Local tabularity without transitivity

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A logic L is *locally tabular* if, for any finite n, there exist only finitely many pairwise nonequivalent formulas in L built from the variables $p_1, \ldots, p_n$.

Equivalently, a logic L is locally tabular if the variety of its algebras is *locally finite*, i.e., every finitely generated L-algebra is finite.

If a logic is locally tabular, then

- it has the finite model property (thus it is Kripke complete);
- all its extensions are locally tabular.

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Segerberg-Maksimova criterion for extensions of $\mathrm{K4}$

Formulas of finite height

$$B_1 = p_1 \to \Box\Diamond p_1, \quad B_{i+1} = p_{i+1} \to \Box(\Diamond p_{i+1} \vee B_i)$$

Theorem (Segerberg, Maksimova)

A logic $\mathrm{L} \supseteq \mathrm{K4}$ *is locally tabular iff* L *contains* B_h *for some* $h > 0$.

New results on local tabularity of normal unimodal logics

- A necessary syntactic condition:

  a logic is locally tabular, only if it is *pretransitive* and is of *finite height*.

- A semantic criterion:

  $Log(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is of uniformly finite height and has the *ripe cluster property*.

- Segerberg – Maksimova syntactic criterion for extensions of logics much weaker than $K4$:

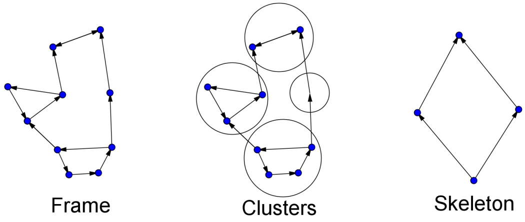
  if $m \geq 1$, $\Diamond^{m+1}p \to \Diamond p \vee p \in \mathrm{L}$, then L is locally tabular iff it is of finite height.

Frames of finite height

A poset $\mathbb{F}$ is of *finite height* $\leq n$ if every its chain contains at most n elements.

R^* denotes the transitive reflexive closure of R.
$\sim_R = R^* \cap R^{*-1}$, an equivalence class modulo $\sim_R$ is a *cluster* in (W, R) (so clusters are maximal subsets where R^* is universal).
The *skeleton of* (W, R) is the poset $(W/{\sim_R}, \leq_R)$, where for clusters $C,\ D$, $C \leq_R D$ iff xR^*y for some $x \in C, y \in D$.

Height of a frame is the height of its skeleton.



Transitive logics of finite height

For any transitive F,

$$\mathrm{F} \vDash B_h \iff ht(\mathrm{F}) \leq h,$$

where

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Pretransitive relations and logics

$R^{\leq m} = \bigcup_{0 \leq i \leq m} R^i$.
R is *m-transitive*, if $R^{\leq m} = R^*$, or equivalently, $R^{m+1} \subseteq R^{\leq m}$.
R is pretransitive, if it is m-transitive for some $m \geq 0$.

$\Diamond^0 \varphi := \varphi$, $\Diamond^{i+1}\varphi := \Diamond\Diamond^i\varphi$,
$\Diamond^{\leq m}\varphi := \bigvee_{i=0}^{m} \Diamond^i \varphi$.

Proposition

R is m-transitive iff $(W, R) \vDash \Diamond^{m+1} p \to \Diamond^{\leq m} p$.

A logic L is *m-transitive*, if $(\Diamond^{m+1} p \to \Diamond^{\leq m} p) \in L$.
L is *pretransitive*, if it is m-transitive for some $m \geq 0$. Pretransitive logics are exactly those logics, where the transitive reflexive closure modality ("master modality") is expressible.

Pretransitive logics of finite height

$\varphi^{[m]}$ is obtained from φ by replacing $\Diamond$ with $\Diamond^{\leq m}$ and $\Box$ with $\Box^{\leq m}$.

Proposition

For an m-transitive frame F, $\mathrm{F} \vDash B_h^{[m]} \iff ht(\mathrm{F}) \leq h$.

A pretransitive L *is of finite height* $\leq h$, if L contains $B_h^{[m]}$ (here m is the least such that L is m-transitive).

