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Completeness results for normal modal logics

Teljességi eredmények normális modális logikákra

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A HKR 346. § ad 76. § (4) c) pontja értelmében:
„… A szakdolgozathoz csatolni kell egy nyilatkozatot arról, hogy a munka a hallgató saját szellemi terméke…”

SZERZŐSÉGI NYILATKOZAT

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Budapest, 20..

a hallgató aláírása

A nyilatkozatot a diplomamunkához kell csatolni.

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Introduction

The general topic of this thesis is modal logic. Modal logic has many applications in different fields of research such as, but not limited to, philosophy, mathematics, linguistics, computer science, law, etc. Let us start by stating that modal logic is not *a* logic, in the sense of it being a single entity, but rather it is a large family of *logics*. For this reason we often talk about (modal) logics, in plural. We only study propositional modal languages. The propositional nature of the considered logics means that their computational complexity is usually fairly low, which is desirable. However they are also very expressive. In fact it has been shown that propositional modal logics can express not only certain first-order properties, but also some that are strictly second-order. Modal correspondence theory provides tools to study these links between modal and first- or second-order languages. It turns out that certain fragments of first- and second-order logic can be modally expressed. Coupled with their propositional nature, and the flexibility with which modal operators can be defined, this often leads to sought after positive decidability results.

These facts are especially interesting since there are many completeness results for modal logics. Completeness and soundness are not simply desired properties of logics, they are often the very reason that makes such systems usable and useful. It is hard to overestimate the importance of completeness results for a logic. The introduction by Kripke in the 1950s, of frame-based semantics, paved the way for the first competeness results for many modal logics (that were only characterized syntactically for many of them, at that time). However it was not until the 1970s that the first important incompleteness results arose, with papers by Thomason [21] and Fine [12], both published in 1974 albeit with independently obtained results.

After introducing basic definitions and tools regarding modal languages and logic in the first chapter, we turn our discussion to Kripke semantics (the most common semantics used for propositional modal logics), focusing mostly on completeness issues. We will mostly discuss *normal* modal logics, a family of systems respecting certain syntactic closure conditions. However, we don't limit our discussion to single modal languages (as is sometimes the case), and give our results for modal languages of arbitrary similarity type. After giving a general method for compeleteness (via canonical models) for Kripke semantics, we prove some of its inherent limitations. Namely, we will show that not every normal modal logic is complete in Kripke semantics. We do so by giving a full proof of the incompleteness of $\mathbf{K}_t\mathbf{ThoM}$.

In the last section of this thesis, we try to heal this failure of completeness by introducing algebraic semantics. This is the most technical part of the thesis, and culminates with a proof of the Jónsson-Tarski Theorem. As we will show, with this theorem, we obtain a general completeness result for any normal modal logic (of arbitrary similarity type).

Chapter 1

Introduction: Modal logic

We recall some basic definitions[1].

1.1 Modal languages

This section introduces modal languages and discusses a few examples.

Definition 1.1 (Modal languages). We let $\Phi = \{p_0, p_1, ...\}$ be a countably infinite set of propositional letters. By a *modal similarity type* we mean a set $\tau = \{\triangle_0, \triangle_1, \triangle_2, ...\}$ whose members we call modal operators, each of which has a fixed arity. As primitive logical symbols we use $\bot, \neg, \vee$. We denote a modal language by $ML(\tau, \Phi)$, or simply by its similarity type τ, since the modalities are what essentially distinguish (propositional) modal languages from each other.

The set Fml_τ of τ-formulas of a language $ML(\tau, \Phi)$ is the smallest set such that: $p \in Fml_\tau$ (for all $p \in \Phi$), $\bot \in Fml_\tau$, if $\varphi \in Fml_\tau$ then $\neg\varphi \in Fml_\tau$, if $\varphi, \psi \in Fml_\tau$ then $\varphi \vee \psi \in Fml_\tau$, and if $\varphi_1, ..., \varphi_n \in Fml_\tau$ and $\triangle \in \tau$ then $\triangle(\varphi_1, ..., \varphi_n) \in Fml_\tau$ (for every n-ary $\triangle \in \tau$).

[1] The basic definitions are (mostly) standard, and we try to stick with some standard notation. The standards we stick most closely with are [9], [8], and [10].

We can write the above inductive rule defining Fml_τ more concisely:

$$\varphi ::= p \in \Phi \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \triangle(\varphi_1, ..., \varphi_n)_{\triangle\in\tau}$$

$\dashv$

Each modal operator has a *dual* defined as $\triangledown(\varphi_0, \varphi_1, ..., \varphi_n) := \neg\triangle(\neg\varphi_0, \neg\varphi_1, ..., \neg\varphi_n)$.

We also use the usual propositional shorthands for conjunction, implication, bi-conditional, and verum. Respectively $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$, $\varphi \to \psi := \neg\varphi \vee \psi$, $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, and $\top := \neg\bot$.

Example 1.2 (Basic modal language). We call *basic modal language* the modal language whose similarity type $\tau = \{\Diamond\}$ is a singleton containing a unary modal operator. We denote this modality as is customary with the symbol $\Diamond$ and call it the diamond operator. Its dual, the box operator $\Box$ is defined as $\Box\varphi := \neg\Diamond\neg\varphi$. By *basic modal similarity type* we mean the similarity type of the basic modal language. $\dashv$

In general, unary $\triangle$ are denoted by $\Diamond$. Duals of unary triangle, unary $\triangledown$ are denoted by $\Box$ (as in the basic modal language).

