The Second Incompleteness Theorem

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10th May 2024

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Be h the Gödel number of $H(v_1)$. $H(\bar{h}) \leftrightarrow F(\overline{d(h)})$ is provable.

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Be h the Gödel number of $H(v_1)$. $H(\bar{h}) \leftrightarrow F(\overline{d(h)})$ is provable.
$H[\bar{h}] \leftrightarrow H(\bar{h})$ is FOL-provable, therefore $H[\bar{h}] \leftrightarrow F(\overline{d(h)})$ is provable.

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$H[\bar{h}] \leftrightarrow H(\bar{h})$ is FOL-provable, therefore $H[\bar{h}] \leftrightarrow F(\overline{d(h)})$ is provable.
But the Gödel number of $H[\bar{h}]$ is $d(h)$, therefore $H[\bar{h}]$ is a fixed point for F.

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The earlier definition can be read as defining the Gödel sentence with respect to $\mathcal{N}$.

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Let $H_h(v_1)$ represent $d^{-1}(A)$. Then $H[\overline{h}]$ is provable iff $H(\overline{h})$ is provable iff $h \in d^{-1}(A)$ iff $d(h) \in A$.
But $d(h)$ is just the Gödel number of $H[\overline{h}]$. Q.e.d.

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But according to Tarski's theorem, there is no such predicate.

Another Tarski-like theorem

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Therefore $\mathcal{S}$ is inconsistent, against the assumption of the theorem.

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$P(v_1)$ is a provability predicate for $\mathcal{S}$, if for any sentences X and Y:

- P_1 If X is provable, then $P(\overline{X})$ is provable, too;
- P_2 $P(\overline{X \to Y}) \to (P(\overline{X}) \to P(\overline{Y}))$ is provable;
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If P.A. is ω-consistent, then $P(v_1)$ represents P, therefore in this case even a biconditional holds: X is provable iff $P(\overline{X})$ is provable. But simple consistency entails that $P(v_1)$ represents some superset of P and this is enough for P_1.

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If P.A. is consistent, then a such $P(v_1)$ satisfies the conditions P_2 and P_3, too, but the proof is more difficult; we skip it.

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By propositional logic, the claim follows.

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The claim follows by propositional logic.

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1. $\neg P(\overline{G})$ would be provable;
2. by P_1, $P(\overline{G})$ would be provable, too.

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In the following, if not declared otherwise, we are working within some given system $\mathcal{S}$ and $P(v_1)$ is a provability predicate for $\mathcal{S}$.

Theorem 5.: If G is a fixed point of $\neg P(v_1)$ and $\mathcal{S}$ is consistent, then G is not provable.

By assumption, $G \leftrightarrow \neg P(\overline{G})$ is provable. If G were provable, then:

1. $\neg P(\overline{G})$ would be provable;
2. by P_1, $P(\overline{G})$ would be provable, too.

Be $\bot$ any sentence refutable in $\mathcal{S}$ (a logical falsity or as you like it; a traditional choice is '$\overline{0} = \overline{1}$'). `consis` is the sentence $\neg P(\overline{\bot}))$.

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If $P(v_1)$ is a correct provability predicate (i.e., it expresses the set P), then `consis` is true iff $\bot$ is not provable iff $\mathcal{S}$ is consistent. In this sense, it 'expresses' the consistency of $\mathcal{S}$.

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$G \leftrightarrow (P(\overline{G}) \rightarrow \perp)$ is provable by propositional logic, hence $G \rightarrow (P(\overline{G}) \rightarrow \perp)$ is provable, too.

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By P_6, $P(\overline{G}) \rightarrow P(\overline{\bot})$, and therefore the sentence $\neg P(\overline{\bot}) \rightarrow \neg P(\overline{G})$ is provable.

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$\neg P(\bar{G}) \rightarrow G$ is provable by assumption.
By propositional logic, the provability of `consis` $\rightarrow G$. follows.

From this lemma and Theorem 5., we can prove an abstract form of the second incompleteness theorem (next slide).

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If $\mathcal{S}$ is P.A., then there is a Σ_1 formula $P(v_1)$ expressing the set P. If P.A. is consistent, then the sentence `consis` is true, but the above argument shows that it is not provable. This consideration needs that $P(v_1)$ is a provability predicate because the key lemma needs it.

Henkin's sentence and Löb's theorem

Since P.A. is diagonalizable, $P(v_1)$ has a fixed point, the sentence H (Henkin, 1952). $H \leftrightarrow P(\overline{H})$, therefore H is true iff it is provable. (Gödel's sentence G is true iff it is not provable.)
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From this and the hypothesis follows that $P(\overline{X}) \to Y$ is provable.
From the fixed point property now follows that X is provable.
By P_1, $P(\overline{X})$ is provable, and by modus ponens, Y is provable.

The 2nd incompleteness theorem from Löb's theorem

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Hence, by Löb's theorem, $\bot$ is provable, therefore $\mathcal{S}$ is inconsistent.