Necessary syntactic condition

Theorem

Every locally tabular logic is pretransitive of finite height:
L *is locally tabular* $\Rightarrow \mathrm{L}$ *contains* $(\Diamond^{m+1}p \to \Diamond^{\leq m}p) \wedge B_h^{[m]}$ *for some* m, h.

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For $m \geq 2$, pretransitive logics are much more complex than $\mathrm{K4}$.
In particular, the FMP (and even the decidability) of the logics $\mathrm{K} + (\Diamond^{m+1}p \to \Diamond^{\leq m}p)$ is unknown for $m \geq 2$.

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All logics $\mathrm{K} + (\Diamond^{m+1}p \to \Diamond^{\leq m}p) \wedge B_h^{[m]}$ have the FMP [Kudinov and Sh, 2015].

However, for $m \geq 2$, none of them are locally tabular: all these logics have Kripke incomplete extensions [Miyazaki, 2004], [Kostrzycka, 2008].

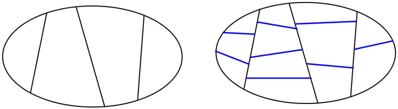
Semantic criterion

Partitions, the finite model property, and local tabularity

The FMP is often proved via constructing partitions of Kripke frames and models (*filtrations*).

Local tabularity in terms of partitions:

If F is an L-frame and $\mathcal{A}$ is a finite partition of F, then there exists a finite refinement of $\mathcal{A}$ with special properties.



As usual, a partition $\mathcal{A}$ of a non-empty set W is a set of non-empty pairwise disjoint sets such that $W = \cup\mathcal{A}$. The corresponding equivalence relation is denoted by $\sim_{\mathcal{A}}$, so $\mathcal{A} = W/{\sim_{\mathcal{A}}}$.
A partition $\mathcal{B}$ refines $\mathcal{A}$, if each element of $\mathcal{A}$ is the union of some elements of $\mathcal{B}$, or equivalently, $\sim_{\mathcal{B}} \subseteq \sim_{\mathcal{A}}$.

Minimal filtrations

The *minimal filtration of* (W, R) *through* $\mathcal{A}$ is the frame $(\mathcal{A}, R_{\mathcal{A}})$, where for $U, V \in \mathcal{A}$

$$U\, R_{\mathcal{A}}\, V \iff \exists u \in U\, \exists v \in V\; uRv.$$

Let $\mathrm{M} = (W, R, \theta)$ be a model, Γ a set of formulas. A partition $\mathcal{A}$ of M *respects* Γ, if for all $x, y \in W$

$$x \sim_{\mathcal{A}} y \quad \Rightarrow \quad \forall \varphi \in \Gamma(\mathrm{M}, x \vDash \varphi \iff \mathrm{M}, y \vDash \varphi).$$

Filtration lemma (late 1960s)

Let Γ be a set of formulas closed under tanking subformulas, $\mathcal{A}$ respect Γ. Then, for all $x \in W$ and all formulas $\varphi \in \Gamma$,

$$\mathrm{M}, x \vDash \varphi \iff (\mathcal{A}, R_{\mathcal{A}}, \theta_{\mathcal{A}}), [x]_{\mathcal{A}} \vDash \varphi.$$

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$$U \, R_{\mathcal{A}} \, V \iff \exists u \in U \, \exists v \in V \; uRv.$$

Fact

Consider a Kripke complete logic $\mathrm{L} = Log(W, R)$. If for every finite partition $\mathcal{A}$ of W there exists a finite $\mathcal{B}$ such that $\mathcal{B}$ refines $\mathcal{A}$ and $(\mathcal{B}, R_{\mathcal{B}}) \models \mathrm{L}$, then L has the FMP.

