Axiomatizing a Real-Valued Modal Logic

George Metcalfe

Mathematical Institute University of Bern

Joint work with Denisa Diaconescu and Laura Schnüriger

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Hansoul and Teheux (2013) consider a Łukasiewicz modal logic with

- standard "crisp" Kripke frames
- connectives defined on the real unit interval [0,1]

$$x \to y = \min(1, 1 - x + y)$$
 $\neg x = 1 - x$
 $x \oplus y = \min(1, x + y)$ $x \odot y = \max(0, x + y - 1)$

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$$\Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi)$$
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$$\frac{\varphi}{\Box \varphi}$$

and a rule with infinitely many premises

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Towards a Solution...

We provide a finitary axiomatizion of a **real-valued modal logic** that extends the multiplicative fragment of Abelian logic.

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The Multiplicative Fragment of Abelian Logic

The multiplicative fragment of Abelian logic is axiomatized by

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$$(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$$

(C)
$$(\varphi \to (\psi \to \chi)) \to (\psi \to (\varphi \to \chi))$$

(I)
$$\varphi \to \varphi$$

(A)
$$((\varphi \to \psi) \to \psi) \to \varphi$$

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \text{ (mp)}$$

and is complete with respect to the logical matrix

$$\langle \mathbb{R}, \mathbb{R}_{\geq 0}, \{ \rightarrow \} \rangle$$
 where $x \to y = y - x$.



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A Modal Language

We define further connectives (for a fixed variable p_0)

$$\overline{0} := p_0 \to p_0
\neg \varphi := \varphi \to \overline{0}
\varphi + \psi := \neg \varphi \to \psi.$$

For our modal language, we add a unary connective \square , and define

$$\Diamond \varphi := \neg \Box \neg \varphi.$$

The set of formulas ${\rm Fm}$ for this language is defined inductively as usual over a countably infinite set of variables ${\rm Var}.$

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Frames and Models

A frame $\mathfrak{F} = \langle W, R \rangle$ consists of

- ullet a non-empty set of worlds W
- an accessibility relation $R \subseteq W \times W$.

 \mathfrak{F} is called **serial** if for all $x \in W$, there exists $y \in W$ such that Rxy.

Models

A K(\mathbb{R})-model $\langle W, R, V \rangle$ consists of

- a serial frame $\langle W, R \rangle$
- an evaluation map $V: \operatorname{Var} \times W \to [-r, r]$ for some r > 0.

The evaluation map is extended to $V \colon \operatorname{Fm} \times W \to \mathbb{R}$ by

$$V(\varphi \to \psi, x) = V(\psi, x) - V(\varphi, x)$$
$$V(\Box \varphi, x) = \inf\{V(\varphi, y) : Rxy\}.$$

It follows also that

$$V(\overline{0},x) = 0$$
 $V(\varphi + \psi,x) = V(\varphi,x) + V(\psi,x)$
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Validity

A formula φ is

- valid in a $K(\mathbb{R})$ -model $\langle W, R, V \rangle$ if $V(\varphi, x) \geq 0$ for all $x \in W$
- $K(\mathbb{R})$ -valid if it is valid in all $K(\mathbb{R})$ -models.

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The following are equivalent for any formula φ :

- (1) φ is $K(\mathbb{R})$ -valid
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The Axiom System $K(\mathbb{R})$

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(K)
$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$$

(P)
$$\Box(\varphi + \cdots + \varphi) \rightarrow (\Box\varphi + \cdots + \Box\varphi)$$

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \text{ (mp)} \qquad \frac{\varphi}{\Box \varphi} \text{ (nec)} \qquad \frac{\varphi + \dots + \varphi}{\varphi} \text{ (con)}$$

The Sequent Calculus $GK(\mathbb{R})$

$$\frac{\Gamma}{\Delta \Rightarrow \Delta} \text{ (ID)} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Sigma, \Delta} \qquad \text{(CUT)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Pi \Rightarrow \Sigma, \Delta} \qquad \text{(MIX)} \qquad \frac{\Gamma, \dots, \Gamma \Rightarrow \Delta, \dots, \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{(SC)}$$

$$\frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma, \varphi \rightarrow \psi \Rightarrow \Delta} \qquad \text{(\rightarrow\Rightarrow$)} \qquad \frac{\Gamma, \varphi \Rightarrow \psi, \Delta}{\Gamma \Rightarrow \varphi \rightarrow \psi, \Delta} \qquad \text{(\Rightarrow\rightarrow$)}$$

$$\frac{\Gamma \Rightarrow \varphi, \dots, \varphi}{\Box \Gamma \Rightarrow \Box \varphi, \dots, \Box \varphi} \qquad \text{(\Box)}$$

Equivalence of Proof Systems

We interpret sequents by

$$(\varphi_1,\ldots,\varphi_n\Rightarrow\psi_1,\ldots,\psi_m)^{\mathcal{I}}:=(\varphi_1+\cdots+\varphi_n)\rightarrow(\psi_1+\ldots+\psi_m),$$

where $\varphi_1 + \cdots + \varphi_n := \overline{0}$ for n = 0.

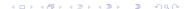
Theorem

The following are equivalent

- (1) $\Gamma \Rightarrow \Delta$ is derivable in $GK(\mathbb{R})$
- (2) $(\Gamma \Rightarrow \Delta)^{\mathcal{I}}$ is derivable in $K(\mathbb{R})$.

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 $\mathrm{GK}(\mathbb{R})$ admits cut elimination



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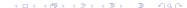
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The Main Result

Theorem

The following are equivalent for any formula φ :

- (1) φ is derivable in $K(\mathbb{R})$.
- (2) φ is $K(\mathbb{R})$ -valid.
- (3) $\Rightarrow \varphi$ is derivable in $GK(\mathbb{R})$.

Proof Idea for $(2) \Rightarrow (3)$

We prove by induction on the complexity of a sequent S that

$$S^{\mathcal{I}}$$
 is $K(\mathbb{R})$ -valid \Longrightarrow S is derivable in $GK(\mathbb{R})$.

The base case where S contains no boxes is easy and the cases where S contains an implication follow using the invertibility of $(\rightarrow \Rightarrow)$ and $(\Rightarrow \rightarrow)$.

If S contains only boxed formulas and variables, then the multisets of variables on the left and right must coincide, and can be cancelled.

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Suppose then that S is $\Box \Gamma \Rightarrow \Box \varphi_1, \ldots, \Box \varphi_n$. We apply the following $GK(\mathbb{R})$ -derivable rule for some k > 0 and $k\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_n$:

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Complexity

Using our labelled tableau rules, we also obtain:

Theorem

Checking $K(\mathbb{R})$ -validity of formulas is in EXPTIME.

Concluding Remarks

There remain many issues to resolve:

- Can we add extend our axiomatization to an "Abelian modal logic" with lattice connectives? Do we obtain a Łukasiewicz modal logic?
- Can we develop useful algebraic semantics for these logics?
- Is the complexity of checking $K(\mathbb{R})$ -validity EXPTIME-complete? What is the complexity of validity in Łukasiewicz modal logic?

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