$ is the provability predicate of $\mathsf{PA}{\restriction_{\mathtt{F}}}$.

Theorem (H. & Shavrukov)

GLT *is the joint provability logic of* $\triangle_{\mathtt{s}}$ *and* $\Box_{\mathsf{PA}}$.

Theorem (Pakhomov, Freund)

There are slow provability predicates $\triangle_1$, $\triangle_2$, *for which*

i. $\mathsf{PA} \vdash \Diamond_{\mathsf{PA}}\varphi \leftrightarrow \nabla_1^{\omega}\varphi$

ii. $\mathsf{PA} \vdash \Diamond_{\mathsf{PA}}\varphi \leftrightarrow \nabla_2^{\varepsilon_0}\varphi$