MODAL AUTOMATA

studying modal fixpoint logics one step at a time

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(largely joint work with Carreiro, Enqvist, Facchini, Fontaine, Seifan, Zanasi, ...)

Fixpoints in modal logic

Examples:

- $U\varphi\psi \equiv \varphi \vee (\psi \wedge \bigcirc U\varphi\psi)$
- $\langle\alpha^*\rangle\varphi \equiv \varphi \vee \langle\alpha\rangle\langle\alpha^*\rangle\varphi$
- $C\varphi \equiv \bigwedge_a K_a\varphi \wedge \bigwedge_a K_a C\varphi$

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Languages:

- LTL, CTL, PDL, CTL*, GL, $\ldots \subseteq \mu$ML
- μML was introduced by Dexter Kozen (1983)
- μML extend basic modal logic with explicit fixpoint operators μ, ν
  - $U\varphi\psi := \mu x.\varphi \vee (\psi \wedge \bigcirc x)$
  - $\langle\alpha^*\rangle\varphi := \mu x.\varphi \vee \langle\alpha\rangle x$
  - $[\alpha^*]\varphi = \nu x.\varphi \wedge [\alpha]x.$
  - $C\varphi := \nu x. \bigwedge_a K_a\varphi \wedge \bigwedge_a K_a x$

The modal μ-calculus μML

- Formulas:

$$\varphi \ ::= \ p \ \mid \ \neg\varphi \ \mid \ \varphi \vee \varphi \ \mid \ \Diamond\varphi \ \mid \ \mu p.\varphi'$$

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$$\begin{array}{lcl} [\![\mu p.\varphi]\!]^{\mathbb{S},V} & := & LFP(\lambda X.[\![\varphi]\!]^{\mathbb{S},V[p\mapsto X]}) \\ [\![\nu p.\varphi]\!]^{\mathbb{S},V} & := & GFP(\lambda X.[\![\varphi]\!]^{\mathbb{S},V[p\mapsto X]}) \end{array}$$

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- Unravelling:
  - $\eta x.\varphi \equiv \varphi[\eta x.\varphi/x]$ for $\eta = \mu, \nu$
  - ν can unravel infinitely often, μ cannot
  - traces in evaluation game and in tableaux

The modal μ-calculus 2

- [+] natural extension of basic modal logic
- [+] expressive
- [+] good computational properties
- [+] nice meta-logical theory
- [−] hard to understand (nested) fixpoint operators
- [−] theory of μML isolated from theory of ML

Logic & Automata

Most results on μML use automata . . .

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Automata in Logic

- long & rich history (Büchi, Rabin, ...)
- mathematically interesting theory
- many practical applications
- automata for μML:
  - Janin & Walukiewicz (1995): μ-automata (nondeterministic)
  - Wilke (2002): modal automata (alternating)

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- Modal automata
- One-step logic
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Kripke structures

- Fix a set X of proposition letters
- Elements of PX are called colors
- Transition system/Kripke structure: pair $\mathbb{S} = (S, \sigma)$ with
  - $\sigma = (\sigma_R, \sigma_V)$,
  - $\sigma_V : S \to \mathrm{P}X$ is a marking/coloring
  - $\sigma_R : S \to \mathrm{P}S$ encodes the binary relation
- $\sigma(s) \in \mathrm{P}X \times \mathrm{P}S$ is the one-step unfolding of s.
- Elements over $\mathrm{P}X \times \mathrm{P}S$ are called one-step frames over S

One-step Logic

- A one-step frame is a pair (Y, U) with $Y \subseteq \mathrm{X}$ and U some set
- Let A (variables) be disjoint from X (proposition letters): $A \cap X = \varnothing$
- One-step formulas: $\neg p \wedge \Diamond(a \wedge b)$, $\Box a \wedge (\Diamond b \vee q)$, ...

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with $p \in \mathrm{X}$ and $\pi \in Latt(A)$

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  with $p \in \mathsf{X}$ and $\pi \in \mathit{Latt}(A)$
- $\mathit{Latt}(A)$: prop. lang. over A ($\pi \ ::= \ a \mid \bot \mid \top \mid \pi \vee \pi \mid \pi \wedge \pi$)
- One-step model (Y, U, m) with $Y \subseteq \mathsf{X}$ and $m : U \to \mathsf{P}A$
- One-step semantics interprets $1\mathrm{ML}(\mathsf{X}, A)$ over one-step models