Special minimal filtrations: tuned partitions

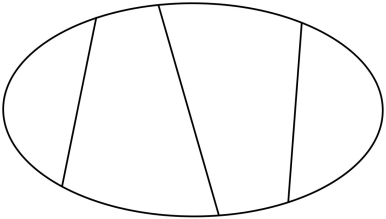
Definition

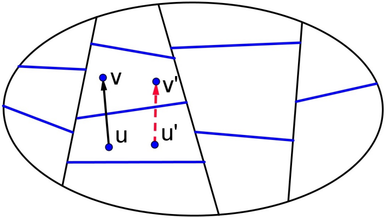
A partition $\mathcal{A}$ of $\mathrm{F} = (W, R)$ is *R-tuned*, if for any $U, V \in \mathcal{A}$

$$\exists u \in U \; \exists v \in V \; uRv \quad \Rightarrow \quad \forall u \in U \; \exists v \in V \; uRv.$$

Fact (Franzen, early 1970s)

- If $\mathcal{A}$ is R-tuned, then $Log(W, R) \subseteq Log(\mathcal{A}, R_{\mathcal{A}})$.
- If for every finite partition $\mathcal{A}$ of W there exists a finite R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$, then $Log(W, R)$ has the FMP.





First semantic criterion

Definition

A frame F is *ripe*, if there exists $f : \mathbb{N} \to \mathbb{N}$, such that for every finite partition $\mathcal{A}$ of W there exists an R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq f(|\mathcal{A}|)$.

A class of frames $\mathcal{F}$ is ripe if all frames $\mathrm{F} \in \mathcal{F}$ are ripe for a fixed f.

Theorem (First criterion)

$Log(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is ripe.

Corollary

The following conditions are equivalent:

- *a logic L is locally tabular;*
- *L is the logic of a ripe class of frames;*
- *L is Kripke complete and the class of all its frames is ripe.*

Semantic criterion. Main result

Definition

A class $\mathcal{F}$ of frames has the *ripe cluster property*, if the class of clusters in its frames $\{\mathrm{C} \mid \exists \mathrm{F} \in \mathcal{F} \text{ s.t. } \mathrm{C} \text{ is a cluster in } \mathrm{F}\}$ is ripe. A logic has the ripe cluster property, if the class of its frames has.

Theorem

A logic $\mathit{Log}(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is of uniformly finite height and has the ripe cluster property.

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Theorem

A logic $Log(\mathcal{F})$ is locally tabular iff $\mathcal{F}$ is of uniformly finite height and has the ripe cluster property.

Theorem

Suppose L_0 is a canonical pretransitive logic with the ripe cluster property. Then for any logic $\mathrm{L} \supseteq \mathrm{L}_0$:

$$\mathrm{L} \textit{ is locally tabular iff it is of finite height.}$$

Syntactic criterion for some logics below $\mathrm{K4}$

$$\mathrm{K4} \supseteq \mathrm{wK4} = \mathrm{K} + \Diamond\Diamond p \to \Diamond p \vee p \supseteq \mathrm{K} + \Diamond\Diamond\Diamond p \to \Diamond p \vee p \supseteq \ldots$$

Theorem

All the above logics have the ripe cluster property. Thus, if L contains $\Diamond^m p \to \Diamond p \vee p$ for some m, then

$$\mathrm{L} \textit{ is locally tabular iff it is of finite height.}$$

Proof.

Recall that a partition $\mathcal{A}$ of $\mathrm{F} = (W, R)$ is *R-tuned*, if for any $U, V \in \mathcal{A}$
$\exists u \in U\ \exists v \in V\ uRv \quad \Rightarrow \quad \forall u \in U\ \exists v \in V\ uRv.$

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$\exists u \in U\ \exists v \in V\ uRv \quad \Rightarrow \quad \forall u \in U\ \exists v \in V\ uRv.$

$\mathrm{K4}$ has the ripe cluster property:

If C is a cluster in a transitive frame, then C is either an irreflexive singleton, or $R = W \times W$. Trivially, any partition of C is R-tuned.

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$\mathrm{wK4}$ has the ripe cluster property:

Let $C = (W, R)$ be a cluster in a $\mathrm{wK4}$-frame. Then $\neq_W \subseteq R \subseteq W \times W$. Consider a partition $\mathcal{A}$. Let $x, y \in U$, $z \in V$, xRz for some $U, V \in \mathcal{A}$. Suppose $z \neq y$; then yRz. Suppose $z = y$; in this case $U = V$, so $x \in V$; since R is symmetric, we have yRx. Thus, any partition of C is R-tuned.

