Russell's Logicism and Ramsey's criticism

Historical introduction to the philosophy of mathematics

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It eliminates the Russell paradox, the Liar paradox, the least number not definable by ... letters, the Richard, the hypergame paradoxes. It doesn't eliminate the Yablo paradox.

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The technical elaboration of predicativity goes through the theory of types.

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Ramified theory of types: types are descending sequences of natural numbers.

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Problem: we cannot use the definition of number in our usual inductive proofs because the properties for which we want to use induction are of higher type than the type of φ.

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Russell and Whitehead, *Principia Mathematica* I-III. (1st edition: 1910, 11, 13).

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Another basic principle of logicism, involving some critique of formalism: Numbers of arithmetics are the same as the numbers used for counting in everyday life, therefore expressions of arithmetics are not just symbols free of any content.

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$\forall x f(x)$ is the logical product [conjunction], $\exists x f(x)$ is the logical sum [disjunction] of all the propositions resulting by substitution from $f\hat{x}$. I.e., they are truth functions.

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A second fundamental thesis: mathematics is essentially extensional. E.g. set equivalence means that there exists a mapping between the two sets, and this is independent of whether the mapping can be expressed (defined) in some way or other.