One-step Semantics: details

- One-step model (Y, U, m) with $Y \subseteq \mathsf{X}$ and $m : U \to \mathsf{P}A$
- Zero-step semantics

$$\begin{array}{lll} [\![a]\!]^0 & := & \{u \in U \mid a \in m(u)\} \\ [\![\bot]\!]^0 & := & \varnothing \\ [\![\top]\!]^0 & := & U \end{array} \qquad \begin{array}{lll} [\![\pi \vee \pi']\!]^0 & := & [\![\pi]\!]^0 \cup [\![\pi']\!]^0 \\ [\![\pi \wedge \pi']\!]^0 & := & [\![\pi]\!]^0 \cap [\![\pi']\!]^0 \end{array}$$

- One-step semantics

$$\begin{array}{lll} (Y, U, m) \Vdash^1 p & \text{if} & p \in Y \\ (Y, U, m) \Vdash^1 \neg p & \text{if} & p \notin Y \\ (Y, U, m) \Vdash^1 \Diamond\pi & \text{if} & U \cap [\![\pi]\!]^0 \neq \varnothing \\ (Y, U, m) \Vdash^1 \Box\pi & \text{if} & U \subseteq [\![\pi]\!]^0 \\ (Y, U, m) \Vdash^1 \bot & & \textit{never} \\ (Y, U, m) \Vdash^1 \top & & \textit{always} \\ (Y, U, m) \Vdash^1 \alpha \vee \alpha' & \text{if} & (Y, U, m) \Vdash^1 \alpha \text{ or } (Y, U, m) \Vdash^1 \alpha' \\ (Y, U, m) \Vdash^1 \alpha \wedge \alpha' & \text{if} & (Y, U, m) \Vdash^1 \alpha \text{ and } (Y, U, m) \Vdash^1 \alpha' \end{array}$$

Modal automata

- A modal automaton is a triple $\mathbb{A} = (A, \Theta, Acc)$, where
  - A is a finite set of states
  - $\Theta : A \to 1\mathrm{ML}(\mathsf{X}, A)$ is the transition map
  - $Acc \subseteq A^\omega$ is the acceptance condition

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  - $Acc \subseteq A^{\omega}$ is the acceptance condition
- An initialized automaton is pair $(\mathbb{A}, a)$ with $a \in A$
- Parity automata: Acc is given by map $\Omega : A \to \omega$
  - Given $\rho \in A^{\omega}$, $Inf(\rho) := \{a \in A \mid a$ occurs infinitely often in $\pi_b\}$
  - $Acc_{\Omega} := \{\rho \in A^{\omega} \mid \max\{\Omega(a) \mid a \in Inf(\rho)\}$ is even $\}$

Acceptance game

Acceptance game $\mathcal{A}(\mathbb{A}, \mathbb{S})$ of $\mathbb{A} = \langle A, \Theta, Acc \rangle$ on $\mathbb{S} = \langle S, \sigma \rangle$:

Position	Player	Admissible moves
$(a, s) \in A \times S$	$\exists$	$\{m : \sigma_R(s) \to \mathsf{P}A \mid \sigma(s), m \models \Theta(a)\}$
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Winning conditions:

- finite matches are lost by the player who gets stuck,
- infinite matches are won as specified by the acceptance condition:
  - match $\pi = (a_0, s_0)m_0(a_1, s_1)m_1 \ldots$ induces list $\pi_A := a_0 a_1 a_2 \ldots$
  - $\exists$ wins if $\pi_A \in Acc$

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Definition $(\mathbb{A}, a)$ accepts $(\mathbb{S}, s)$ if $(a, s) \in Win_{\exists}(\mathcal{A}(\mathbb{A}, \mathbb{S}))$.

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- Which properties of modal parity automata are determined
  - already at one-step level
  - by the interaction of combinatorics and dynamics

Fragments/Variations

Fix automaton $\mathbb{A} = (A, \Theta, \Omega)$

- Write $a \rightsquigarrow b$ if b occurs in $\Theta(a)$, and $\triangleright := (\rightsquigarrow)^+$
- A cluster is an equivalence relation of $\bowtie := \triangleright \cup \triangleleft \cup \Delta_A$
- $\mathbb{A}$ is weak if $a \bowtie b$ implies $\Omega(a) = \Omega(b)$ so WLOG $\Omega : A \to \{0, 1\}$
- A PDL-automaton is a weak parity automaton $\mathbb{A}$ s.t. for $a \in A$:
  - if $\Omega(a) = 1$ then $\Theta(a) \in ADD^1(\mathsf{X}, A, C)$ given by

    $$\alpha \ ::= \ \beta \ \mid \ \langle d \rangle c \ \mid \ \alpha \vee \alpha.$$

    where $\beta \in 1ML(\mathsf{X}, A \setminus C)$ and $c \in C$
  - if $\Omega(a) = 0$ then $\Theta(a) \in MUL^1(\mathsf{X}, A, C)$ defined dually

Proposition (Carreiro & Venema) test-free PDL $\equiv$ PDL-automata

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Proposition If $(Y, U, m) \leftrightarroweq^1 Y', U', m')$ then $(Y, U, m) \equiv^1 Y', U', m')$.

