The incompleteness of **CC**

András Máté

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Theories, languages, calculi

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The theorems can be generated by a canonical calculus Σ^* (details omitted). The alphabet of Σ^* can be encoded again in the two-letter alphabet $\mathcal{A}_1 = \{\alpha, \beta\}$ and the whole calculus will be encoded/translated as an $\mathcal{A}_{cc}$-string σ^*.
$(\mathcal{A}_{cc} = \{\alpha,\ \beta,\ \xi,\ \gg,\ *\})$.

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The theorems of $\mathbf{CC}^*$ are generated by the calculus Σ^*. Its auxiliary letters partly overlap in meaning with the auxiliary letters of $\mathbf{H}_3$ and therefore with the non-logical constants of $\mathcal{L}^{1*}$ and because of this, we will use the same letter (V for variable, T for term, F for formula, etc.). But to avoid ambiguity, the auxiliary letters of Σ^* will be written in boldface.

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To produce the code σ^* of Σ^* (and other codes of the expressions occurring in it), we need to translate each letter of its alphabet into $\mathcal{A}_1^\circ$. The translation of any letter C will be denoted by C$'$. Therefore we will see S as expressing the four-argument substitution relation in $\mathcal{L}^{1*}$, **S** as an auxiliary letter expressing substitution in Σ^* and $\mathbf{S}'$ as the code of **S**.

Some simplifications and a new axiom

We shall study the theory **CC** instead of **CC***. Its language is $\mathcal{L}^{10}$ which differs from $\mathcal{L}^{1*}$ by omitting the predicates A, F and G. So **CC** may be regarded as the transformation of the hypercalculus $\mathbf{H}_2$ into a first-order theory.

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The calculus producing the theorems of **CC** is Σ and its code in $\mathcal{L}^{10}$ is σ. The truth definiton for the formulas of $\mathcal{L}^{10}$ is the same as the truth definition for them in $\mathcal{L}^{1*}$. The notational convention introduced on the previous slide also remains valid.

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The axioms are just the axioms of **CC*** minus the axioms concerning the omitted predicates plus the axiom SUD (Substitution Uniquely Determinded):

$$\forall \mathfrak{x}_1 \forall \mathfrak{x}_2 \forall \mathfrak{x}_3 \forall \mathfrak{x}_4$$
$$(D(\sigma)(\mathfrak{x}_3 \mathbf{S}' \mathfrak{x}_2 \mathbf{S}' \mathfrak{x}_1 \mathbf{S}' \mathfrak{x}) \supset D(\sigma)(\mathfrak{x}_4 \mathbf{S}' \mathfrak{x}_2 \mathbf{S}' \mathfrak{x}_1 \mathbf{S}' \mathfrak{x}) \supset \mathfrak{x}_3 = \mathfrak{x}_4)$$

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It follows from the truth definition of the previous class that this axiom is true.

Diagonalization in **CC** (preparatory steps)

We have seen last week that all the theorems of $\mathbf{CC}^*$ are true. This statement trivially extends to the theorems of **CC**. The converse of this latter statement – that every true closed formula is provable – would be the completeness statement for **CC**. We will prove the falsity of this statement roughly by the standard Gödelian methods. At first, we show that the simplest true propositions are provable.

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Lemma 2.: If a string f is derivable in Σ, then $\sigma D f'$ is derivable in $\mathbf{H}_2$. Therefore, $D(\sigma)(f')$ is a true atomic formula of $\mathcal{L}^{10}$. According to Lemma 1., it is a theorem of **CC**.

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Let us now assume that $Diag_\sigma(a, b)$ is a theorem. Then each conjunct is a theorem, too, so they are true according our truth definition. The third conjunct says that the calculus with the code σ derives the string with the code b, i.e., B is a theorem of $\mathbf{CC}$.

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Now we have proven

Lemma 3. B is a theorem of **CC** iff $Diag_\sigma(a, b)$ is a theorem.

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According to Lemma 3., G is a theorem of **CC** iff $Diag_\sigma(a_0, g)$ is a theorem.

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According to Lemma 3., G is a theorem of **CC** iff $Diag_\sigma(a_0, g)$ is a theorem.

But from G follows $\neg Diag_\sigma(a_0, g)$. Therefore, if G is a theorem, then **CC** is inconsistent. Hence, G is not a theorem.

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To sum up: G is not a theorem, but if it were false, then it would be provable. Therefore, it is a true but unprovable sentence.

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Theorem: Be T a first-order theory such that

i. all the theorems of **CC** are provable in T;
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Then T is incomplete. There is a sentence in the language of T which is true but not provable.

Be $K' = k$. If K derives a string f, then $D(k)(f')$ is provable in T (because it is provable in **CC**). So we have an analogue of Lemma 2. Then we can introduce $Diag_k(a/x, b)$ exactly as we have introduced $Diag_\sigma$. We can prove Lemma 3. for theorems of T instead of **CC**, and produce a Gödel sentence for T.