A survey on Kearns' modal semantics without possible worlds

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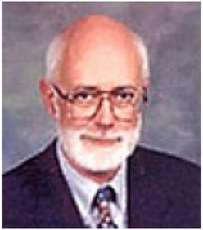
What is Kearns' semantics?



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- John Kearns develops some semantics without possible worlds for modal logics **T, S4** and **S5**.
- Motivation: "I do not think there are such things as possible worlds, or even that they constitute a useful fiction."
- Technically: (i) Non-deterministic semantics combined with (ii) additional hierarchy of valuations.

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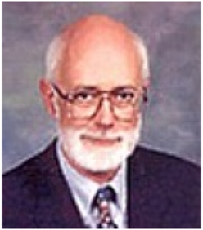

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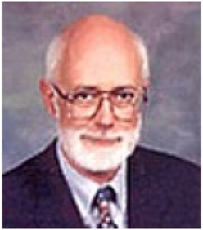

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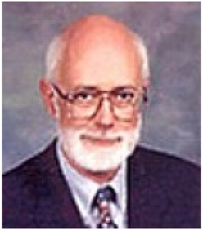

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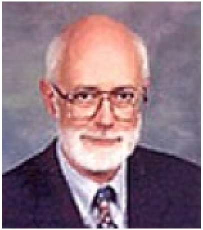

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Four truth values

Read intuitively (through the canonical model construction)

$$v_0(\beta) := \begin{cases} \mathbf{T} & \text{if } \Sigma \vdash \Box\beta \text{ and } \Sigma \vdash \beta & \text{Necessarily true} \\ \mathbf{t} & \text{if } \Sigma \nvdash \Box\beta \text{ and } \Sigma \vdash \beta & \text{Contingently true} \\ \mathbf{f} & \text{if } \Sigma \vdash \Diamond\beta \text{ and } \Sigma \nvdash \beta & \text{Contingently false} \\ \mathbf{F} & \text{if } \Sigma \nvdash \Diamond\beta \text{ and } \Sigma \nvdash \beta & \text{Necessarily false} \end{cases}$$

Non-determinacy in Kearns' semantics

$\alpha \tilde{\to} \beta$	**T**	**t**	**f**	**F**
T	**T**	**t**	**f**	**F**
t	**T**	???	**f**	**f**
f	**T**	???	???	**t**
F	**T**	**T**	**T**	**T**

Non-determinacy in Kearns' semantics

$\alpha \tilde{\to} \beta$	**T**	**t**	**f**	**F**
T	**T**	**t**	**f**	**F**
t	**T**	???	**f**	**f**
f	**T**	???	???	**t**
F	**T**	**T**	**T**	**T**

Question

How shall we assign the values for the blanks?

Non-determinacy in Kearns' semantics

$\alpha \tilde{\to} \beta$	T	t	f	F
T	T	t	f	F
t	T	t	f	f
f	T	t	t	t
F	T	T	T	T

Kearns' example

- $\alpha \to \beta$

Non-determinacy in Kearns' semantics

$\alpha \tilde{\to} \beta$	**T**	**t**	**f**	**F**
T	**T**	**t**	**f**	**F**
t	**T**	**T, t**	**f**	**f**
f	**T**	**T, t**	**T, t**	**t**
F	**T**	**T**	**T**	**T**

Kearns' example

- $\alpha \to \alpha$
- $(\alpha \to \beta) \to \alpha$

Non-determinacy in Kearns' semantics

$\alpha \tilde{\rightarrow} \beta$	**T**	**t**	**f**	**F**
T	**T**	**t**	**f**	**F**
t	**T**	**T**, **t**	**f**	**f**
f	**T**	**T**, **t**	**T**, **t**	**t**
F	**T**	**T**	**T**	**T**

Now...

How can we spell out this idea formally?

Non-deterministic Semantics


"Non-Deterministic Multiple-valued Structures" (JLC, 2005)

The first systematic presentation of non-deterministic semantics which generalizes the many-valued semantics very nicely. Since then, it has been applied to semantics for systems of paraconsistent logic and so on.

