On the Lambek Calculus with the Kleene Star and the Exponential

Stepan Kuznetsov,
Steklov Mathematical Institute (Moscow)

AiML 2016, Budapest, August 30 – September 2, 2016

The Lambek Calculus with the Unit Constant ($\mathbf{L}^{\mathbf{1}}$)

$$\frac{}{A \to A}\ (\mathrm{id})$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, B \,/\, A, \Pi, \Delta_2 \to C}\ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A}\ (\to /)$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \to C}\ (\setminus \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus)$$

$$\frac{\Delta_1, A, B, \Delta_2 \to C}{\Delta_1, A \cdot B, \Delta_2 \to C}\ (\cdot \to) \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to B}{\Pi_1, \Pi_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Delta_1, \Delta_2 \to C}{\Delta_1, \mathbf{1}, \Delta_2 \to C}\ (\mathbf{1} \to) \qquad \frac{}{\to \mathbf{1}}\ (\to \mathbf{1})$$

Lambek Grammar

John loves Mary

Lambek Grammar

John	loves	Mary
np	$(np \setminus s) / np$	np

Lambek Grammar

John	loves	Mary	
np	$(np \setminus s) / np$	np	$\rightarrow s$

Lambek Grammar

$$\begin{array}{lccccc} & \text{John} & & \text{loves} & \text{Mary} & \\ \mathbf{L^1} \vdash & np & & (np \setminus s) \,/\, np & np & \rightarrow s \end{array}$$

Lambek Grammar

	John	loves	Mary	
$\mathbf{L^1} \vdash$	np	$(np \setminus s) / np$	np	$\rightarrow s$

the	girl	whom	John	loves	

Lambek Grammar

$$\begin{array}{lccccc} & \text{John} & \text{loves} & \text{Mary} & \\ \mathbf{L^1} \vdash & np & (np \setminus s) \,/\, np & np & \rightarrow s \end{array}$$

$$\begin{array}{cccccc} \text{the} & \text{girl} & \text{whom} & \text{John} & \text{loves} \\ np \,/\, n & n & (n \setminus n) \,/ (s \,/\, np) & np & (np \setminus s) \,/\, np \end{array}$$

Lambek Grammar

$$\begin{array}{lccccc} & \text{John} & \text{loves} & \text{Mary} & \\ \mathbf{L^1} \vdash & np & (np \setminus s) \,/\, np & np & \rightarrow s \end{array}$$

$$\begin{array}{ccccccc} \text{the} & \text{girl} & \text{whom} & \text{John} & \text{loves} & \\ np \,/\, n & n & (n \setminus n) \,/ (s \,/\, np) & np & (np \setminus s) \,/\, np & \\ & & & \underbrace{\qquad\qquad\qquad\qquad}_{\rightarrow s \,/\, np} & \end{array}$$

Lambek Grammar

$$\begin{array}{lccccc} & \text{John} & \text{loves} & \text{Mary} & \\ \mathbf{L^1} \vdash & np & (np \setminus s) \,/\, np & np & \rightarrow s \end{array}$$

$$\begin{array}{lccccccc} & \text{the} & \text{girl} & \text{whom} & \text{John} & \text{loves} & \\ \mathbf{L^1} \vdash & np \,/\, n & n & (n \setminus n) \,/ (s \,/\, np) & np & (np \setminus s) \,/\, np & \rightarrow np \\ & & & & \underbrace{\qquad\qquad\qquad\qquad}_{\rightarrow s \,/\, np} & & \end{array}$$

Lambek Grammar

$$\begin{array}{lccccc} & \text{John} & \text{loves} & \text{Mary} & \\ \mathbf{L^1} \vdash & np & (np \setminus s) \,/\, np & np & \rightarrow s \end{array}$$

$$\begin{array}{lccccccc} & \text{the} & \text{girl} & \text{whom}_i & \text{John} & \text{loves} & []_i & \\ \mathbf{L^1} \vdash & np \,/\, n & n & (n \setminus n) \,/ (s \,/\, np) & np & (np \setminus s) \,/\, np & & \rightarrow np \end{array}$$

$$\underbrace{np \quad (np \setminus s) \,/\, np}_{\rightarrow s \,/\, np}$$

Extending Lambek Grammar: Multiple Extraction

the person whom the friends of admire

Extending Lambek Grammar: Multiple Extraction

the person whom$_i$ the friends of []$_i$ admire []$_i$

Extending Lambek Grammar: Multiple Extraction

the person	whom$_i$	the friends of $[]_i$ admire $[]_i$
...	$(n \setminus n) / (s / !np)$	$\rightarrow s / !np$

