The beginnings: Frege

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A substantial change of the views about the objects of mathematics and mathematical truth.

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Foundational research: supports or refutes such answers.

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Motivation: there are eternal truths.

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- The rules of the calculus should be defined in purely syntactical terms and each step of a deduction can be checked algorithmically.

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Definition of the successor-relation in a given sequence from the relation „immediate successor" (in other words: the definition of the transitive closure of a relation).

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The definitions are based on a critical examination of the notions of number in philosophy and mathematics and a philosophical analysis of the concept of number.

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Formal research in his work (consistent fragments, roots of inconsistency): from the 1980's (Boolos, Heck, Wright, etc.)

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Judgment: a sentence can be asserted as a judgment, but it can occur as a part of a more complex of a more complex sentence. In this latter case the part-sentence is not asserted.

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