Kearns' truth tables for **S5**, **S4** and **T**

à la Avron

A **S5**-*Nmatrix* for a propositional language $\{\neg, \Box, \Diamond, \to\}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

(a) $\mathcal{V} = \{\mathbf{T}, \mathbf{t}, \mathbf{f}, \mathbf{F}\}$,

(b) $\mathcal{D} = \{\mathbf{T}, \mathbf{t}\}$,

(c) For every *n*-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding *n*-ary function $\tilde{*} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

α	$\tilde{\neg}\alpha$	$\tilde{\Box}\alpha$	$\tilde{\Diamond}\alpha$
T	**F**	**T**	**T**
t	**f**	**F**	**T**
f	**t**	**F**	**T**
F	**T**	**F**	**F**

$\alpha \tilde{\to} \beta$	**T**	**t**	**f**	**F**
T	**T**	**t**	**f**	**F**
t	**T**	**T**, **t**	**f**	**f**
f	**T**	**T**, **t**	**T**, **t**	**t**
F	**T**	**T**	**T**	**T**

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A **S4**-*Nmatrix* for a propositional language $\{\neg, \Box, \Diamond, \rightarrow\}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

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α	$\tilde{\neg}\alpha$	$\tilde{\Box}\alpha$	$\tilde{\Diamond}\alpha$
T	**F**	**T**	**T**
t	**f**	**f**, **F**	**T**, **t**
f	**t**	**f**, **F**	**T**, **t**
F	**T**	**F**	**F**

$\alpha \tilde{\rightarrow} \beta$	**T**	**t**	**f**	**F**
T	**T**	**t**	**f**	**F**
t	**T**	**T**, **t**	**f**	**f**
f	**T**	**T**, **t**	**T**, **t**	**t**
F	**T**	**T**	**T**	**T**

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A **T**-*Nmatrix* for a propositional language $\{\neg, \Box, \Diamond, \to\}$ is a tuple $M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

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(c) For every n-ary connective $*$ of $\mathcal{L}$, $\mathcal{O}$ includes a corresponding n-ary function $\tilde{*} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ as follows (we omit the brackets for sets):

α	$\tilde{\neg}\alpha$	$\tilde{\Box}\alpha$	$\tilde{\Diamond}\alpha$
T	**F**	**T**, **t**	**T**, **t**
t	**f**	**f**, **F**	**T**, **t**
f	**t**	**f**, **F**	**T**, **t**
F	**T**	**f**, **F**	**f**, **F**

$\alpha \tilde{\to} \beta$	**T**	**t**	**f**	**F**
T	**T**	**t**	**f**	**F**
t	**T**	**T**, **t**	**f**	**f**
f	**T**	**T**, **t**	**T**, **t**	**t**
F	**T**	**T**	**T**	**T**

Non-determinacy is not enough!

An example in **S5**

α	$\alpha \to \alpha$
T	**T**
t	**T**
t	**t**
f	**T**
f	**t**
F	**T**

Note the non-determinacy in red!

Non-determinacy is not enough!

An example in **S5**

α	$\alpha \to \alpha$
T	**T**
t	**T**
t	**t**
f	**T**
f	**t**
F	**T**

And here, too!

Non-determinacy is not enough!

An example in **S5**

α	$\alpha \to \alpha$	$\Box(\alpha \to \alpha)$
T	**T**	**T**
t	**T**	**T**
t	**t**	**F**
f	**T**	**T**
f	**t**	**F**
F	**T**	**T**

So, $\Box(\alpha \to \alpha)$ is not valid!

Hierarchy introduced

Definition

Let v be a function $v : L \to \mathcal{V}$. Then,

- v is a *0th-level-**S5**-valuation* if v is a legal-**S5**-valuation.
- v is a *$m+1$st-level-**S5**-valuation* iff v is a mth-level-**S5**-valuation and assigns value **T** to every sentence α if $v'(\alpha) \in \mathcal{D}$ for any mth-level-**S5**-valuation v'.

Remark

The 'height' of the hierarchy might be very large!

Definition

Based on these, we define

- v to be a **S5**-*valuation* iff v is *mth-level-**S5**-valuation* for any $m \geq 0$.
- α is a **S5**-*tautology* ($\models_{\mathbf{S5}} \alpha$) iff $v(\alpha) = \mathbf{T}$ for any **S5**-valuation v.

