# Intuitionism continued

## Historical introduction to the philosophy of mathematics

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**B**rouwer–**H**eyting–**K**olmogorov interpretation: Not a (formal) definition of the logical constants of intuitionistic logic, but just an informal descripition of their meaning because it is based on an informal notion of construction.

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is decidable again, therefore $\forall x(A(x) \vee \neg A(x))$ holds, too. But $\forall x A(x) \vee \neg \forall x A(x)$ does not hold because we don't know whether Goldbach's conjecture is true or not and therefore we are not in the position to assert either member of the disjunction.

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Another example: $B(x) \Longleftrightarrow_{def} \exists y(y > x \wedge P(y) \wedge P(y + 2))$ is not a decidable predicate. Therefore $\forall x(B(x) \vee \neg B(x))$ does not hold.

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$$((A \rightarrow B) \wedge (A \rightarrow \neg B)) \rightarrow \neg A$$

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Indirect refutation

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holds.
But indirect proof

$$((\neg A \to B) \wedge (\neg A \to \neg B)) \to A$$

is not generally valid.

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With predicate logic, the situation is a bit more difficult, but there is a negative translation function $\mathbf{g}$ from FOL to intuitionist predicate logic s.t. for any first-order formula $A$, FOL proves $A \leftrightarrow \mathbf{g}(A)$, intuitionist predicate logic proves $\mathbf{g}(A) \leftrightarrow \neg\neg\mathbf{g}(A)$ and if FOL proves $A$, then intuitionist predicate logic proves $\mathbf{g}(A)$.

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Consequence: FOL and intuitionist predicate logic are equiconsistent.

Intuitionist logic has several different semantics. Perhaps the most important one, with soundness and completeness theorems: Kripke-structures. In case of propositional logic: Kripke-structures are trees and nodes on a branch of a tree represent (by and far) the consecutive stands of research.

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**HA** is capable of Gödelisation, therefore incompleteness theorems are valid for it.

# Real numbers

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'Let us consider the concept: "real number between 0 and 1." For the formalist this concept is equivalent to "elementary series of digits after the decimal point," for the intuitionist it means "law for the construction of an elementary series of digits after the decimal point, built up by means of a finite number of operations." And when the formalist creates the "set of all real numbers between 0 and 1," these words are without meaning for the intuitionist, even whether one thinks of the real numbers of the formalist, determined by elementary series of freely selected digits, or of the real numbers of the intuitionist, determined by finite laws of construction.' (Brouwer)

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Intuitionist theory of real numbers is *incomparable* with classical real analysis. Some true propositions of classical analysis are not true intuitionistically, but there are theorems of intuitionist analysis which are not true classically.

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Be $A(n)$ is a decidable predicate of natural numbers for which we don't know whether $\forall n A(n)$ is true or not; say, '$2n$ is the sum of two prime numbers'. Let us define a sequence of real numbers:

$$r_n = \begin{cases} 2^{-n} & \text{if } \forall m \leq n A(m) \\ 2^{-m} & \text{if } \neg A(m) \wedge m \leq n \wedge \forall k < m A(k) \end{cases}$$

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Therefore, the function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is not totally defined (it is undefined for the above $r$).

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Most of the classical concepts have an intuitionistic counterpart based on choice sequences. E. g. the intuitionistic counterpart of the (sufficiently small) neighborhood of a real number is the set of choice sequences having a (sufficiently long) common initial segment with the given choice sequence.

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In classical mathematics, we postulate that every non-empty set of real numbers with an upper bound has a least upper bound (Dedekind-completeness). In intuitionistic mathematics, continuity axioms have a similar role.

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Every total real function is continuous.
"Funny" functions are eliminated from intuitionistic analysis.