Addenda to the first incompleteness theorem

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If $\mathcal{S}$ is consistent, every true Σ_0 sentence is provable, the $A(v_1, v_2)$ Σ_0 formula enumerates P^* and the Gödel number of $\forall v_2 \neg A(v_1, v_2)$ is a, then $G = \forall v_2 \neg A(\bar{a}, v_2)$ is true. Because:

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G is not provable, therefore $a \notin P^*$. Because $A(v_1, v_2)$ enumerates P^*, for any n, the sentences $A(\bar{a}, \bar{n})$ are refutable and therefore the sentences $\neg A(\bar{a}, \bar{n})$ are provable. So they are all true and hence $G = \forall v_2 \neg A(\bar{a}, v_2)$ is true, too.

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Consequence: if P.A. is consistent, then G is true.

Let us extend P.A. with the axiom $\neg G$. This system is not correct, it is consistent but not ω-consistent (if P.A. was consistent).

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Theorem: If P.A. is consistent, then it is ω-incomplete.

A plausible generalization: if $\mathcal{S}$ is consistent, axiomatizable and every true Σ_0-sentence is provable, then it is ω-incomplete.

Homeworks

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1. If X is a sentence with the Gödel number x, be $P(\bar{X})$ the sentence $P(\bar{x})$ (just another notation). The Σ_1 formula $P(v_1)$ expresses P, the set of Gödel numbers of provable sentences. I.e., for any sentence X, $P(\bar{X})$ is true iff X is provable. If X is a Σ_0 sentence and it is true, then it is provable. Therefore, for any X Σ_0 sentence, $X \to P(\bar{X})$ is true. Show that it is provable, too.

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2. Show that every true Σ_1 sentence is provable in P.A. (Therefore, for any X Σ_1 sentence, the sentence $X \rightarrow P(\bar{X})$ is true.)

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3. Prove that not every sentence of the form $X \to P(\bar{X})$ (where X is any sentence) is provable in P.A., if P.A. is correct.

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After László Kalmár.

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Enumerate the one-variable open terms of our language:

$$k_0(x),\ k_1(x),\ \ldots, k_n(x),\ \ldots$$

Each of them receives an ordinal (i.e. Gödel) number.

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Consider the inequalities of the form $k_n(l) \neq m$ and arrange them in a two-dimensional table on the following way:

$$\begin{array}{cccccc} k_0(x) \neq 0 & k_0(x) \neq 1 & \ldots & k_0(x) \neq n & \ldots \\ k_1(x) \neq 0 & k_1(x) \neq 1 & \ldots & k_1(x) \neq n & \ldots \\ \vdots & & & & \\ k_n(x) \neq 0 & k_n(x) \neq 1 & \ldots & k_n(x) \neq n & \ldots \end{array}$$

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\end{array}$$

The red formulas are the diagonal sentences. The nth of them says that the value of the nth term never equals to its own Gödel number.

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Assumed that our arithmetics does not prove false Σ_1 sentences, incompleteness follows. If G were false, then it would be provable – therefore it is true. But in this case, it is not provable and its negation is not provable, either, because $\neg G$ is false.

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This consideration proves incompleteness assuming truth and needs expressibility. To prove it from ω-consistency we needed representability and from simple consistency, we will need separability.