It ain't necessarily so!

Basic sequent systems for negative modalities

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Frame: $\mathcal{F} = \langle W, R \rangle$, where $W \neq \varnothing$ and $R \subseteq W \times W$

Model: $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is a frame,
and $V : W \times \mathcal{L} \to \{f, t\}$ respects

(we use $\mathcal{M}, z \Vdash \alpha$ to denote $V(z, \alpha) = t$)

[S⊃] $\mathcal{M}, w \Vdash \varphi \supset \psi$ iff $\mathcal{M}, w \nVdash \varphi$ or $\mathcal{M}, w \Vdash \psi$

[S$\smile$] $\mathcal{M}, w \Vdash \smile\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for <u>some</u> $v \in W$ such that wRv

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Recovering the (now standard) basic modal languages

$\sim\alpha := \alpha \supset \smile(\alpha \supset \alpha)$ behaves as the **classical negation**
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<u>Note 0</u>. Conversely, the (paraconsistent) negation $\smallsmile$ might be recovered through ${\sim}\Box$.

<u>Note 1</u>. It is reasonable to expect $\smallsmile$ to be, in general, weaker than $\sim$, i.e.:

$${\sim}\alpha \vDash {\smallsmile}\alpha, \quad \text{yet} \quad {\smallsmile}\alpha \nvDash {\sim}\alpha$$

<u>Note 2</u>. Our minimal language, in what follows, will be:

$\mathcal{L}_{\wedge\vee\top\bot}$, classically interpreted

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	$\curlyvee$	$+$	$-$	$\curlywedge$
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Negations?

[J.M., J Appl Log 2005]

$[\![\textit{falsificatio}]\!]$ $\quad {}_{\forall k \exists p}\, \#^k p \not\succ \#^{k+1} p$

$[\![\textit{verificatio}]\!]$ $\quad {}_{\forall k \exists p}\, \#^{k+1} p \not\succ \#^k p$

[Boxes] and <Diamonds>

Let $\sim$ represent **classical negation**. Then:

$$[+]{\sim}\alpha \equiv {\sim}\langle + \rangle\alpha \qquad \langle + \rangle{\sim}\alpha \equiv {\sim}[+]\alpha$$

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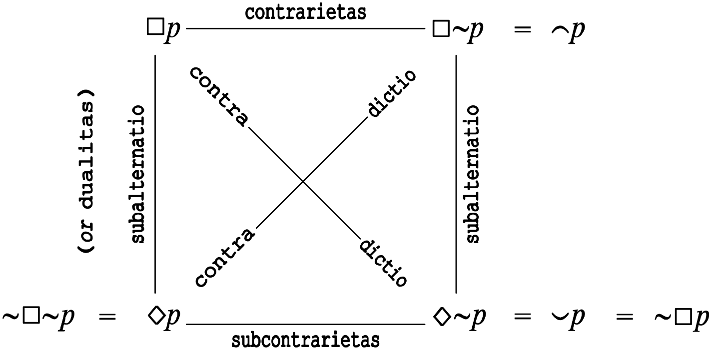

$\Box p$ contrarietas $\Box{\sim}p \; = \; \frown p$

(or dualitas) subalternatio

contra dictio

contra dictio

subalternatio

${\sim}\Box{\sim}p \; = \; \Diamond p$ subcontrarietas $\Diamond{\sim}p \; = \; \smile p \; = \; {\sim}\Box p$

Figure: *Square of Modalities* (som)

[J.M., Log Anal 2005]

Non-classical behavior and derivability adjustment

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The two sides of **negation**:

⟦#-explosion⟧ $p, \# p \succ q$ ⟦#-implosion⟧ $q \succ \# p, p$

Non-classical behavior and derivability adjustment

The two sides of **negation**, and their *failures*:

$[\![\#\text{-explosion}]\!]$ $p, \# p \nsucc q$ $[\![\#\text{-implosion}]\!]$ $q \succ \# p, p$

paraconsistency

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Some (strong) **gentler** versions:

[C1#]	$\textcircled{\#} p, p, \# p \succ$	[C2#]	$\succ p, \textcircled{\#} p$	[C3#]	$\succ \# p, \textcircled{\#} p$
[D1#]	$\succ \# p, p, \textcircled{\#} p$	[D2#]	$\textcircled{\#} p, p \succ$	[D3#]	$\textcircled{\#} p, \# p \succ$

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Enriching the object language through adjustment connectives

The [Cn#] clauses (strongly) internalize the 'consistency assumption', and the [Dn#] clauses (strongly) internalize the 'determinacy assumption'.

