Burali-Forti Paradox

Zalán Molnár

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Preliminary Definitions

- **Asymmetry**
  $\mathsf{asym}_X(R) \overset{\text{def}}{\Leftrightarrow} \neg\exists x \exists y(x \in X \land y \in X \land xRy \land yRx)$
- **Transitivity**
  $\mathsf{trans}_X(R) \overset{\text{def}}{\Leftrightarrow} \forall x \forall y \forall z((x \in X \land y \in X \land z \in X \land xRy \land yRz) \to xRz)$
- **Connectedness**
  $\mathsf{Conn}_X(R) \overset{\text{def}}{\Leftrightarrow} \forall x \forall y((x \in X \land y \in X) \to (xRy \lor yRx \lor x = y))$
- **Well-foundness**
  $\mathsf{WF}_X(R) \overset{\text{def}}{\Leftrightarrow} \forall Y((Y \subseteq X \land Y \neq \emptyset) \to \exists z \forall x(z \in Y \land x \in Y \land$
  $\land z \neq x \land \neg zRx))$
- **Partial ordering**
  $\mathsf{PO}_X(R) \overset{\text{def}}{\Leftrightarrow} \mathsf{asym}_X(R) \land \mathsf{trans}_X(R)$
- **Total ordering**
  $\mathsf{TO}_X(R) \overset{\text{def}}{\Leftrightarrow} \mathsf{PO}_X(R) \land \mathsf{Conn}_X(R)$
- **Well-ordering**
  $\mathsf{WO}_X(R) \overset{\text{def}}{\Leftrightarrow} \mathsf{TO}_X(R) \land \mathsf{WF}_X(R)$

Order isomorphism/similarity

Let $X_<$ and $Y_\sqsubset$ be partially/well-ordered classes. $X_<$ and $Y_\sqsubset$ are order isomorphic/similar iff $\exists i$ s.t. $i : X \to Y$ and $\mathsf{bij}(i)$ and for any $x, y \in X$ we have:

$$x < y \Leftrightarrow i(x) \sqsubset i(y)$$

with symbols $X \cong Y$.

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Order Type

Two sets X and Y have the same *order type* iff $X \cong Y$. An order type is an equivalence class of the order isomorphic classes.

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Initial segment

Let $\mathsf{WO}_X(<)$ and $a \in X$. Then the class:

$$\mathsf{seg}_{X,<}(a) = \{x \in X : x < a\}$$

is an *initial segment* of X with respect to $<$, generated by a.

Preliminary Theorems

- **Restriction of a well-ordering is a well-ordering**

  $\forall X \forall R (\forall Y (Y \subseteq X \wedge \mathsf{WO}_X(R)) \rightarrow (\mathsf{WO}_Y(R_X \restriction Y)))$

- **No well-ordering is similar to an initial segment of itself**

  $\neg \exists X \exists a (a \in X \wedge \mathsf{WO}_X(<) \wedge \mathsf{seg}_{X,<}(a) \wedge X \cong \mathsf{seg}_{X,<}(a))$

- **Well-orderings are comparable**
  - $X \cong Y$
  - $X \cong \mathsf{seg}_{Y,<}(a)$
  - $Y \cong \mathsf{seg}_{X,<}(a)$

Ordinal Numbers I.

- (Neumann) ordinals are the generalized concept of Neumann numbers.
- Ordinal numbers measure the "length" of the finite/transfinite well-orderings.
- Ordinal numbers are well-order sets that represent their order type.
- The *elementhood* relation well-orders the ordinal numbers.

Ordinal Numbers II.

