Distributive mereotopology

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This talk is in the field of spatial theories and logics. Region-based theory of space (RBTS) which in a sense is another name of **mereotopology** takes as a primary notion the notion of *region* as an abstraction of physical body instead of point, line and plane. The motivation for this is that points, lines and planes do not have separate existence in the reality. RBTS has simpler way of representing of qualitative spatial information.

Contact algebra

Contact algebra is one of the main tools in RBTS.
Contact algebra is a Boolean algebra $\underline{B} = (B, \leq, 0, 1, \cdot, +, *, C)$ with an additional binary relation C called *contact*, and satisfying the following axioms:

(C1) If aCb, then $a \neq 0$ and $b \neq 0$,
(C2) If aCb and $a \leq a'$ and $b \leq b'$, then $a'Cb'$,
(C3) If $aC(b+c)$, then aCb or aCc,
(C4) If aCb, then bCa,
(C5) If $a \cdot b \neq 0$, then aCb.

The elements of contact algebra are called regions.

Extended distributive contact lattice

There is a problem in the motivation of the operation of Boolean complementation. A question arises: if a represents some physical body, what kind of body represents a^*. To avoid this problem, we drop the operation $*$. The topological relations of dual contact and nontangential inclusion cannot be defined without $*$ and because of this we take them as primary in the language. So we consider the language $\mathcal{L}(0, 1; +, \cdot; \leq, C, \widehat{C}, \ll)$ which is an extension of the language of distributive lattice with the predicate symbols for the relations of contact, dual contact and nontangential inclusion. We obtain an axiomatization of the theory consisting of the universal formulas in the language $\mathcal{L}$ true in all contact algebras. The structures in $\mathcal{L}$, satisfying the axioms in question, are called extended distributive contact lattices (EDC-lattices).

Extended distributive contact lattice

Let $\underline{D} = (D, \leq, 0, 1, \cdot, +, C, \widehat{C}, \ll)$ be a bounded distributive lattice with three additional relations $C, \widehat{C}, \ll$, called respectively **contact**, **dual contact** and **nontangential part-of.** The obtained system is called **extended distributive contact lattice** (EDC-lattice, for short) if it satisfies the following axioms:

Extended distributive contact lattice

Axioms for C alone: The axioms (C1)-(C5) mentioned above.

Axioms for $\widehat{C}$ alone:
($\widehat{C}$1) If $a\widehat{C}b$, then $a, b \neq 1$,
($\widehat{C}$2) If $a\widehat{C}b$ and $a' \leq a$ and $b' \leq b$, then $a'\widehat{C}b'$,
($\widehat{C}$3) If $a\widehat{C}(b \cdot c)$, then $a\widehat{C}b$ or $a\widehat{C}c$,
($\widehat{C}$4) If $a\widehat{C}b$, then $b\widehat{C}a$,
($\widehat{C}$5) If $a + b \neq 1$, then $a\widehat{C}b$.

Extended distributive contact lattice

Axioms for $\ll$ alone:

($\ll$ 1) $0 \ll 0$,

($\ll$ 2) $1 \ll 1$,

($\ll$ 3) If $a \ll b$, then $a \leq b$,

($\ll$ 4) If $a' \leq a \ll b \leq b'$, then $a' \ll b'$,

($\ll$ 5) If $a \ll c$ and $b \ll c$, then $(a+b) \ll c$,

($\ll$ 6) If $c \ll a$ and $c \ll b$, then $c \ll (a \cdot b)$,

($\ll$ 7) If $a \ll b$ and $(b \cdot c) \ll d$ and $c \ll (a+d)$, then $c \ll d$.

Mixed axioms:

$(MC1)$ If aCb and $a \ll c$, then $aC(b \cdot c)$,

$(MC2)$ If $a\overline{C}(b \cdot c)$ and aCb and $(a \cdot d)\overline{C}b$, then $d\widehat{C}c$,

$(M\widehat{C}1)$ If $a\widehat{C}b$ and $c \ll a$, then $a\widehat{C}(b + c)$,

$(M\widehat{C}2)$ If $a\overline{\widehat{C}}(b + c)$ and $a\widehat{C}b$ and $(a + d)\overline{\widehat{C}}b$, then dCc,

$(M \ll 1)$ If $a\overline{\widehat{C}}b$ and $(a \cdot c) \ll b$, then $c \ll b$,

$(M \ll 2)$ If $a\overline{C}b$ and $b \ll (a + c)$, then $b \ll c$.