Extending Lambek Grammar: Multiple Extraction

the person	whom$_i$	the friends of $[]_i$ admire $[]_i$
$\ldots$	$(n \setminus n) / (s \,/\, !np)$	$\underbrace{\qquad\qquad\qquad\qquad}_{\rightarrow s \,/\, !np}$

(Sub)exponential:

$$\frac{\Delta_1, A, \Delta_2 \rightarrow C}{\Delta_1, !A, \Delta_2 \rightarrow C}\ (! \rightarrow) \qquad \frac{\Delta_1, !A, !A, \Delta_2 \rightarrow C}{\Delta_1, !A, \Delta_2 \rightarrow C}\ (\mathrm{contr})$$

$$\frac{\Delta_1, !A, \Delta_2, \Delta_3 \rightarrow C}{\Delta_1, \Delta_2, !A, \Delta_3 \rightarrow C}\ (\mathrm{perm}_1) \qquad \frac{\Delta_1, \Delta_2, !A, \Delta_3 \rightarrow C}{\Delta_1, !A, \Delta_2, \Delta_3 \rightarrow C}\ (\mathrm{perm}_2)$$

Extending Lambek Grammar: Iterated Coordination

John,	Bill,	Mary,	and	Suzy	
np	np	np	$np^* \setminus np / np$	np	$\rightarrow np$

Extending Lambek Grammar: Iterated Coordination

John,	Bill,	Mary,	and	Suzy	
np	np	np	$np^* \setminus np / np$	np	$\rightarrow np$

Kleene star:

$$\frac{\Gamma_1 \rightarrow A \quad \ldots \quad \Gamma_n \rightarrow A}{\Gamma_1, \ldots, \Gamma_n \rightarrow A^*} \; (\rightarrow^*)_n$$

$\mathbf{L}^1$

$$\frac{}{A \to A}\ (\mathrm{id})$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, B \,/\, A, \Pi, \Delta_2 \to C}\ (/ \to) \qquad \frac{\Pi, A \to B}{\Pi \to B \,/\, A}\ (\to /)$$

$$\frac{\Pi \to A \quad \Delta_1, B, \Delta_2 \to C}{\Delta_1, \Pi, A \setminus B, \Delta_2 \to C}\ (\setminus \to) \qquad \frac{A, \Pi \to B}{\Pi \to A \setminus B}\ (\to \setminus)$$

$$\frac{\Delta_1, A, B, \Delta_2 \to C}{\Delta_1, A \cdot B, \Delta_2 \to C}\ (\cdot \to) \qquad \frac{\Pi_1 \to A \quad \Pi_2 \to B}{\Pi_1, \Pi_2 \to A \cdot B}\ (\to \cdot)$$

$$\frac{\Delta_1, \Delta_2 \to C}{\Delta_1, \mathbf{1}, \Delta_2 \to C}\ (\mathbf{1} \to) \qquad \frac{}{\to \mathbf{1}}\ (\to \mathbf{1})$$

The Extended Calculus

$$(\mathrm{id}) \quad (\backslash \to) \quad (\to \backslash) \quad (/ \to) \quad (\to /) \quad (\cdot \to) \quad (\to \cdot) \quad (\mathbf{1} \to) \quad (\to \mathbf{1})$$

$$\frac{\Delta_1, A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (! \to) \qquad \frac{!A_1, \ldots, !A_n \to C}{!A_1, \ldots, !A_n \to !C}\ (\to !)$$

$$\frac{\Delta_1, !A, \Delta_2, \Delta_3 \to C}{\Delta_1, \Delta_2, !A, \Delta_3 \to C}\ (\mathrm{perm}_1) \qquad \frac{\Delta_1, \Delta_2, !A, \Delta_3 \to C}{\Delta_1, !A, \Delta_2, \Delta_3 \to C}\ (\mathrm{perm}_2)$$