This defines (strong versions of) the so-called **LFI**s and **LFU**s.

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[S$\frown$] $\mathcal{M}, w \nVdash {\frown}\varphi$ iff $\mathcal{M}, v \Vdash \varphi$ for <u>some</u> $v \in W$ such that wRv

equivalently:

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It should be noted that, for non-degenerate classes of frames:

$\smile$ is a **paraconsistent negation**,
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Consider also the following *adjustment connectives*:

[SC$\circledsmile$] $\mathcal{M}, w \Vdash \circledsmile\varphi$ iff $\mathcal{M}, w \nVdash \varphi$ or $\mathcal{M}, w \nVdash {\smile}\varphi$

[SD$\circledfrown$] $\mathcal{M}, w \nVdash \circledfrown\varphi$ iff $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash {\frown}\varphi$

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It should be noted that:

$\mathbin{\bigcirc\mkern-15mu\smile}$ **expresses $\smile$-consistency**,
and allows for $\smile$-**explosiveness** to be recovered

$\mathbin{\bigcirc\mkern-15mu\frown}$ **expresses $\frown$-determinancy**,
and allows for $\frown$-**implosiveness** to be recovered

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How does classical negation relate to the non-classical ones?

Recall that: $[-]{\sim}\alpha \equiv {\sim}{<}{-}{>}\alpha$ $\qquad$ ${<}{-}{>}{\sim}\alpha \equiv {\sim}[-]\alpha$

In other words: $\frown{\sim}\alpha \equiv {\sim}\smile\alpha$ $\qquad$ $\smile{\sim}\alpha \equiv {\sim}\frown\alpha$

Moreover, in general: $\frown\alpha \succ {\sim}\alpha$ $\qquad$ ${\sim}\alpha \succ \smile\alpha$

but <u>the converses fail</u>

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[S∼] $\mathcal{M}, w \Vdash {\sim}\varphi$ iff $\mathcal{M}, w \nVdash \varphi$

Intuition: $\boxed{\sim\, = \,\frown + \smile}$

Note indeed that ∼ would be at once full type `[-]` and full type `<->`.

How does classical negation relate to the non-classical ones?

Recall that: $[-]{\sim}\alpha \;\equiv\; {\sim}\langle-\rangle\alpha$ $\qquad \langle-\rangle{\sim}\alpha \;\equiv\; {\sim}[-]\alpha$

In other words: ${\frown}{\sim}\alpha \;\equiv\; {\sim}{\smile}\alpha$ $\qquad {\smile}{\sim}\alpha \;\equiv\; {\sim}{\frown}\alpha$

Moreover, in general: ${\frown}\alpha \succ {\sim}\alpha$ $\qquad {\sim}\alpha \succ {\smile}\alpha$

but the converses fail

What if classical negation is *not* taken as a primitive connective?

To investigate: In which situations is it even *definable* in $\mathcal{L}_{\wedge\vee\top\bot\smile\frown\circledsmile\circledfrown}$?

(we'll answer this later on!)

A sequent calculus for PK

A sequent calculus for PK

A sequent calculus for the weakest normal modal logic over $\mathcal{L}_{\wedge\vee\top\bot\smile\frown\circledsmile\circledfrown}$:

[A. Dodó & J.M., ENTCS 2014]

$$[id]\quad \frac{}{\varphi \Rightarrow \varphi} \qquad\qquad [cut]\quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \Delta}$$

$$[W\Rightarrow]\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta} \qquad\qquad [\Rightarrow W]\quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \varphi, \Delta}$$

$$[\bot\Rightarrow]\quad \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad\qquad [\Rightarrow\top]\quad \frac{}{\Gamma \Rightarrow \top, \Delta}$$

$$[\wedge\Rightarrow]\quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad\qquad [\Rightarrow\wedge]\quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}$$