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An example for hierarchy in **T**

0th-level

$\Box$	$\Box$	$(\alpha$	$\to$	$\alpha)$
T, **t**, **f**, **F**	**T**, **t**	**T**	**T**	**T**
T, **t**, **f**, **F**	**T**, **t**	**t**	**T**	**t**
f, **F**	**f**, **F**	**t**	**t**	**t**
T, **t**, **f**, **F**	**T**, **t**	**f**	**T**	**f**
f, **F**	**f**, **F**	**f**	**t**	**f**
T, **t**, **f**, **F**	**T**, **t**	**F**	**T**	**F**

An example for hierarchy in **T**

1st-level

$\Box$	$\Box$	$(\alpha$	$\rightarrow$	$\alpha)$
T, **t**, **f**, **F**	**T**, **t**	**T**	**T**	**T**
T, **t**, **f**, **F**	**T**, **t**	**t**	**T**	**t**
T, **t**, **f**, **F**	**T**, **t**	**f**	**T**	**f**
T, **t**, **f**, **F**	**T**, **t**	**F**	**T**	**F**

An example for hierarchy in $\mathbf{T}$

2nd-level

$\Box$	$\Box$	$(\alpha$	$\to$	$\alpha)$
$\mathbf{T}, \mathbf{t}$	$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$	$\mathbf{T}$
$\mathbf{T}, \mathbf{t}$	$\mathbf{T}$	$\mathbf{t}$	$\mathbf{T}$	$\mathbf{t}$
$\mathbf{T}, \mathbf{t}$	$\mathbf{T}$	$\mathbf{f}$	$\mathbf{T}$	$\mathbf{f}$
$\mathbf{T}, \mathbf{t}$	$\mathbf{T}$	$\mathbf{F}$	$\mathbf{T}$	$\mathbf{F}$

An example for hierarchy in $\mathbf{T}$

3rd-level

$\Box$	$\Box$	$(\alpha$	$\rightarrow$	$\alpha)$
T	**T**	**T**	**T**	**T**
T	**T**	**t**	**T**	**t**
T	**T**	**f**	**T**	**f**
T	**T**	**F**	**T**	**F**

Hierarchy, reduced

Proposition

In defining valuations, Kearns introduced a whole hierarchy of *m*th-level-valuations. But, it turns out that this is not necessary in the case for **S4** and **S5**. Here we only need to make use of 2 levels.

Remark

However, this is not possible for **T**. Take for instance the following set formulas with iterated modalities:

$(\alpha \to \alpha)$, $\Box(\alpha \to \alpha)$, $\Box\Box(\alpha \to \alpha)$, $\Box\Box\Box(\alpha \to \alpha)$, $\Box\Box\Box\Box(\alpha \to \alpha)$, $\Box\Box\Box\Box\Box(\alpha \to \alpha)$, $\Box\Box\Box\Box\Box\Box(\alpha \to \alpha)$, $\Box\Box\Box\Box\Box\Box\Box(\alpha \to \alpha)$, $\ldots$

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Kearns' semantics for modal logic **K** and beyond

Read intuitively (through the canonical model construction for **K**)

$$v_0(\beta) := \begin{cases} \mathbf{T} & \text{if } \Sigma \vdash \Box\beta \text{ and } \Sigma \vdash \beta \text{ and } \Sigma \vdash \Diamond\beta \\ \mathbf{t_1} & \text{if } \Sigma \vdash \Box\beta \text{ and } \Sigma \vdash \beta \text{ and } \Sigma \nvdash \Diamond\beta \\ \mathbf{t} & \text{if } \Sigma \nvdash \Box\beta \text{ and } \Sigma \vdash \beta \text{ and } \Sigma \vdash \Diamond\beta \\ \mathbf{t_2} & \text{if } \Sigma \nvdash \Box\beta \text{ and } \Sigma \vdash \beta \text{ and } \Sigma \nvdash \Diamond\beta \\ \mathbf{f_2} & \text{if } \Sigma \vdash \Box\beta \text{ and } \Sigma \nvdash \beta \text{ and } \Sigma \vdash \Diamond\beta \\ \mathbf{f} & \text{if } \Sigma \nvdash \Box\beta \text{ and } \Sigma \nvdash \beta \text{ and } \Sigma \vdash \Diamond\beta \\ \mathbf{f_1} & \text{if } \Sigma \vdash \Box\beta \text{ and } \Sigma \nvdash \beta \text{ and } \Sigma \nvdash \Diamond\beta \\ \mathbf{F} & \text{if } \Sigma \nvdash \Box\beta \text{ and } \Sigma \nvdash \beta \text{ and } \Sigma \nvdash \Diamond\beta \end{cases}$$

