Digression: recapitulate some facts about first-order logic and first-order theories

Historical introduction to the philosophy of mathematics

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Γ is <u>consistent</u> iff $Thm(\Gamma)$ is a proper subset of the sentences of the language.

Γ is <u>negation complete</u> iff for every closed sentence A of the language, either A or $\neg A$ is in $Thm(\Gamma)$.

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Semantic completeness is the property of the logical calculus that every semantically valid inference can be proved by derivation in the calculus.

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  - no pair of sentences A, $\neg A$.

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In case II, Γ is inconsistent.

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Löwenheim-Skolem: If a set of sentences has a model, then it has a countable model, too.

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Not a contradiction; but it implies that some important notions (e.g. countability) are incurably relative, model-dependent. (Putnam: 'Models and reality', 1980)

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Every finite subset of this set has a model (namely the standard one extended by a suitable interpretation of 'a'). Therefore, (by compactness) the whole set has a model, too, and it is a model of the axioms.

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BTW., nonstandard models of Peano arithmetics can be characterized by the following 2-order sentence:

$$\exists X(\exists x Xx \land \forall x(Xx \to x > 0) \land$$
$$\forall y[\forall x(Xx \to x > y) \to \forall x(Xx \to x > y')])$$