- A one-step morphism $f : (Y, U, m) \to (Y', U', m')$ is
  - a surjection $f : U \to U'$
  - such that $m = m' \circ f$
  - but it only exists if $Y = Y'$

One-step soundness and completeness

- Given $\alpha, \alpha' \in$ 1ML define $\models^1 \alpha \leq \alpha'$ if for all (Y, U, m):

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  - monotonicity rule for $\Diamond$: $\pi \leq \pi' \ / \ \Diamond\pi \leq \Diamond\pi'$
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- For more on this, check the literature on coalgebra (Pattinson, Schröder,...)

Chromatic automata

Separate X from A

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- Point: (σ_R, m) is an A-structure in the sense of model theory, i.e. a pair (D, I) with $I : A \to \mathrm{P}D$ interpreting each $a \in A$

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Proposition Modal automata $\sim \mathrm{Aut}(FO)$

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Hence Aut(FO) is the bisimulation-invariant fragment of Aut(FOE).

Corollary (Janin & Walukiewicz) μML $\equiv$ MSO$/\underline{\leftrightarrow}$.

Proof (1) μML $\equiv$ Aut(FO)
(2) MSO $\equiv$ Aut(FOE) (on trees)

Bisimulation invariance

Bisimulation invariance

Theorem Let $\mathcal{L}$ and $\mathcal{L}'$ be two one-step languages. Then

$$\mathcal{L}' \equiv_s \mathcal{L}/\underline{\leftrightarrow}^1 \text{ implies } \mathsf{Aut}(\mathcal{L}') \equiv_s \mathsf{Aut}(\mathcal{L})/\underline{\leftrightarrow}$$

This result allows

- variations/generalizations of the Janin-Walukiewicz Theorem

Overview

- Introduction
- Modal automata
- One-step logic
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Model theory of modal automata

- normal form theorems
- characterization theorems
- (uniform) interpolation
- ...

Normal forms

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Normal forms

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- α is disjunctive if for all $(Y, U, m) \Vdash^1 \alpha$
  there is (Y, U', m') and a fr morphism $f : (Y, U') \to (Y, U)$ s.t.
  - $m' \circ f \subseteq m$
  - $(Y', U', m') \Vdash^1 \alpha$ and
  - $|m(u)| \leq 1$ for all $u \in U$.
- Example $\nabla B := \bigwedge \Diamond B \wedge \Box \bigvee B$ for $B \subseteq A$
- $\mathbb{A} = (A, \Theta, \Omega)$ is disjunctive if $\Theta(a)$ is disjunctive for all $a \in A$

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Simulation Theorem (Janin & Walukiewicz)
Every modal automaton has a disjunctive equivalent:

$$\mathrm{Aut}(1\mathrm{ML}) \equiv \mathrm{Aut}(1\mathrm{ML}^d)$$

Uniform Interpolation

Theorem (D'Agostino & Hollenberg) μML enjoys uniform interpolation

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Theorem Aut($\mathcal{L}$) enjoys uniform interpolation if
(1) $\mathcal{L}$ consists of disjunctive formulas
(2) $\mathcal{L}$ is closed under disjunctions

Łos-Tarski Theorem

- φ has the LT-property if the truth of φ is preserved under taking submodels.

Theorem (D'Agostino & Hollenberg)
$\xi \in \mu$ML has LT iff $\xi \equiv \varphi \in \mu\text{ML}_\forall$

$$\mu\text{ML}_\forall \ni \varphi \ ::= \ p \mid \neg p \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mu x.\varphi \mid \nu x.\varphi$$

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- $\mathcal{L}' \equiv_s \mathcal{L}/LT$ if there is a map $(\cdot)^{LT} : \mathcal{L} \to \mathcal{L}'$ such that

$$\alpha \in \mathcal{L} \text{ has LT iff } \alpha \equiv_s \alpha^{LT}$$

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Proposition If $\mathcal{L}' \equiv_s \mathcal{L}/LT$ then $\mathrm{Aut}(\mathcal{L}') \equiv_s \mathrm{Aut}\mathcal{L}/LT$

