Exponentiation is arithmetic First Incompleteness Theorem and Tarski's Theorem for P.A.

András Máté

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Σ_0 sentences are effectively decidable.

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4. If F and G are Σ formulas, then so are $F \vee G$ and $F \wedge G$. If F is Σ_0, then $F \to G$ is Σ, too.

The relations expressible by Σ_1 resp. Σ formulas are Σ_1 resp. Σ relations. Every Σ relation is Σ_0 or Σ_1 (later). They are the recursively enumerable relations.

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$x *_p y = z \leftrightarrow x \cdot p^{l_p(y)} + y = z \leftrightarrow (\exists w_1 \leq z)(\exists w_2 \leq z)(w_1 = p^{l_p(y)} \wedge w_2 = x \cdot w_1 \wedge w_2 + y = z)$

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$x_1 *_p x_2 *_p \ldots *_p x_n = y$ and $x_1 *_p x_2 *_p \ldots *_p x_n P_p y$ are both Σ_0 (for $n \geq 2$). On the same way.

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It follows that $\tilde{P_E}$ is arithmetic, too (although not Σ).

To prove that the adjoint (A^*) of every arithmetic set (A) is arithmetic, too, we need to prove that $x^y = z$ is arithmetic.

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Finite Set Lemma: There is a Σ_0 relation $K(x, y, z)$ s. t.

- for any finite sequence of ordered pairs of natural numbers $((a_1, b_1), \ldots (a_n, b_n))$, there is a number z s.t. $K(x, y, z)$ iff (x, y) is one of the (a_i, b_i)-s;
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$$1(x) \leftrightarrow (\forall y \leq x)(yPx \rightarrow 1Py)$$

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If z is a sequence number$_{new}$ of θ, then $(K(x, y, z)$ holds iff (x, y) is a member of $\theta)$.
Obviously, for any triple of natural numbers (x, y, z), if $K(x, y, z)$ holds, then $x, y \leq z$.

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We have now proven that exponentiation is arithmetic with the help of the Σ_0 relation K encoding finite sequences of ordered pairs. But things become simpler if we have a function encoding the finite sequences of numbers.

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Be w a sequence number$_{new}$ for $(0, a_0), (1, a_1), \ldots (n, a_n)$. For each $i \leq n$, $K(i, a_i, w)$ holds and there is no other m s.t. $K(i, m, w)$. Therefore $\beta(w, i) = a_i$.

Theorems

E-theorem via Beta-function:

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The T_A set of the G?l numbers of true arithmetic sentences is not arithmetic.

If it were, then $\tilde{T}_A$ and $\tilde{T}_A^*$ were arithmetic, too. Therefore, $\tilde{T}_A$ would have a G?l sentence and this sentence were true iff it were not true.

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Theorem P.A. is incomplete.
Because P_E and R_E are Σ, P^*_E and R^*_E are Σ, too. $\tilde{P}^*_E$ is arithmetic, therefore $\tilde{P}_E$ has an *arithmetic* G?l sentence $H(\bar{h})$ (where $H(v_1)$ is the formula expressing $\tilde{P}^*_E$). It is true iff it is not provable in P.E. By correctness, it is true and not provable in P.E. – even less in P.A. $\neg H(\bar{h})$ is false, therefore it is not provable in P.A. Q.e.d.

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An excercise for homework (easy but important):
We know that the above sentence $H(\bar{h})$ is true (let us call it G). Let us add it to the axioms of P.A. The resulting system P.A $+G$ is correct. Is it complete?

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Recursively enumerable and recursive sets and relations

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Intuitively, a set is recursively enumerable if there is an automata (recursive function, Turing-machine, Markov algorithm) that produces all and only its members as outputs. In other words, every member of the set occurs as its output after a finitely long time.
Recursive sets are *decidable*: after a finite time, each member of our domain occurs either as the output of the automata enumerating the set or as the output of the automata enumerating its complement.

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Hilbert's program was: let us prove theorems about mathematical theories *by finitary means* ($\approx$ using only bounded quantifiers). Obvious candidate for a suitable framework: a finitary fragment of P.A.

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Nothing. It would be something like the Truth-teller.