Principle of duality

For the language we can introduce the following principle of duality: dual pairs $(0, 1), (\cdot, +), (\leq, \geq), (C, \widehat{C}), (\ll, \gg)$. For each statement A of the language we can define in an obvious way its dual $\widehat{A}$. For each axiom Ax from the list of axioms of EDCL its dual $\widehat{Ax}$ is also an axiom.

The following statements are well known in the representation theory of distributive lattices.

Lemma

Let F_0 be a filter, I_0 be an ideal and $F_0 \cap I_0 = \varnothing$. Then:

1. **Filter-extension Lemma.** *There exists a prime filter F such that $F_0 \subseteq F$ and $F \cap I_0 = \varnothing$.*
2. **Ideal-extension Lemma.** *There exists a prime ideal I such that $I_0 \subseteq I$ and $F_0 \cap I = \varnothing$.*

There are also stronger filter-extension lemma and ideal-extension lemma. We do not know if these two statements for distributive lattices are new, but we use them in the representation theorem of EDC-lattices.

Lemma

Let F_0 be a filter, I_0 be an ideal and $F_0 \cap I_0 = \varnothing$. Then:

1. **Strong filter-extension Lemma.** *There exists a prime filter F such that $F_0 \subseteq F$, $F \cap I_0 = \varnothing$ and $(\forall x \notin F)(\exists y \in F)(x \cdot y \in I_0)$.*
2. **Strong ideal-extension Lemma.** *There exists a prime ideal I such that $I_0 \subseteq I$, $F_0 \cap I = \varnothing$ and $(\forall x \notin I)(\exists y \in I)(x + y \in F_0)$.*

Canonical relational structure

Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice and let $PF(D)$ denote the set of prime filters of $\underline{D}$. We construct a canonical relational structure (W, R) related to $\underline{D}$ putting $W = PF(D)$ and defining the canonical relation R for $\Gamma, \Delta \in PF(D)$ as follows:

$\Gamma R \Delta \leftrightarrow_{def} (\forall a, b \in D)((a \in \Gamma, b \in \Delta \rightarrow aCb)\&(a \notin \Gamma, b \notin \Delta \rightarrow a\widehat{C}b)\&(a \in \Gamma, b \notin \Delta \rightarrow a \not\ll b)\&(a \notin \Gamma, b \in \Delta \rightarrow b \not\ll a))$

Let $h(a) = \{\Gamma \in PF(D) : a \in \Gamma\}$ be the well known Stone embedding mapping. It turns out that h is an embedding from $\underline{D}$ into the EDC-lattice over (W, R).

Corollary

Every EDC-lattice can be isomorphically embedded into a contact algebra.

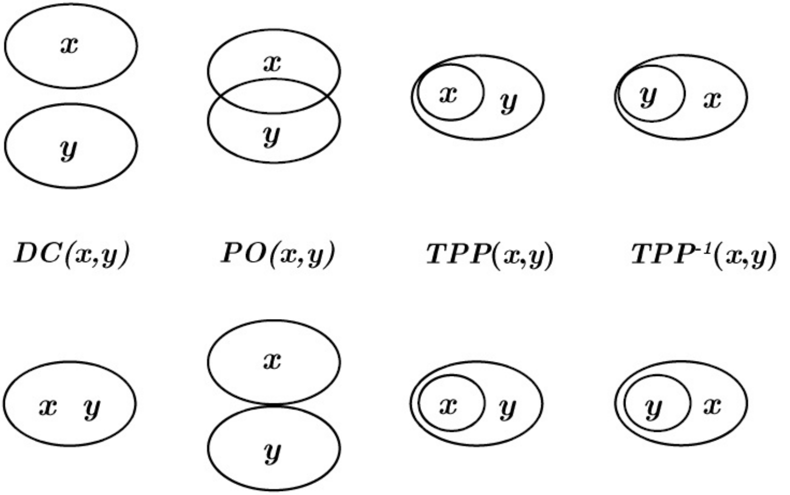
Relations with other mereotopologies

One of the most popular systems of topological relations in Qualitative Spatial Representation and Reasoning is RCC-8. It consists of 8 relations between non-empty regular closed subsets of arbitrary topological space.

