Frege's work(continued)

András Máté

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2. Never ask for the meaning of a word in isolation, but only in the context of sentences.
3. Never forget about the distinction between concept and object.

(Concept is the semantical value of a unary predicate)

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Frege's question: Are the units distinguishable or not?

The *Grundlagen*: lessons from the critical analysis

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2. (Hume's principle:) Two concepts have the same cardinality iff there is a one-to-one mapping between the objects falling under them.

Hume's principle: an abstraction principle

Let $Nx : F(x)$ denote the number belonging to the concept F (to the extension of the predicate F), or the number of the F-s.
$[1-1](f)$ should mean that the function f is a one-to-one correspondence.
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Could we get to the number 2 by considering two cats and disregarding their individual properties?

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Frege's example is the introduction of directions on the plane by the relation of parallelism: Two straight lines have the same direction iff they are parallel to each other.

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We could proceed either on the way that we introduce natural numbers by Hume's principle (this is neo-Fregeanism) or (as Frege did) introduce value ranges by an evident-looking abstraction principle (axiom V. of the *Basic Laws of Arithmetics*) and deduce Hume's principle from it.

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Unlimited comprehension, axiom V., Hume's principle and the definition of direction via parallelism are all abstraction principles. The difference between them is only that the first two are both inconsistent while the third and the fourth are not.

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Then let us identify numbers with (by and far) the equivalence classes of concept(extension)s for this equivalence relation. Let $\breve{x}H(x)$ the extension of the concept H. The definition of the number belonging to the concept F:

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$$Nx : F(x) =_{def} \breve{G}(Equinum(F,G))$$

$Equinum(F,G)$ is a concept of second grade (with fixed F and variable G)

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‘Frege’s theorem’: The Frege-numbers satisfy the axioms of primitive Peano-arithmetics. I.e., 0 is not an immediate successor, $ISucc$ is one-to-one and mathematical induction holds.