$$[\vee\Rightarrow]\quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad\qquad [\Rightarrow\vee]\quad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi \vee \psi, \Delta}$$

$$[\smile\Rightarrow]\quad \frac{\Gamma \Rightarrow \varphi, \Delta}{\frown\Delta, \smile\varphi \Rightarrow \smile\Gamma} \qquad\qquad [\Rightarrow\frown]\quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\frown\Delta \Rightarrow \frown\varphi, \smile\Gamma}$$

$$[\circledsmile\Rightarrow]\quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \smile\varphi, \Delta}{\Gamma, \circledsmile\varphi \Rightarrow \Delta} \qquad\qquad [\Rightarrow\circledsmile]\quad \frac{\Gamma, \varphi, \smile\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \circledsmile\varphi, \Delta}$$

$$[\circledfrown\Rightarrow]\quad \frac{\Gamma \Rightarrow \varphi, \frown\varphi, \Delta}{\Gamma, \circledfrown\varphi \Rightarrow \Delta} \qquad\qquad [\Rightarrow\circledfrown]\quad \frac{\Gamma, \varphi \Rightarrow \Delta \quad \Gamma, \frown\varphi \Rightarrow \Delta}{\Gamma \Rightarrow \circledfrown\varphi, \Delta}$$

The technology of **Basic Sequents**

The technology of Basic Sequents

A **basic rule**: *main* sequent + *context* sequent [O. Lahav & A. Avron 2013]

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Examples:

$$[\wedge\Rightarrow] \quad \frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad [\Rightarrow\wedge] \quad \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi, \Delta}$$

are described as:

$$[\wedge\Rightarrow] \quad \langle p_1, p_2 \Rightarrow ; \pi_0 \rangle \; / \; p_1 \wedge p_2 \Rightarrow \qquad [\Rightarrow\wedge] \quad \langle \Rightarrow p_1 \; ; \; \pi_0 \rangle , \langle \Rightarrow p_2 \; ; \; \pi_0 \rangle \; / \; \Rightarrow p_1 \wedge p_2$$

where $\pi_0 = \{\langle q_1 \Rightarrow \; ; \; q_1 \Rightarrow \rangle , \langle \Rightarrow q_1 \; ; \; \Rightarrow q_1 \rangle\}$.

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while

$$[\smile\Rightarrow] \quad \frac{\Gamma \Rightarrow \varphi, \Delta}{\frown\Delta, \smile\varphi \Rightarrow \smile\Gamma}$$

are described as:

$$[\smile\Rightarrow] \quad \langle \Rightarrow p_1 ; \pi_1 \rangle \; / \smile p_1 \Rightarrow$$

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The **main sequent** is made to match an appropriate *semantic condition*.
For instance, $[\smile\Rightarrow]$ induces:

"if $\mathcal{M}, v \Vdash \ \Rightarrow \varphi$ for every world v such that wRv, then $\mathcal{M}, w \Vdash \smile\varphi \Rightarrow$"

and the **context sequent** is also made to match a *semantic condition*.
For instance, π_1 induces:

"if wRv then $\mathcal{M}, w \Vdash \ \Rightarrow \smile\varphi$ whenever $\mathcal{M}, v \Vdash \varphi \Rightarrow$"

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Together, these correspond precisely to:

$[\mathrm{S}\smile]$ $\quad \mathcal{M}, w \Vdash \smile\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for some $v \in W$ such that wRv

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Together, these correspond precisely to:

$[S\smile]$ $\quad \mathcal{M}, w \Vdash \smile\varphi \quad$ iff $\quad \mathcal{M}, v \nVdash \varphi$ for <u>some</u> $v \in W$ such that wRv

For our convenience, we rewrite this as:

$[\mathbf{F}\smile]$ if $\mathbf{T}_v(\varphi)$ for every $v \in W$ such that wRv, then $\mathbf{F}_w(\smile\varphi)$

$[\mathbf{T}\smile]$ if $\mathbf{F}_v(\varphi)$ for some $v \in W$ such that wRv, then $\mathbf{T}_w(\smile\varphi)$

where we take '$\mathbf{T}_u(\alpha)$' as abbreviating '$V(u, \alpha) = t$', and '$\mathbf{F}_u(\alpha)$' as abbreviating '$V(u, \alpha) = f$'.