Definition

The system **RCC-8.**

- disconnected – DC(a, b): $a\overline{C}b$,
- external contact – EC(a, b): aCb and $a\overline{O}b$,
- partial overlap – PO(a, b): aOb and $a \not\leq b$ and $b \not\leq a$,
- tangential proper part – TPP(a, b): $a \leq b$ and $a \not\ll b$ and $b \not\leq a$,
- tangential proper part^{-1} – TPP^{-1}(a, b): $b \leq a$ and $b \not\ll a$ and $a \not\leq b$,
- nontangential proper part NTPP(a, b): $a \ll b$ and $a \neq b$,
- nontangential proper part^{-1} – NTPP^{-1}(a, b): $b \ll a$ and $a \neq b$,
- equal – EQ(a, b): $a = b$.



$DC(x,y)$ $PO(x,y)$ $TPP(x,y)$ $TPP^{-1}(x,y)$

$EQ(x,y)$ $EC(x,y)$ $NTPP(x,y)$ $NTPP^{-1}(x,y)$

Figure: **RCC-8 relations**

Relations with other mereotopologies

The *RCC*-8 relations are definable in the language of *EDC*-lattices.

Additional axioms

We formulate several additional axioms for EDC-lattices which are adaptations for the language of EDC-lattices of some known axioms considered in the context of contact algebras. First we formulate the so called extensionality axioms for the definable predicates of overlap - $aOb \leftrightarrow_{def} a \cdot b \neq 0$ and underlap - $a\widehat{O}b \leftrightarrow_{def} a + b \neq 1$.

(Ext O) $a \not\leq b \rightarrow (\exists c)(a \cdot c \neq 0$ and $b \cdot c = 0)$ - *extensionality of overlap*,
(Ext $\widehat{O}$) $a \not\leq b \rightarrow (\exists c)(a + c = 1$ and $b + c \neq 1)$ - *extensionality of underlap*.

Additional axioms

We consider also the following axioms.

(Ext C) $a \neq 1 \rightarrow (\exists b \neq 0)(a\overline{C}b)$ - *C-extensionality*,

(Ext $\widehat{C}$) $a \neq 0 \rightarrow (\exists b \neq 1)(a\overline{\widehat{C}}b)$ - *$\widehat{C}$-extensionality*.

(Con C) $a \neq 0, b \neq 0$ and $a + b = 1 \rightarrow aCb$ - *C-connectedness axiom* ,

(Con $\widehat{C}$) $a \neq 1, b \neq 1$ and $a \cdot b = 0 \rightarrow a\widehat{C}b$ - *$\widehat{C}$-connectedness axiom* .

Additional axioms

(Nor 1) $a\overline{C}b \rightarrow (\exists c, d)(c + d = 1, a\overline{C}c \text{ and } b\overline{C}d)$,

(Nor 2) $a\overline{\widehat{C}}b \rightarrow (\exists c, d)(c \cdot b = 0, a\overline{\widehat{C}}c \text{ and } b\overline{\widehat{C}}d)$,

(Nor 3) $a \ll b \rightarrow (\exists c)(a \ll c \ll b)$.

and the so called rich axioms:

(U-rich $\ll$) $a \ll b \rightarrow (\exists c)(b + c = 1 \text{ and } a\overline{C}c)$,

(U-rich $\widehat{C}$) $a\overline{\widehat{C}}b \rightarrow (\exists c, d)(a + c = 1, b + d = 1 \text{ and } c\overline{C}d)$.

(O-rich $\ll$) $a \ll b \rightarrow (\exists c)(a \cdot c = 0 \text{ and } c\overline{\widehat{C}}b)$,

(O-rich C) $a\overline{C}b \rightarrow (\exists c, d)(a \cdot c = 0, b \cdot d = 0 \text{ and } c\overline{\widehat{C}}d)$.

Let $(D_1, C_1, \widehat{C}_1, \ll_1)$ and $(D_2, C_2, \widehat{C}_2, \ll_2)$ be two EDC-lattices and D_1 be a substructure of D_2. It is valuable to know under what conditions we have equivalences of the form:

D_1 satisfies some additional axiom iff D_2 satisfies the same axiom.

The importance of such conditions is related to the representation theory of EDC-lattices satisfying some additional axioms. In general, if we have some embedding theorem for EDC-lattice D satisfying a given additional axiom, it is not known in advance that the lattice in which D is embedded also satisfies this axiom. That is why it is good to have such conditions which automatically guarantee this. We formulate several such "good conditions": dense and dual dense sublattice, C- and $\widehat{C}$-separable sublattice.

