The unprovability of the consistency of **CC**

András Máté

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$Diag_{\sigma}(a, b)$ is the abbreviaton of the formula

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Lemma: $Diag_\sigma(a, b)$ is a theorem of **CC** iff B is a theorem of it.

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If G were provable in **CC**, then **CC** would be inconsistent.
* If G were false, then it would be provable in **CC**.
Therefore, G is a true but unprovable sentence of $\mathcal{L}^{10}$ (first incompleteness theorem).

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Using the result of Step 1., we get
(Step 2.) $\mathbf{CC} \vdash \neg G \supset Th(c_0)$

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and consequently,
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$\mathbf{CC} \vdash Cons_\sigma \supset G$.
It means that if $Cons_\sigma$ were provable, then G, the Gödel sentence would be provable, too. But from the first incompleteness theorem we know that the Gödel sentence is not provable, and therefore $Cons_\sigma$ can't be provable, either. Q.e.d.