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Proposition $\mathrm{FO}_\forall \equiv_s \mathrm{FO}/LT$

Corollary (1) $\mathrm{Aut}(\mathrm{FO}_\forall) \equiv_s \mathrm{Aut}(\mathrm{FO})/LT$
(2) it is decidable whether $\mathbb{A} \in \mathrm{Aut}(\mathrm{FO})/\varphi \in \mu\mathrm{ML}$ has LT

Continuity

- A formula φ is (Scott) p-continuous if

$$\mathbb{S}, s \Vdash \varphi \text{ iff } \mathbb{S}[p \mapsto U], s \Vdash \varphi \text{ for some finite } U \subseteq V(p)$$

  or equivalently

$$\varphi_p(W) = \bigcup \{\varphi_p(U) \mid U \subseteq_\omega W\}$$

Theorem (Fontaine) $\xi \in \mu$ML is p-continuous iff $\xi \equiv \varphi \in CONT_p(\mu\text{ML})$

$$CONT_P(\mu\text{ML}) \ni \varphi \ ::= \ p \mid \psi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \mu x.\varphi'$$

where $p \in P$, $\psi \in \mu$ML is p-free, and $\varphi' \in CONT_{P\cup\{x\}}(\mu\text{ML})$.

Continuity continued

- φ is horizontally p-continuous if

$$\mathbb{S}, s \Vdash \varphi \text{ iff } \mathbb{S}[p \mapsto U], s \Vdash \varphi \text{ for some finitely branching } U \subseteq V(p)$$

- φ is vertically p-continuous if

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Observations

- p-continuity = horizontal p-continuity + vertical p-continuity
- horizontal p-continuity is easily determined at one-step level
- vertical p-continuity is easily determined at level of priority map Ω

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Theorem (Fontaine & Venema)
Syntactic characterizations of automata that are (hor/vert) continuous.

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Theorem (Fontaine & Venema)
Syntactic characterizations of automata that are (hor/vert) continuous.
All three are decidable properties.

Continuity 3

Sublanguages of μML:

- μML

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle d \rangle \varphi \mid \mu x.\varphi'$$

  where φ' is monotone in x

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Theorem (Carreiro, Facchini, Venema & Zanasi) $\mu_c\mathrm{ML} \equiv \mathrm{WMSO}/\underline{\leftrightarrow}$
Proof
(1) $\mathrm{WMSO} \equiv \mathrm{Aut}_{cw}(\mathrm{FO}^\infty)$
(2) careful analysis of FO^∞ as a one-step language
(3) $\mathrm{Aut}_{cw}(\mathrm{FO}^\infty) \equiv_s \mathrm{Aut}_{cw}(\mathrm{FO})$

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Completeness

Kozen Axiomatisation:

- complete calculus for modal logic
- $\varphi(\mu p.\varphi) \vdash_K \mu p.\varphi$ ($\alpha \vdash_K \beta$ abbreviates $\vdash_K \alpha \to \beta$)
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$\vdash_K$ is sound and complete for all formulas.

Questions (2015)
How to generalise this to similar logics, eg, the monotone μ-calculus?
How to generalise this to restricted frame classes?
Does completeness transfer to fragments of μML?

Walukiewicz' Proof: Evaluation

Why is Walukiewicz' proof hard?

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1. complex combinatorics of traces
2. incorporate simulation theorem into derivations
3. mix of $\vdash_K$-derivations, tableaux and automata
4. tableau rules for boolean connectives complicate combinatorics
5. ...

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5. ...

content vs wrapping

Our Approach: Principles

- separate the combinatorics from the dynamics
- focus on automata rather than formulas
- make traces first-class citizens

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Dynamics: coalgebra

- one step at a time
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Automata

- uniform, 'clean' presentation of fixpoint formulas
- excellent framework for developing trace theory
- direct formulation of simulation theorem

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- direct formulation of simulation theorem
- bring automata into proof theory

Automata & Formulas

Theorem
There are maps $\mathbb{B}_{-} : \mu\mathsf{ML} \to \mathsf{Aut}(\mathsf{ML}_1)$ and $\xi : \mathsf{Aut}(\mathsf{ML}_1) \to \mu\mathsf{ML}$ that
(1) preserve meaning: $\varphi \equiv \mathbb{B}_{\varphi}$ and $\mathbb{A} \equiv \xi(\mathbb{A})$