Definition

Dense and dual dense sublattice. Let D_1 be a distributive sublattice of D_2. D_1 is called a *dense* sublattice of D_2 if the following condition is satisfied:

(Dense) $(\forall a_2 \in D_2)(a_2 \neq 0 \Rightarrow (\exists a_1 \in D_1)(a_1 \leq a_2$ and $a_1 \neq 0))$.

Dually we define a *dual dense* sublattice.

If h is an embedding of the lattice D_1 into the lattice D_2 then we say that h is a *dense* (*dually dense*) embedding if the sublattice $h(D_1)$ is a dense (dually dense) sublattice of D_2.

Definition

C-separability. Let D_1 be a substructure of D_2; we say that D_1 is a *C-separable EDC-sublattice of* D_2 if the following conditions are satisfied:

(C-separability for C) - $(\forall a_2, b_2 \in D_2)(a_2 \overline{C} b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 \leq b_1, a_1 \overline{C} b_1))$.

(C-separability for $\widehat{C}$) - $(\forall a_2, b_2 \in D_2)(a_2 \overline{\widehat{C}} b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 + a_1 = 1, b_2 + b_1 = 1, a_1 \overline{C} b_1))$.

(C-separability for $\ll$) - $(\forall a_2, b_2 \in D_2)(a_2 \ll b_2 \Rightarrow (\exists a_1, b_1 \in D_1)(a_2 \leq a_1, b_2 + b_1 = 1, a_1 \overline{C} b_1))$.

If h is an embedding of the lattice D_1 into the lattice D_2 then we say that h is a *C-separable embedding* if the sublattice $h(D_1)$ is a C-separable sublattice of D_2.

Theorem

Topological representation theorem for EDC-lattices. *Let* $\underline{D} = (D, C, \widehat{C}, \ll)$ *be an EDC-lattice. Then:*
(i) There exists a topological space X *and an embedding of* $\underline{D}$ *into the contact algebra* $RC(X)$ *of regular closed subsets of* X*.*
(ii) There exists a topological space Y *and an embedding of* $\underline{D}$ *into the contact algebra* $RO(Y)$ *of regular open subsets of* Y*.*

Topological representation theory

The following lemma relates topological properties to the properties of the relations C, $\widehat{C}$ and $\ll$ and shows the importance of the additional axioms for EDC-lattices.

Lemma

(i) If X is semiregular, then X is weakly regular iff $RC(X)$ satisfies any of the axioms (Ext C), (Ext $\widehat{C}$).
(ii) X is κ-normal iff $RC(X)$ satisfies any of the axioms (Nor 1), (Nor 2) and (Nor 3).
(iii) X is connected iff $RC(X)$ satisfies any of the axioms (Con C), (Con $\widehat{C}$).
(iv) If X is compact and Hausdorff, then $RC(X)$ satisfies (Ext C), (Ext $\widehat{C}$) and (Nor 1), (Nor 2) and (Nor 3) .

Definition

U-rich and O-rich EDC-lattices. Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an EDC-lattice. Then:
(i) $\underline{D}$ is called U-rich EDC-lattice if it satisfies the axioms (Ext $\widehat{O}$), (U-rich $\ll$) and (U-rich $\widehat{C}$).
(ii) $\underline{D}$ is called O-rich EDC-lattice if it satisfies the axioms (Ext O), (O-rich $\ll$) and (O-rich $\widehat{C}$).

Theorem

Topological representation theorem for U-rich EDC-lattices
Let $\underline{D} = (D, C, \widehat{C}, \ll)$ be an U-rich EDC-lattice. Then there exists a compact semiregular T_0-space X and a dually dense and C-separable embedding h of $\underline{D}$ into the Boolean contact algebra $RC(X)$ of the regular closed sets of X. Moreover:
(i) $\underline{D}$ satisfies (Ext C) iff $RC(X)$ satisfies (Ext C); in this case X is weakly regular.
(ii) $\underline{D}$ satisfies (Con C) iff $RC(X)$ satisfies (Con C); in this case X is connected.
(iii) $\underline{D}$ satisfies (Nor 1) iff $RC(X)$ satisfies (Nor 1); in this case X is κ-normal.

There is also a topological representation theorem of U-rich EDC-lattices, satisfying (Ext C), in T_1-spaces.

Adding the axiom (Nor 1), we obtain representability in compact T_2-spaces.

Thank you very much!