Formalism, Hilbert's program

Historical introduction to the philosophy of mathematics

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Standard way to investigate theories in the metatheory: formalize them in first-order logic.

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Hilbert's school: young mathematicians working in the 1920's in Göttingen on this program (Wilhelm Ackermann, Paul Bernays, John von Neumann, Jacques Herbrand).

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Propositions may have meaning, may formulate true claims about some sort of objects, but this is irrelevant for mathematics.

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Bernays (1928): 'Making us methodologically free from the intuition of space is not the same as to ignore the fact that the starting points of geometry lay in the intuition of space.'

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Reduction to logic cannot guarantee consistency. (This is the moral from the paradoxes.)

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The main goal of the investigation is to prove that the risky, transfinite constituents of the theory don't make it inconsistent.

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E.g. the instances of the scheme

$$\forall x A(x) \vee \exists x \neg A(x)$$

are trivially valid for a finite domain because they can be verified in finitely many steps. But on an infinite domain, after finitely many steps, it is always possible that we didn't find an object a for which $\neg A(a)$ holds but we didn't verify $\forall x A(x)$, either.

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Certainty does not lie in logic, but in experience and intuition (as the framework of experience).
Metamathematics is more reliable than other mathematical theories because the reference to infinity is minimalized.

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We can extend our system (of real elements) by the ideal element ω.

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'No one will drive us from the paradise which Cantor created for us.'

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By relative consistency proofs, we can reduce the problem of consistency of mathematical theories to the consistency of ‘more fundamental’ ones. The proofs should be purely formal and must not use anything but the axioms.

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This first link should be some limited fragment of the arithmetics of natural numbers, with some limited logic (bounded quantifiers). In such a theory we should prove the consistency of the full Peano arithmetics, and then we can move forward.

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BTW. tacit universal quantification is allowed in general. Universal instantiation and existential generalization don't count as risky.

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And then comes Gödel ...