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Say that $w, v \in W$ **agree with respect to the formula** α**, according to** V,
if either ($\mathbf{T}_w(\alpha)$ and $\mathbf{T}_v(\alpha)$) or ($\mathbf{F}_w(\alpha)$ and $\mathbf{F}_v(\alpha)$).

Call $\mathcal{M}$ a **differentiated** model
if $w = v$ whenever w and v agree with respect to every $\alpha \in \mathcal{L}$, according to V.

Call $\mathcal{M}$ a **strengthened** model iff wRv

if ($\mathbf{T}_v(\alpha)$ implies $\mathbf{F}_w(\frown\alpha)$) and ($\mathbf{F}_v(\alpha)$ implies $\mathbf{T}_w(\smile\alpha)$), for every $\alpha \in \mathcal{L}$.

The technology of Basic Sequents

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Adequacy Theorem.

(corollary of [O. Lahav & A. Avron 2013])

PK is *sound* and *complete* with respect to any class of Kripke models that: (*i*) contains only models that satisfy all the appropriate $[\mathbf{T}\#]$ and $[\mathbf{F}\#]$ conditions; and (*ii*) contains all strengthened differentiated models that satisfy all the appropriate $[\mathbf{T}\#]$ and $[\mathbf{F}\#]$ conditions.

Cut-elimination and analyticity, almost for free

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A strategy in two steps:

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Step 1. Present an adequate semantics for the cut-free fragment of PK.

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We now build models with **quasi valuations**

$QV : W \times \mathcal{L} \rightarrow \{\{f\}, \{t\}, \{f, t\}\}$ such that:

[$\mathbf{F}\smile$] if $\mathbf{T}_v(\varphi)$ for every $v \in W$ such that wRv, then $\mathbf{F}_w(\smile\varphi)$

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where we take '$\mathbf{T}_u(\alpha)$' as abbreviating '$t \in QV(u, \alpha)$', and '$\mathbf{F}_u(\alpha)$' as abbreviating '$f \in QV(u, \alpha)$'.

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Note that these are in principle **non-deterministic**!

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Step 2. Show that the existence of a **countermodel in the form of a strengthened differentiated quasi model** implies the existence of an ordinary countermodel.

Let an **instance** of a quasi model $\mathcal{QM} = \langle\langle W, R\rangle, QV\rangle$ be any model $\mathcal{M} = \langle\langle W, R'\rangle, V\rangle$ such that $\mathbf{X}^Q_w(\varphi)$ whenever $\mathbf{X}_w(\varphi)$, for every $\mathbf{X} \in \{\mathbf{T}, \mathbf{F}\}$, every $w \in W$ and every $\varphi \in \mathcal{L}$.

Cut-elimination and analyticity, almost for free

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Theorem.

Every quasi model has an instance.

Cut-elimination and analyticity, almost for free

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Consider the **well-founded relation** $\ll$ on $\mathcal{L}$ such that $\alpha \ll \beta$ iff either:
(*i*) α is a proper subformula of β, or
(*ii*) $\alpha = \smile\gamma$ and $\beta = {\bigcirc\hspace{-0.9em}\smile}\gamma$ for some $\gamma \in \mathcal{L}$, or
(*iii*) $\alpha = \frown\gamma$ and $\beta = {\bigcirc\hspace{-0.9em}\frown}\gamma$ for some $\gamma \in \mathcal{L}$.

Since the class of all quasi models contains the strengthened differentiated quasi models, it follows that:

Corollary

PK enjoys cut-admissibility.

Corollary

PK is $\ll$-analytic:
If a sequent *s* is derivable from a set *S* of sequents in PK, then there is a derivation of *s* from *S* such that every formula φ that occurs in the derivation satisfies $\varphi \ll \psi$ for some ψ in $S \cup s$.