□

Syntactic criterion for some logics below $\mathrm{K4}$

$\mathrm{K4} \supseteq \mathrm{wK4} = \mathrm{K} + \Diamond\Diamond p \to \Diamond p \vee p \supseteq \mathrm{K} + \Diamond\Diamond\Diamond p \to \Diamond p \vee p \supseteq \dots$

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All the above logics have the ripe cluster property. Thus, if L contains $\Diamond^m p \to \Diamond p \vee p$ for some m, then

L *is locally tabular iff it is of finite height.*

Proof.

Recall that a partition $\mathcal{A}$ of $\mathrm{F} = (W, R)$ is R-*tuned*, if for any $U, V \in \mathcal{A}$
$\exists u \in U \ \exists v \in V \ uRv \quad \Rightarrow \quad \forall u \in U \ \exists v \in V \ uRv.$

$\mathrm{K} + \Diamond^{m+1} p \to \Diamond p \vee p$ has the ripe cluster property:

Let $C = (W, R)$ be a cluster in a frame validating this logic. If $\mathcal{A}$ is a finite partition of C, then there exists an R-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ such that $|\mathcal{B}| \leq m|\mathcal{A}|$. (The proof is a bit more tricky.)

□

Intuitionistic case

$Log(\mathbb{N}, \leq)$ is not locally tabular: it is of infinite height.
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In terms of partitions:

For every finite partition $\mathcal{A}$ of $\mathbb{N}$ there exists a finite $\leq$-tuned refinement $\mathcal{B}$ of $\mathcal{A}$. So $Log(\mathbb{N}, \leq)$ have the fmp.

But $(\mathbb{N}, \leq)$ is not ripe enough: for any natural n there exists a two-element partition of $\mathbb{N}$ such that for every $\leq$-tuned refinement $\mathcal{B}$ of $\mathcal{A}$ we have $|\mathcal{B}| > n$. So $Log(\mathbb{N}, \leq)$ is not locally tabular.

Still, if $\mathcal{A}$ is induced by upward-closed sets, then $\mathcal{A}$ consists of intervals, so it is $\leq$-tuned already.

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In the intuitionistic case, locally tabular logics are logics of ripe frames, where partitions supposed to be generated by upward-closed sets.

Two problems

Problem

A syntactic criterion for local tabularity over K.

Problem

A syntactic criterion for local tabularity of intermediate logics.

A few concluding remarks

Every locally tabular logic is pretransitive of finite height. But it is not a sufficient condition.

If a logic contains $\Diamond^m p \to \Diamond p \vee p$ for some m and is of finite height, then it is locally tabular. But it is not a necessary condition:

logics axiomatized by Chagrov's formulas corresponding to the first-order properties

$$\forall x_0, \ldots, x_{m+1} \left(x_0 R x_1 \ldots R x_{m+1} \to \bigvee_{i<j} x_i = x_j \right)$$

are locally tabular [Shehtman, 2014].

A few concluding remarks

For $m \in \mathbb{N}$, consider the first-order property

$$\mathrm{P}_m = \forall x_0, \ldots, x_{m+1} \left(x_0 R x_1 \ldots R x_{m+1} \to \bigvee_{i<j} x_i = x_j \vee \bigvee_{i+1<j} x_i R x_j \right).$$

Note that P_m implies m-transitivity.

These properties correspond to modal formulas
$\neg(A_0 \wedge \Diamond(A_1 \wedge \ldots \wedge A_{m+1}))$, where
$A_i = p_i^+ \wedge \Box q_i \wedge \bigwedge_{j<i-1} \neg q_j$ (for $i > 1$), $A_i = p_i^+ \wedge \Box q_i$ (for $i = 0, 1$).

Theorem

If $\mathcal{F}$ is a ripe class, then $\mathcal{F}$ satisfies P_m for some m.

Problem

Suppose that $\mathcal{F}$ is a class of clusters satisfying P_m for some m. Is $\mathcal{F}$ ripe?

The positive solution of the above problem will provide us with a syntactic criterion of local tabularity over K.

Thank you!