$$\frac{\Delta_1, !A, !A, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{contr}) \qquad \frac{\Delta_1, \Delta_2 \to C}{\Delta_1, !A, \Delta_2 \to C}\ (\mathrm{weak})$$

$$\frac{\Gamma, A^n, \Delta \to C \qquad \text{for all } n \geq 0}{\Gamma, A^*, \Delta \to C}\ (^* \to)_\omega \qquad \frac{\Gamma_1 \to A \quad \ldots \quad \Gamma_n \to A}{\Gamma_1, \ldots, \Gamma_n \to A^*}\ (\to ^*)_n$$

Theorem

The cut rule, $\frac{\Pi \to A \quad \Delta_1, A, \Delta_2 \to C}{\Delta_1, \Pi, \Delta_2 \to C}$*, is admissible.*

Complexity

Some linguistically motivated extensions of $\mathbf{L}^1$ look natural and harmless, but lead to high complexity!

Complexity

Some linguistically motivated extensions of $\mathbf{L}^1$ look natural and harmless, but lead to high complexity!

! and *	Π^0_2-hard (Π^1_1-complete?)
!	r.e.-complete [P. Lincoln, J. Mitchell, A. Scedrov, N. Shankar 1992; M. Kanovich, S. K., A. Scedrov 2016]
*, $\cap$, and $\cup$	Π^0_1-hard [W. Buszkowski, E. Palka 2008]
pure $\mathbf{L}^1$	NP-complete [M. Pentus 2003]
*	**?**

Complexity

Some linguistically motivated extensions of $\mathbf{L}^1$ look natural and harmless, but lead to high complexity!

! and *	Π^0_2-hard (Π^1_1-complete?)
!	r.e.-complete [P. Lincoln, J. Mitchell, A. Scedrov, N. Shankar 1992; M. Kanovich, S. K., A. Scedrov 2016]
*, $\cap$, and $\cup$	Π^0_1-hard [W. Buszkowski, E. Palka 2008]
pure $\mathbf{L}^1$	NP-complete [M. Pentus 2003]
*	**?**

Technique: encoding Lambek *theories* using the (sub)exponential connectives. For * use results of D. Kozen on Horn theories for Kleene algebra.

Complexity

Some linguistically motivated extensions of $\mathbf{L}^1$ look natural and harmless, but lead to high complexity!

! and *	Π^0_2-hard (Π^1_1-complete?)
!	r.e.-complete [P. Lincoln, J. Mitchell, A. Scedrov, N. Shankar 1992; M. Kanovich, S. K., A. Scedrov 2016]
*, $\cap$, and $\cup$	Π^0_1-hard [W. Buszkowski, E. Palka 2008]
pure $\mathbf{L}^1$	NP-complete [M. Pentus 2003]
*	**?**

Technique: encoding Lambek *theories* using the (sub)exponential connectives. For * use results of D. Kozen on Horn theories for Kleene algebra.

Finite systems (for example, cyclic proofs) are strictly weaker... and are worth studying.

Some References

- J. Lambek. *The mathematics of sentence structure.* Amer. Math. Monthly 65 (3), 1958, 154–170.
- G. Morrill, O. Valentín. *Computational coverage of TLG: Nonlinearity*. Proc. NLCS '15 (EPiC Series, vol. 32), 2015, 51–63.
- P. Lincoln, J. Mitchell, A. Scedrov, N. Shankar. *Decision problems for propositional linear logic*. APAL 56, 1992, 239–311.
- D. Kozen. *On the complexity of reasoning in Kleene algebra.* Information and Computation 179, 2002, 152–162.
- W. Buszkowski, E. Palka. *Infinitary action logic: complexity, models, and grammars.* Studia Logica 89, 2008, 1–18.
- M. Kanovich, S. K., A. Scedrov. *Undecidability of the Lambek calculus with a relevant modality.* Proc. Formal Grammar 2015 and 2016, LNCS 9804, 240–256. (arXiv: 1601.06303)

!(Thanks*)