Seriality, Reflexivity, Functionality, Symmetry

Seriality, Reflexivity, Functionality, Symmetry

sequent system	:	frames	:	some distinguishing **features**
PKD	:	serial	:	$\frown p \models \smile p$
PKT	:	reflexive	:	$p, \frown p \models q$ and $q \models \smile p, p$
PKF	:	total functional	:	$\smile p \equiv \frown p$
PKB	:	symmetric	:	$\smile\smile p \models p$ and $p \models \frown\frown p$

$$[\mathbf{D}] \quad \frac{\Gamma \Rightarrow \Delta}{\frown\Delta \Rightarrow \smile\Gamma}$$

$$[\mathbf{T}_1] \quad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \smile\varphi, \Delta} \qquad\qquad [\mathbf{T}_2] \quad \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma, \frown\varphi \Rightarrow \Delta}$$

$$[\mathbf{Fun}] \quad \frac{\Gamma \Rightarrow \Delta}{\smile\Delta \Rightarrow \smile\Gamma}$$

$$[\mathbf{B}_1] \quad \frac{\Gamma, \smile\Gamma', \varphi \Rightarrow \Delta, \frown\Delta'}{\frown\Delta, \Delta' \Rightarrow \frown\varphi, \smile\Gamma, \Gamma'} \qquad\qquad [\mathbf{B}_2] \quad \frac{\Gamma, \smile\Gamma' \Rightarrow \varphi, \Delta, \frown\Delta'}{\frown\Delta, \Delta', \smile\varphi \Rightarrow \smile\Gamma, \Gamma'}$$

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The full story may be checked in the paper!

On the definability of classical negation

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When $\sim$ is not a primitive connective

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1. Classical negation *is definable* in the logics:
   PKT-$\{\smile, \circledsmile\}$, **(set** $\sim\varphi := \frown\varphi \vee \circledfrown\varphi$**)**
   PKT-$\{\frown, \circledfrown\}$, **(set** $\sim\varphi := \smile\varphi \wedge \circledsmile\varphi$**)**
   PKD, and PKF. **(set** $\sim\varphi := (\frown\varphi \wedge \circledsmile\varphi) \vee \circledfrown\varphi$**)**

On the definability of classical negation

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   PKD, and PKF. **(set** $\sim\varphi := (\frown\varphi \wedge \circledsmile\varphi) \vee \circledfrown\varphi$**)**
2. Classical negation *is not definable* in the logics:
   PK, PKB,
   PKT-$\{\circledsmile, \circledfrown\}$,
   PKD-$\{\circledsmile\}$, PKF-$\{\circledsmile\}$,
   PKD-$\{\circledfrown\}$, and PKF-$\{\circledfrown\}$.

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More on the study of *negative modalities*:

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More on the study of *negative modalities*:

- Studying **properties of negation** through other classes of frames
  Examples:

Church-Rosser : $\smile\smile p \models \frown\frown p$

transitive : $\smile p, \smile(\smile p) \models q$ and $q \models \frown(\frown p), \frown p$

euclidean : $\frown p, \smile(\frown p) \models q$ and $q \models \frown(\smile p), \smile p$

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More on the study of *negative modalities*:

- Studying properties of negation through other classes of frames
  Examples:
  Church-Rosser : $\smile\smile p \models \frown\frown p$
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  euclidean : $\frown p, \smile(\frown p) \models q$ and $q \models \frown(\smile p), \smile p$

- Studying combinations of negations of the same type.
  Examples: add the **backward-looking** modalities
  [S$\smile^{-1}$] $\mathcal{M}, w \Vdash \smile^{-1}\varphi$ iff $\mathcal{M}, v \nVdash \varphi$ for <u>some</u> $v \in W$ such that $wR^{-1}v$
  [S$\frown^{-1}$] $\mathcal{M}, w \nVdash \frown^{-1}\varphi$ iff $\mathcal{M}, v \Vdash \varphi$ for <u>some</u> $v \in W$ such that $wR^{-1}v$
  Note the validity in *PK* of **pure** consecutions such as:
  $\smile^{-1}\smile p \models p$ and $\smile\smile^{-1}p \models p$ (as well as $p \models \frown^{-\mathbf{1}}\frown p$ and $p \models \frown\frown^{-\mathbf{1}}p$)
  and the validity in *PKB* of **mixed** consecutions such as
  $\frown^{-1}\smile p \models p$ and $\frown\smile^{-1}p \models p$ (as well as $p \models \smile^{-\mathbf{1}}\frown p$ and $p \models \smile\frown^{-\mathbf{1}}p$)