Zermelo's First Proof of the Well-ordering Theorem

Zalán Molnár

November 20, 2017

WOT, AC

WOT – Well-ordering Theorem

Claim: for any set X, there is an ordering, which well-orders X.

$$\forall X \exists R(\mathsf{WO}_X(R))$$

WOT, AC

WOT – Well-ordering Theorem

CLAIM: for any set X, there is an ordering, which well-orders X.

$$\forall X \exists R(\mathsf{WO}_X(R))$$

Choice Function

A *choice function* f is defined on a collection of nonempty sets X, such that, for all A in X, $f(A)$ is an element of A.

$$f : X \to \bigcup X \text{ s.t. } \forall A \in X(f(A) \in A)$$

WOT, AC

WOT – Well-ordering Theorem

CLAIM: for any set X, there is an ordering, which well-orders X.

$$\forall X \exists R(\mathsf{WO}_X(R))$$

Choice Function

A *choice function* f is defined on a collection of nonempty sets X, such that, for all A in X, $f(A)$ is an element of A.

$$f : X \to \bigcup X \text{ s.t. } \forall A \in X(f(A) \in A)$$

AC – Axiom of Choice

For any collection of nonempty sets X, there is a *choice function* f defined on X.

$$\forall X(\emptyset \notin X \to \exists f(f : X \to \bigcup X \text{ s.t. } \forall A \in X(f(A) \in A))$$

AC $\rightarrow$ WOT

- WOT is proved by invoking the new mathematical tool, AC.
- What we prove exactly is that, AC $\longrightarrow$ WOT.

AC $\to$ WOT

Definitions

- Let M be any arbitrary set, the cardinality of M is denoted by $|M|$ and let m be an arbitrary element of M.
- Let $M' \subseteq M$, s.t. $M' \neq \emptyset$ (so $m \in M'$ for some $m \in M$).
- Let $M - M'$ denote the subset complementary to M'.
- $\forall M' \forall M'' (\forall X (X \in M' \leftrightarrow X \in M'') \to M' = M'')$, where $M', M'' \subseteq M$. Otherwise M' and M'' are different.
- Set of all subsets M' is denoted by $\wp(M)$.

Aim is to prove, that M can be well-ordered!

AC $\to$ WOT

- **Distinguished element:**
  For every M', there is associated an arbitrary element $m'_1 \in M'$. Such m'_1 is the *distinguished* element of M'. How can we define it?

AC $\to$ WOT

- **Distinguished element:**
  For every M', there is associated an arbitrary element $m'_1 \in M'$. Such m'_1 is the *distinguished* element of M'. How can we define it?

Define distinguished element m'_1

Invoke AC and define γ to a choice function as follows:

$$\gamma : \{\wp(M) - \{\emptyset\}\} \to M \text{ s.t. } \forall M' \in \wp(M)\ (\gamma(M') \in M').$$

AC → WOT

- **Distinguished element:**
  For every M', there is associated an arbitrary element $m'_1 \in M'$. Such m'_1 is the *distinguished* element of M'. How can we define it?

Define distinguished element m'_1

Invoke AC and define γ to a choice function as follows:

$$\gamma : \{\wp(M) - \{\emptyset\}\} \to M \text{ s.t. } \forall M' \in \wp(M)\ (\gamma(M') \in M').$$

Definition of "γ-set"

Using a fixed γ, let M_γ be defined as follow:

a) $M_\gamma \subseteq M$

b) M_γ is well-ordered by some ordering $\prec$

c) if a is an arbitrary element of M_γ, then a determines a set A where $A = \{x \in M : x \prec a\}$ s.t. $a = \gamma(M - A)$.

AC $\to$ WOT

1) Whenever M'_γ and M''_γ are any two distinct set:

   $M'_\gamma \cong seg_{M''_\gamma,\prec}(a)$ for some $a \in M''_\gamma$

   or

   $M''_\gamma \cong seg_{M'_\gamma,\prec}(a)$ for some $a \in M'_\gamma$

2) If two γ-sets have an element in common, say a, then $seg_{M'_\gamma,\prec}(a) = seg_{M''_\gamma,\prec}(a)$

3) If two γ-sets have two common elements a and b, then in both set $a \prec b \vee b \prec a$

REMARK: x is a γ-element iff $x \in M_\gamma$ for some M_γ.

AC $\to$ WOT

Proof

Let $L_\gamma = \bigcup_{i \in I} M_{\gamma_i}$. We claim, that L_γ is well-ordered and $L_\gamma = M$.

i) $\mathsf{WO}(L_\gamma)$

ii) L_γ is a γ-set and the largest such

i) $\mathsf{WO}(L_\gamma)$ set:
   a) $\mathsf{Conn}(L_\gamma)$
   b) $\mathsf{TO}(L_\gamma)$
   c) $\mathsf{WF}_{L_\gamma}(\prec)$

AC → WOT

Proof

ii) L_γ is a γ set:

Let a be an arbitrary γ-element and $A = \{x : x \prec a\}$ In any M_γ containing a, $A = seg_{M_\gamma,\prec}(a)$. According to def. of γ-set $a = \gamma(M - A)$, so L_γ is a γ set.

L_γ is the largest:

Clearly $L_\gamma \subseteq M$. We have to prove, that $M \subseteq L_\gamma$. Suppose $\exists x \in M$ s.t. $x \notin L_\gamma$. Then $M - L_\gamma \neq \emptyset$. But then, $\exists m'$ s.t. $m' = \gamma(M - L_\gamma)$. Now let $L'_\gamma = L_\gamma \cup \{m'\}$ and define the well-ordering s.t. $x \prec m'$ for all $x \in L_\gamma$. But then L'_γ would be a γ-set, and m' would be one of its γ-element, which contradict to the assumption, that L_γ is the set of all γ elements.