Kearns' semantics for modal logic **K** and beyond

Truth-tables for $\Box$ for **K**

A	$\tilde{\Box}A$
$\mathbf{T}$	$\mathbf{T}, \mathbf{t}_1, \mathbf{t}, \mathbf{t}_2$
$\mathbf{t}_1$	$\mathbf{T}, \mathbf{t}_1, \mathbf{t}, \mathbf{t}_2$
$\mathbf{t}$	$\mathbf{f}_2, \mathbf{f}, \mathbf{f}_1, \mathbf{F}$
$\mathbf{t}_2$	$\mathbf{f}_2, \mathbf{f}, \mathbf{f}_1, \mathbf{F}$
$\mathbf{f}_2$	$\mathbf{T}, \mathbf{t}_1, \mathbf{t}, \mathbf{t}_2$
$\mathbf{f}$	$\mathbf{f}_2, \mathbf{f}, \mathbf{f}_1, \mathbf{F}$
$\mathbf{f}_1$	$\mathbf{T}, \mathbf{t}_1, \mathbf{t}, \mathbf{t}_2$
$\mathbf{F}$	$\mathbf{f}_2, \mathbf{f}, \mathbf{f}_1, \mathbf{F}$

Kearns' semantics for modal logic **K** and beyond

Truth-tables for $\Box$ for **D**

A	$\tilde{\Box} A$
$\mathbf{T}$	$\mathbf{T}, \mathbf{t}, \mathbf{t_2}$
$\mathbf{t}$	$\mathbf{f_2}, \mathbf{f}, \mathbf{F}$
$\mathbf{t_2}$	$\mathbf{f_2}, \mathbf{f}, \mathbf{F}$
$\mathbf{f_2}$	$\mathbf{T}, \mathbf{t}, \mathbf{t_2}$
$\mathbf{f}$	$\mathbf{f_2}, \mathbf{f}, \mathbf{F}$
$\mathbf{F}$	$\mathbf{f_2}, \mathbf{f}, \mathbf{F}$

Kearns' semantics for modal logic **K** and beyond

Truth-tables for $\Box$ for **T**

A	$\tilde{\Box}A$
T	**T**, **t**
t	**f**, **F**
f	**f**, **F**
F	**f**, **F**

Future avenues (I)

Analyticity

Any partial valuation which seems to refute a given formula can be extended to a full valuation (which necessarily refutes that formula too).

Counterexample to analyticity

$\Box$	$(p$	$\to$	$q)$	$\to$	$\Box$	$(\Box$	p	$\to$	$\Box$	$q)$
T	**t**	**T**	**t**	**F**	**F**	**f**	**t**	**t**	**f**	**t**

Question

In order to evaluate a formula in Kearns' semantic one has to take into account *every* formula.

Is it possible to simplify the requirement that we have to take every formula into account? E.g., in order to evaluate a formula only take a special set of formulas into account?

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Counterexample to analyticity

$$\frac{\Box \quad (p \quad \to \quad q) \quad \to \quad \Box \quad (\Box \quad p \quad \to \quad \Box \quad q)}{\mathbf{T} \quad \mathbf{t} \quad \mathbf{T} \quad \mathbf{t} \quad \mathbf{F} \quad \mathbf{F} \quad \mathbf{f} \quad \mathbf{t} \quad \mathbf{t} \quad \mathbf{f} \quad \mathbf{t}}$$

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Future avenues (II)

Kearn's semantics for (sub)intuitionistic logics?

In view of the close connection between the modal logic S4 and intuitionistic logic, one may also ask if we can devise a Kearns-style semantics for intuitionistic logic. To this end, one may introduce an Nmatrix for the intuitionistic conditional based on the translation between S4 with the persistence condition and intuitionistic logic.

Difficulty

We require further consideration since it is not clear at all how to rule out valuations to include, for example, $\alpha \to_{\Box} \alpha$ as a tautology.

$\alpha \tilde{\to}_{\Box} \beta$	**T**	**t**	**f**	**F**
T	**T**	**f**, **F**	**f**, **F**	**F**
t	**T**	**T**, **f**, **F**	**f**, **F**	**f**, **F**
f	**T**	**T**, **f**, **F**	**T**, **f**, **F**	**f**, **F**
F	**T**	**T**	**T**	**T**

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$\alpha \tilde{\to}_{\Box} \beta$	**T**	**t**	**f**	**F**
T	**T**	**f**, **F**	**f**, **F**	**F**
t	**T**	**T**, **f**, **F**	**f**, **F**	**f**, **F**
f	**T**	**T**, **f**, **F**	**T**, **f**, **F**	**f**, **F**
F	**T**	**T**	**T**	**T**

Köszönöm!

The details of the talk are available at:

`https://sites.google.com/site/hitoshiomori/home/publications`

`http://www.ruhr-uni-bochum.de/philosophy/logic/team/skurt.html.en`