The Gödel-Rosser theorem

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Remember the last two axiom schemes of the system (R):

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Ω_5 $v_1 \leq \bar{n} \vee \bar{n} \leq v_1$

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Consequences:

1. If a system is a consistent axiomatizable extension of Ω_4 and Ω_5 in which all true Σ_0 sentences are provable, then it is incomplete.
2. If a system is a consistent axiomatizable extension of (R), then it is incomplete.
3. If P.A. is consistent, then it is incomplete.

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$H(\bar{h})$ is provable iff $h \in A$ (representation). But $H(\bar{h})$ is provable iff $h \in P^*$ (diagonal formula property). Therefore, $h \in A$ iff $h \in P^*$. But A and P^* are disjoint, hence $h \notin P^*$ and $h \notin A$. R^* is a subset of A, therefore $h \notin R^*$. $H(\bar{h})$ is neither provable nor refutable.

Separation, separability

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Theorem 2.: If $E_h = H(v_1)$ separates R^* from P^* in $\mathcal{S}$ and $\mathcal{S}$ is consistent, then $H(\bar{h})$ is undecidable.

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Obvious consequence of Theorem 1 and the Lemma.

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Separability generalized for relations:
$F(v_1, \ldots, v_n)$ separates the n-ary relation R_1 from R_2 in $\mathcal{S}$ if for all numbers $k_1, \ldots, k_n$, if $R_1(k_1, \ldots, k_n)$ holds, then $R_1(\bar{k}_1, \ldots, \bar{k}_n)$ is provable and if $R_2(k_1, \ldots, k_n)$ holds, then $R_1(\bar{k}_1, \ldots, \bar{k}_n)$ is refutable.

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$\mathcal{S}$ is a Rosser system for sets resp. n-ary relations if for any two Σ_1 sets A and B resp. any two n-ary Σ_1 relations R_1 and R_2 $A - B$ is separable from $B - A$ resp. $R_1 - R_2$ is separable from $R_2 - R_1$.

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$\mathcal{S}$ is a Rosser system if it is a Rosser system for sets and for all n-ary relations.

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Be $A(x, y)$ and $B(x, y)$ the formulas enumerating A resp. B. The formula separating $B - A$ from $A - B$ is this:

$$\forall y(A(x, y) \to (\exists z \leq y)B(x, z))$$

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We proved: If every true Σ_0 sentence is provable in $\mathcal{S}$, then every Σ_1 set and relation is enumerable. (12.04., proposition A_2).

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Together with lemma S, this gives the following
Theorem 3.: Any extension of Ω_4 and Ω_5 in which all true Σ_0 sentences are provable is a Rosser system.

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Therefore, (R), (Q) and P.A. are Rosser systems.

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If the conditions hold, then P^* and R^* are disjoint enumerable sets, hence according to Lemma S, they are separable. Incompleteness follows from Theorem 2. of this class.

If $A(x, y)$ enumerates P^* and $B(x, y)$ enumerates R^*, then – according to the proof of Lemma S – the formula

$$\forall y(A((x, y) \to (\exists z \leq y)B(x, z))$$

separates R^* from P^*.

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separates R^* from P^*.

If the Gödel number of this formula is h, then by Theorem 2. of this class, the following sentence is undecidable:

$$\forall y(A((\bar{h}, y) \to (\exists z \leq y)B(\bar{h}, z))$$

Rosser's theorem

Now we can prove Theorem R on the first slide of this class: Every simply consistent axiomatizable extension of Ω_4 and Ω_5 in which all Σ_1 sets are enumerable is incomplete.

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The undecidable sentence is the same as on the previous slide.

An (informal) interpretation of the Gödel sentence

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Gödel's undecidable sentence:

$$\forall v_2 \neg A(\bar{a}, v_2), \tag{G}$$

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Then (G) says: 'No number witnesses that the ath diagonal sentence is provable', where the ath diagonal sentence is (G) itself. In other words: 'I am not provable'.

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where $A(x, y)$ enumerates P^* and $B(x, y)$ enumerates R^* (i.e., the ordinals of refutable diagonal sentences) and (R) is the hth diagonal sentence.

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Interpretation (on the same line): 'If any number witnesses that I am provable, than there is a number less or equal to it witnessing that I am refutable'. I.e., 'If I'm provable, then the system is inconsistent.'