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(3) interact nicely with Booleans, modalities, fixpoints, and substitution:

$$\xi(\mathbb{A}[\mathbb{B}/x]) \equiv_{K} \xi(\mathbb{A})[\xi(\mathbb{B})/x].$$

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As a corollary, we may apply proof-theoretic concepts to automata

Framework

Satisfiability Game $\mathcal{S}(\mathbb{A})$ (Fontaine, Leal & Venema 2010)

- basic positions: binary relations $R \in \mathsf{P}(A \times A)$
- R corresponds to $\bigwedge\{\Delta(a) \mid a \in R\}$
- direct representation of $\mathbb{A}$-traces through $R_0 R_1 \cdots$
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- basic positions: pair of binary relations (R, R')
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Special Automata

Modal Automaton: $\mathbb{A} = \langle A, a_I, \Delta, \Omega \rangle$, with $\Delta : A \to \mathsf{ML}_1(P, A)$

- *Latt*(A) $\alpha ::= p \mid \alpha \vee \alpha \mid \bot \mid \alpha \wedge \alpha \mid \top$
- $\mathsf{ML}_1(P, A)$ $\varphi ::= p \mid \neg p \mid \Diamond\alpha \mid \Box\alpha \mid \varphi \vee \varphi \mid \bot \mid \varphi \wedge \varphi \mid \top$

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Disjunctive Automaton $\Delta : A \to \mathsf{ML}_1^d(P, A)$

- *List*(*P*) $\pi ::= \bot \mid \top \mid p \wedge \pi \mid \neg p \wedge \pi$
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  where $B \subseteq A$.

Semi-disjunctive Automaton $\Delta(a) \in \mathrm{ML}_1^{s, C_a}(P, A)$

- $List(P)$ $\pi \ ::= \ \bot \mid \top \mid p \wedge \pi \mid \neg p \wedge \pi$
- $\mathrm{ML}_1^{s,C}(P, A)$ $\varphi \ ::= \ \bot \mid \top \mid \pi \wedge \nabla\{\bigwedge B \mid B \in \mathcal{B}\} \mid \varphi \vee \varphi$,
  where for all $B \in \mathcal{B}$, all $b, b' \in B$ with $b \neq b'$, b or b' is a maximal even element of C.

Key Lemmas

Strong Simulation Theorem (cf W39)
For every modal automaton $\mathbb{A}$ there is an equivalent disjunctive simulation $\overline{\mathbb{A}}$ such that

$$\mathbb{A} \models_G \overline{\mathbb{A}}$$

$$\overline{\mathbb{A}} \models_G \mathbb{A}$$

$$\mathbb{B}[\overline{\mathbb{A}}/x] \models_G \mathbb{B}[\mathbb{A}/x]$$

for all automata $\mathbb{B}$.

Lemma (cf W36)
Let $\mathbb{A}, \mathbb{B}$ be respectively a semidisjunctive and an arbitrary automaton. If $\mathbb{A} \models_G \mathbb{B}$, then $\mathbb{A} \wedge \neg\mathbb{B}$ has a thin refutation.

Lemma (cf Kozen)
If $\mathbb{A}$ is a consistent automaton, then $\exists$ has a winning strategy in $\mathcal{S}_{thin}$.

Corollary If $\mathbb{A}$ is a consistent (semi-)disjunctive automaton, then $\mathbb{A}$ is satisfiable.

Proof of Kozen-Walukiewicz Theorem

Main Proposition
For every $\varphi \in \mu\text{ML}$ there is an equivalent disjunctive automaton $\mathbb{D}$ such that

$$\varphi \vdash_K \mathbb{D}.$$

Proof
Induction on φ: similar to Walukiewicz' proof, but using the above lemmas.

Work in progress

Theorem Assume that

- $\mathcal{L}$ is a one-step language with an adequate disjunctive base
- **H** is a one-step sound and complete axiomatization for $\mathcal{L}$

Then $\mathbf{H} + \mathit{Koz}$ is a sound and complete axiomatization for $\mu\mathcal{L}$.

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Examples:

- linear time μ-calculus
- k-successor μ-calculus
- standard modal μ-calculus
- graded μ-calculus
- monotone modal μ-calculus
- game μ-calculus
- ...

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Sample results:

- R1 one-step bisimulation invariance implies bisimulation invariance
- R2 one-step disjunctiveness implies uniform interpolation
- R3 systematic characterization of continuity, complete additivity, ...
- R4 one-step completeness + disjunctive basis implies completeness

Sample questions/problems:

- Q1 Does J-W Thm hold on finite models?
- Q2 Which fragments of μML have interpolation? (PDL!)
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Modal automata are too nice to leave them to computer science alone!

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