The incompleteness of P.E.

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If $a_1,\ a_2,\ \ldots,\ a_n \in K_{11}$, $\delta a_1 \delta a_2 \delta \ldots \delta a_n \delta$ is the sequence number of the sequence of numbers $(a_1,\ a_2,\ \ldots, a_n)$ or of the sequence of expressions $(E_{a_1},\ E_{a_2},\ \ldots,\ E_{a_n})$.

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A formation sequence for terms is a finite sequence $X_1,\ X_2,\ \ldots,\ X_n$ of expressions s.t. every X_i is either a variable, or a numeral, or for some $j, k < i$ $\mathcal{R}_t(X_j,\ X_k,\ X_i)$ holds.

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Formulas: the procedure is the same. We define a formation relation for formulas, construction sequences for formulas and an expression is a formula iff there is a formation sequence for it.

Arithmetization of P.E.

Use the following abbreviations:

- x imp y for the Gödel number of $(E_x \to E_y)$;
- neg(x) for the Gödel number of $\neg E_x$;
- x pl y for the Gödel number of $(E_x + E_y)$;
- x tim y for the Gödel number of $(E_x \cdot E_y)$;
- x exp y for the Gödel number of $(E_x \mathbf{E} E_y)$;
- s(x) for the Gödel number of E_x';
- x id y for the Gödel number of $E_x = E_y$;
- x le y for the Gödel number of $E_x \leq E_y$.

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$\mathrm{N}_{12}(x)$: a bit difficult because of the difficult structure of the axiom N_{12}.

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$$R_E(x) \leftrightarrow P_E(neg(x))$$

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Hence; R_E has a Gödel sentence. This sentence is true iff it is refutable. By correctness, the sentence is false but not refutable. Its negation can't be refutable because it is true. Q.e.d.