### A budget of paradoxes Historical introduction to the philosophy of mathematics

András Máté

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We have unlimited comprehension:

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for any open sentence A(x).

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We have proved a logical falsity from the (unlimited) comprehension using only logical rules.

### An embarrassing analogy

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Be H any set,  $\mathcal{P}(H)$  its power set, and f an injective mapping from H to  $\mathcal{P}(H)$ . We show that there is at least one member of  $\mathcal{P}(H)$  that is not in the range of f:

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Suppose (for contradiction) that  $H_0 = f(h)$ .

$$h\in f(h)\leftrightarrow h\not\in f(h)$$

#### Russell's paradox in Frege's *Grundgesetze* system

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Because of the first conjunct in the scope of  $\exists$ , any concept F which makes the existential quantification true is true for just the same objects as R (because of Axiom V). Therefore, the right side is true iff  $\neg R({}^{\vee}R)$ .

#### The central problem: paradoxes

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Let me introduce a collection of relevant paradoxes. (A budget of paradoxes: De Morgan 1872.)

## The Liar paradox

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#### (L) The sentence in the first line of this frame is false.

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If L is false, then the sentence that claims that L is false is true, therefore L is true.

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If L is false, then the sentence that claims that L is false is true, therefore L is true.

 $\mathbf{L}\leftrightarrow\neg\mathbf{L}$ 

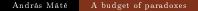
#### Variants for the Liar

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 $p_1 \leftrightarrow \neg p_2, p_2 \leftrightarrow \neg p_3, \dots p_{2n-1} \leftrightarrow \neg p_1.$ 



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Strenghtened Liar:

Let us allow that 'is false' and 'is not true' are not the same. I.e., there are sentences that are neither true nor false ("gappy").

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 $L_S \leftrightarrow (L_S \text{ is not true})$ 

#### Burali-Forti paradox

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# Let BF the class of all ordinals, well-ordered by the relation < (i.e., $\in).$

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- It is an ordinal. It is larger than any ordinal because any ordinal is a member of it.

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- It is an ordinal. It is larger than any ordinal because any ordinal is a member of it.
- It is smaller than its successor.

#### Two more famous paradoxes

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Let us call a one-place predicate F <u>heterological</u> iff F(F) is false. E. g. 'abstract' is abstract, but 'red' is not red. Is 'heterological' heterological? Known as Grelling-Nelson, Weyl, or simply heterological-paradox.

The smallest number not definable in English by 72 characters

3 1 4 3

#### Richard's paradox

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Let us enumerate all these numbers in the sequence  $a_k$ . Consider the following real number  $a = 0.d_1d_2...d_n...:$  $d_n = 6$  if the *n*th digit after the decimal point of  $a_n$  is 5 and

d = 5 otherwise.

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a differs from any member of our sequence, but it is defined.

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The students write the test on Wednesday and they get really surprised.

G is an ordinary game between two players iff it finishes in finitely many steps. H is the following hypergame: the first player chooses an ordinary game, and then they play it. Is H an ordinary game or not? 'Whatever involves *all* of a collection must not be one of the collection' or, conversely: 'If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total.'

Self-reference: a sentence refers for itself, i.e. its truth conditions contain some condition about its own truth resp. falsity.

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Russell's principle forbids self-reference. It is apparently enough to avoid the previous paradoxes.

#### Yablo's paradox

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Stephen Yablo, 1989

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Stephen Yablo, 1989

It is a liar-like, but <u>infinitary</u> paradox that does not violate the vicious circle principle and does not contain any sort of self-reference.

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Three ways out of the trap of paradoxes:

- Improve logic and produce a unique general theory free of risks (logicism)
- **2** Risky theories but a reliable metatheory (formalism)
- Abandon the priority of logic in favor of a more reliable basis (intuitionism)