Recursivity of the diagonal function

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3rd May 2024

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(1) If $R(a_1, \dots, a_n)$, then $F(\bar{a}_1, \dots, \bar{a}_n)$ is provable;

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Proof: If $F(\bar{a}_1, \ldots, \bar{a}_n)$ is provable, then, by consistency, it is not refutable. Therefore, by (2), $\tilde{R}(a_1, \ldots, a_n)$ does not hold, i. e., $R(a_1, \ldots, a_n)$ holds. So the converse of the conditional (1) is proved. The converse of (2) goes on the same way.

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Therefore:

Theorem 1. All recursive relations are definable in (R), and, *a fortiori*, in every extension of (R), including (Q) and P.A.

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F strongly defines f if F weakly defines f and in addition, if $f(a_1, \ldots, a_n) = b$, then the following sentence is provable:

$$\forall v_{n+1}(F(\bar{a}_1, \ldots, \bar{a}_n, v_{n+1}) \to v_{n+1} = \bar{b}).$$

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We will show that $\exists v_2(F(v_1, v_2) \wedge G(v_2))$ works as $H(v_1)$. Be n and m such that $f(n) = m$. We should prove that $H(\bar{n}) \leftrightarrow G(\bar{m})$ is provable.

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We could generalize the theorem for n-ary functions.

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Therefore $H(v_1)$ represents $f^{-1}(A)$.

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2. To prove that definability of A implies definability of $f^{-1}(A)$ we need to show that in that case $H(v_1)$ represents $f^{-1}(A)$ and $\neg H(v_1)$ represents $\widetilde{f^{-1}(A)}$. The former was proven above, the latter goes on the same way.

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To prove Theorem 3., we need the following lemma: If all formulas of Ω_4 and Ω_5 are provable in $\mathcal{S}$, then any function weakly definable is strongly definable, too.

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1. For any $k \leq m$, $F(\bar{n}, \bar{k}) \to \bar{m} \leq \bar{k}$ is provable. Because for $k < m$, $F(\bar{n}, \bar{k})$ is refutable, and for $k = m$, $\bar{m} \leq \bar{k}$ is provable (by Ω_5).

Strong definability of recursive functions

A function $f(x_1, \ldots, x_n)$ is recursive if the relation $f(x_1, \ldots, x_n) = x_{n+1}$ is recursive.

Theorem 3. All recursive functions are strongly definable in (R) and in its extensions.

To prove Theorem 3., we need the following lemma: If all formulas of Ω_4 and Ω_5 are provable in $\mathcal{S}$, then any function weakly definable is strongly definable, too.

We prove it for functions of one argument only; the generalization is easy.

Be $F(x, y)$ a formula that weakly defines f. Be $G(x, y)$ the formula $F(x, y) \wedge \forall z(F(x, z) \to y \leq z)$. We show that G strongly defines f. Be n an arbitrary number and $m = f(n)$.

1. For any $k \leq m$, $F(\bar{n}, \bar{k}) \to \bar{m} \leq \bar{k}$ is provable. Because for $k < m$, $F(\bar{n}, \bar{k})$ is refutable, and for $k = m$, $\bar{m} \leq \bar{k}$ is provable (by Ω_5).
Hence, $z \leq \bar{m} \to (F(\bar{n}, z) \to \bar{m} \leq z)$ is provable (using Ω_4).

Proof of the lemma continued

$\bar{m} \leq z \to (F(\bar{n}, z) \to \bar{m} \leq z)$ is provable (by propositional logic).

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$F(\bar{n}, \bar{m})$ is provable because F weakly defines f, and therefore
$F(\bar{n}, \bar{m}) \wedge \forall z(F(\bar{n}, z) \to \bar{m} \leq z)$, i.e., $G(\bar{n}, \bar{m})$ is provable.

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2. For any $k \neq m$, $F(\bar{n}, \bar{k})$ is refutable, but $G(\bar{n}, \bar{k}) \to F(\bar{n}, \bar{k})$ is provable by propositional logic, therefore $G(\bar{n}, \bar{k})$ is refutable.

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By 1. and 2., we proved that G weakly defines f. For the additional condition of strong definition, we show first that $G(\bar{n}, y) \rightarrow y \leq \bar{m}$ is provable.

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$F(\bar{n}, \bar{m})$ is provable (weak def.), and therefore $G(\bar{n}, y) \to y \leq \bar{m}$ is provable (propositional logic).

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Then $y \leq m \to (G(\bar{n}, y) \to y = \bar{m})$ is provable (using Ω_4).

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From this formula and the formula proved on the previous slide, $G(\bar{n}, y) \rightarrow y = \bar{m}$ follows by propositional logic.
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From this lemma and Theorem 1., Theorem 3. follows.

A proposition, a consequence and a homework

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Proposition: For any function $f(x_1, \ldots, x_n)$, if the relation $f(x_1, \ldots, x_n) = x_{n+1}$ is Σ_1, then f is recursive.

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We need only that the complement of the relation is Σ_1, too. But $f(x_1, \ldots, x_n) \neq x_{n+1}$ is equivalent with $\exists y(f(x_1, \ldots, x_n) = y \wedge y \neq x_{n+1})$ and the latter is a Σ, therefore Σ_1 formula.

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The diagonal function $d(x)$ is Σ_1. Therefore, by the above proposition, it is recursive, and by Theorem 3., it is strongly definable in (R) and in its extensions.

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Homework: Show that for the complete theory $\mathcal{N}$, representability, definability and complete representability all coincide. Is this true for P.A., too? (Is the set P^* completely representable in P.A.?)