Peano-arithmetics: incompleteness and the problem of consistency
László Kalmár's proofs

Historical introduction to the philosophy of mathematics

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Second Incompleteness Theorem: The sentence expressing the consistency of Peano arithmetics is neither provable nor refutable (under the same conditions and with the same generalizations).

Kalmár's proof of the first incompleteness theorem

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Numerical terms are the terms containing no variable.
We suppose that we can calculate the value of any numerical term.
To calculate a numerical term t is to prove some equality $t = n$ (where n is a numeral).

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k_0(x) \neq 0 & k_0(x) \neq 1 & \ldots & k_0(x) \neq n & \ldots \\
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Lemma (not proved) : $f(x)$ can be expressed in our language by a term with one variable.

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If (G) is false, then for some n, $k_g(n) = f(n) = g$, and therefore P_n proves (G).
To sum up, (G) is provable iff it is false.

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If our arithmetics (that can be Peano arithmetics or any effective extension of it) calculates every numerical term and *proves only true equalities with at most one variable*, then the Gödel sentence (G) is true and not provable, and its negation is not provable because it is false. Therefore it is negation incomplete.

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A plausible reading of the (G) sentence: For every x, $k_g(x)$ (i.e., $f(x)$) is different from g. It means that the diagonal formula numbered with g has no proof. But the g-th diagonal formula is (G) itself!! Therefore (G) says: 'I am not provable'.

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A consistent theory is <u>ω-inconsistent</u> iff there is some property P s.t. the theory proves $P(0)$, $P(1)$, ... $P(n)$, ... for each numeral n, but it proves $\exists x \neg P(x)$, too.

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CPA is a deductively undecidable sentence of PA. (Second incompleteness theorem.) It is true on the standard model but false on some non-standard models.

PA + $\neg$ CPA is an example for consistent, but ω-inconsistent theory (if Peano arithmetics is consistent).

Impact of the second incompleteness theorem

- Gödel: 'I wish to note expressly that [this theorem] does not contradict Hilbert's formalistic viewpoint. For this viewpoint presupposes only the existence of a consistency proof in which nothing but finitary means of proof is used and it is conceivable that there exist finitary proofs that *cannot* be expressed in the formalism of [first-order Peano arithmetics].' (Original paper on the incompleteness theorems)

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- von Neumann: 'Thus I am today of the opinion that
  1. Gödel has shown the unrealizability of Hilbert's program.
  2. There is no more reason to reject intuitionism (if one disregards the aesthetic issue, which in practice also for me be the decisive factor).'

  (Letter to Carnap, 1931)

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Our axioms except of induction axioms are verifiable formulas and that is all we need about them.

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  - The root is the closing formula of the deduction.
  - Each leaf is of one of the following sorts:
    1. Truths of propositional logic (tautologies)
    2. $\exists$-axioms: $A(t) \to \exists r A(r)$
    3. Equality formulas: $r = s \to (A(r) \to A(s))$
    4. Verifiable formulas
    5. Induction axioms

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Transformation of the proof tree

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The closing formula is deduced by this transformed tree from verified numerical equalities (substitutions of the axioms) using propositional logic only. So there is no reason to doubt in it.

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- Elimination of I-inferences. We use an I-inference to prove a truth about some concrete number, e.g. 3 only. So we can substitute it by inferences from 0 to 1, from 1 to 2, from 2 to 3.
- Elimination of forks. A fork is the following configuration in the proof tree: An existentially quantified formula is introduced somewhere by using an ∃-scheme, and the same formula is the consequent of some ∃-axiom at some leaf. The idea is that relevant existentially quantified formulas occur in such pairs. The paths from these two formulas to the closing formula must met at some node before the closing formula because otherwise the closing formula would contain quantification. Forks can be substituted by propositional proof trees, too.

What is remaining?

We should prove yet that from any proof of a numerical formula we can reach *by a finite number of* iterated use of the I-inference elimination and fork elimination such a transformed proof tree. This is the part of our proof which can't be formalized within 1-order Peano Arithmetic.

$0 - \omega$-figures

Recursive definition of the $0 - \omega$-figures together with their ordering $<$ and their classification into degrees:[1]

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- The first and smallest figure is '0', the single member of the degree 0.
- The members of the first degree are (nonempty) sum(expression)s of the form $\omega^0 + \omega^0 + \ldots + \omega^0$. The shorter one is the smaller one, and 0 is smaller than any of them. Let us write 1 instead of ω^0 and r instead of the sum of the length r.

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0 – ω-figures, continued

- Let us have introduced the figures up to the degree k together with their ordering. An expression of the form

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belongs to the degree $k+1$ iff

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  - or else iff it is a continuation of b.

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We begin with the leaves and follow step by step the proof. The ordinal of each node depends on the ordinals of its immediate ancestor(s) in a rather simple way. On the very end, we arrive at the ordinal of the closing formula - this is the ordinal of the proof.

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Therefore, we can reach the transformed tree in finitely many steps.

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- Let us have a decreasing sequence from ω^a. Its first member is $c = \omega^{a_1} + \omega^{a_2} + ... + \omega^{a_r}$, where $a_1 < a$. We should prove that c is descending finite.

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- c is not larger than $\omega^{a_1} + \omega^{a_1} + ... + \omega^{a_1}$ (shortly, $\omega^{a_1} \cdot r$). Therefore, if we have a descending chain from c, we can get a descending chain from $\omega^{a_1} \cdot r$ putting this latter ordinal to the beginning of the sequence. Therefore, if $\omega^{a_1} \cdot r$ is descending finite, then ω^a is descending finite, too.

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- Therefore we reduced the descending finiteness of ω^a to that of ω^{a_1}, where $a_1 < a$. Iterating this consideration, we can get a decreasing sequence $a > a_1 > a_2 ...$ where the first member has the degree k and therefore by hypothesis the sequence is finite.

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Q. e. d.

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BTW. we did not use the other transfinite tool (∃-inference or equivalently, existential instantiation) in the proof.