Definitions

- **Transitive Class**
  $\mathsf{TR}(X) \overset{\text{def}}{\Leftrightarrow} \forall x(x \in X \to x \subseteq X)$
- **Ordinal Number**
  $\mathsf{On}(X) \overset{\text{def}}{\Leftrightarrow} \mathsf{TR}(X) \wedge \mathsf{WO}_X(\in)$

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$$\mathsf{Ord} = \{x : \mathsf{On}(x)\}$$

Does Ord exist as a set? $\Rightarrow$ The collection of all ordinals is a proper class. (Burali-Forti Paradox)

Theorem 0

$\mathsf{On}(\emptyset)$

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Theorem 1

$\forall X(\mathsf{On}(X) \rightarrow (\forall y(y \in X \rightarrow (\mathsf{On}(y) \wedge y_{\in} = seg_{X,\in}(y)))$

Proof – We have to prove that y is:

a) $\mathsf{TR}(y)$
b) $\mathsf{WO}(y)$

a) Assume that $\neg\mathsf{TR}(y) \Leftrightarrow \exists x(x \in y \wedge x \nsubseteq y)$. So there is a $z \in x$ s.t. $z \notin y$. Since $\mathsf{TR}(X) \Rightarrow x, z \in X$ and by $\mathsf{Conn}_X(\in)$ we have that $y \in z$ or $y = z$.
   i) Assume that $y \in z$. Then $z \in x$ and $x \in y$ and $y \in z$. Then we have $z \in x$ and by transitivity $x \in z$ which contradicts to asymmetry.
   ii) Assume that $y = z$. Then $y \in x$ and $x \in y$, which contradicts again to asymmetry.

b) *Lemma* – Restriction of a well-ordering is a well-ordering
Since $\mathsf{Tr}(X)$, we have that $y \subseteq X$, but then y is well-ordered by $\in_y$.

Theorem 2

$\forall X(\mathsf{On}(X) \to \mathsf{On}(sX))$

Theorem 3

$\forall X \forall Y((\mathsf{On}(X) \land \mathsf{On}(Y) \land X \subsetneq Y) \to (X \in Y)$

Theorem 4

$\forall X \forall Y((\mathsf{On}(X) \land \mathsf{On}(Y)) \to (X \subseteq Y \lor Y \subseteq X)$

Theorem 5

$\forall X \forall Y((\mathsf{On}(X) \wedge \mathsf{On}(X) \wedge X \neq Y) \rightarrow (X \in Y \vee Y \in X)$

Theorem 6.

$\forall X(\forall x(x \in X \wedge \mathsf{On}(x)) \rightarrow (\mathsf{On}(\bigcap X))$

Theorem 7.

$\forall X(\forall x(x \in X \wedge \mathsf{On}(x)) \rightarrow (\mathsf{WO}_X(\in))$

Burali-Forti Paradox I

To be proved:

a) Ord is an ordinal number
b) Ord is a proper class
c) any other *ordinal class* is a set

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a) Ord is an ordinal number:
   i) Ord is a transitive class: for any $x \in \mathsf{Ord}$ we have $\mathsf{On}(x)$. By *Theorem 1* for any $y \in x$ we have $\mathsf{On}(y)$ too. Since $\mathsf{On}(y)$, $y \in \mathsf{Ord}$, which means $x \subseteq \mathsf{Ord}$.
   ii) Since for any $x \in \mathsf{Ord}$ is $\mathsf{On}(x)$ by *Theorem 7* we have, that $\mathsf{WO}_{\mathsf{Ord}}(\in)$

Burali-Forti Paradox II

b) Ord is a proper class: Suppose Ord is a set. By a) we know that Ord is an ordinal number, so in the one hand $\mathsf{Ord} \in \mathsf{Ord}$, but by the definition of well-ordering $\mathsf{Ord} \notin \mathsf{Ord}$. So we get the contradiction: $(\mathsf{Ord} \in \mathsf{Ord} \wedge \mathsf{Ord} \notin \mathsf{Ord})$, therefore Ord is a proper class.

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c) Any other ordinal is a set: Let X be an ordinal class s.t. $X \neq \mathsf{Ord}$. By *Theorem 5* $X \in \mathsf{Ord} \vee \mathsf{Ord} \in X$. If $\mathsf{Ord} \in X$, then Ord is a set, which contradicts to b), so $X \in \mathsf{Ord}$, but then X is a set. $\square$

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Theorem of GB

$\forall X((\mathsf{On}(X) \wedge \neg M(X)) \leftrightarrow X = \mathsf{Ord})$