The basic modal language is the most common modal language, and there are many ways one can understand $\Diamond$. For instance, it has been widely understood as an *alethic* modality[2], in which case $\Diamond\varphi$ is read as "it is possible that φ". Under that reading $\Diamond$'s dual $\neg\Diamond\neg$ is then understood as "it is not possible that not", which is the same as the more succint "it is necessary that". The same way we use the word "necessary" in English to mean that "it is not possible that (something) is not the case", we defined $\Box$ as a abbreviation to mean $\neg\Diamond\neg$. Therefore $\Box\varphi$ is read as "it is necessary that φ" under the alethic understanding.

There are many other ways the basic modal operators $\Diamond$ and $\Box$ have been understood. We can note that even though we only have $\Diamond$ as a primitive

[2]See e.g. [3]

operator in the basic modal language, we can always give a reading for a $\Box$ operator, since it can always be defined from $\Diamond$. In some modal logics, it is sometimes easier to use one of $\Diamond$ or $\Box$ more often (in axioms for example), because it is easier to read one or the other given the intended semantics of said logic.

In *provability logic*[3] for instance, $\Box\varphi$ is read as "it is provable that φ". We don't have a nice shorthand in English with the meaning of "not provable that not ...", which is how we would read $\Diamond$ in the context of provability logic. We can still have $\Diamond$ as our primitive operator in such a case, and only give a way to read $\Box$. In the end, it doesn't matter whether we use $\Diamond$ or $\Box$ (or both) as our primitive modality for the basic modal language since they are interdefinable. In any case we get $\Box = \neg\Diamond\neg$ and $\Diamond = \neg\Box\neg$.[4] In the basic modal language, eventhough it might be the case that only one modality is used as a primitive, its dual is always available to make our life easier if we wish so (and it is often the case).

Some of the other common ways the basic modalities (ie. the modalities of the basic modal language; ie. $\Diamond$ or $\Box$) have been read include: "it is permitted to" for $\Diamond$ in the context of *deontic* logic[5] (its dual "not permitted not to" is what we mean by "obligated to" in English, therefore the deontic reading of $\Box\varphi$ is "it is obligatory to φ"); in *epistemic* logic $\Box\varphi$ can be read as "it is known that φ".

Of course the reading of $\Diamond$ is not limited to any of the above mentioned cases. In fact modalities don't need to have an intuitive way to read them in

[3]See e.g. [6]

[4]This latter equality is justified striaghtforwardly, although we haven't yet introduced the (very basic) tools used to make such a claim, the main idea is our languages will all contain the tautology $\varphi \leftrightarrow \neg\neg\varphi$. Thus starting from the definition of $\Box = \neg\Diamond\neg$ we get $\neg\Box = \neg\neg\Diamond\neg$, whence we eliminate the double negation $\neg\Box = \Diamond\neg$, then $\neg\Box\neg = \Diamond\neg\neg$, and finally $\neg\Box\neg = \Diamond$.

[5]See e.g. [13]

English. The meaning is given mathematically by the semantics we choose. The semantics for some language often models some aspect of reality (maybe it was chosen for that reason, in which case we can talk of intended meaning), but it is not necessary.

We will see different examples of specific languages later on. For instance, we will introduce temporal languages when we discuss incompleteness results for Kripke semantics.

1.2 Semantics

We can interpret modal languages in relational structures. This kind of semantics is due to Saul Kripke and called *Kripke semantics*[6]. It is quite nice since it is fairly intuitive, allows for model theoretic explorations, and, as we will see is complete in many cases (although there are some limitations). In Kripke semantics, modal languages are interpreted on frames.

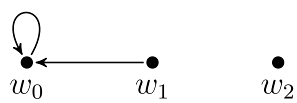
A *relational structure* is a set W with some relations R on it. We always take W to be non empty. We denote such a relational structure by $(W, R_i)_{i\in I}$ and call it a *frame*.

Definition 1.3 (Kripke frames). Given a modal similarity type τ, a τ-frame $\mathfrak{F}$ is a tuple containing a non empty set W and for each n-ary $\triangle \in \tau$, an $(n+1)$-ary relation $R_\triangle$ on W, denoted $\mathfrak{F} = (W, R_\triangle)_{\triangle\in\tau}$. $\dashv$

The elements of W are called *points, states, worlds, nodes*, but not exclusively. We can think of the relations $R_\triangle$ as telling us which points *see*

[6]Some logicians (such as Blackburn, Rijke, and Venema [8] for example) refer to Kripke semantics as relational semantics, for obvious reasons since languages are interpreted in relational structures. Other logicians (such as Chagrov and Zakharyaschev [10]) refer to another kind of semantics when they talk about relational semantics, namely general frame semantics (an intermediary semantics between Kripke semantics and algebraic semantics). To avoid confusion, we will not use the expression ‘relational semantics’, and instead talk about Kripke semantics, and general frame semantics.

which, or which points are *accessible* from which. For that reason, they are often called *accessibility relations*. For example, let's assume we have a frame (W, R), such that $W = \{w_0, w_1, w_2\}$ and R is the binary relation $R = \{(w_0, w_0), (w_1, w_0)\}$. Then we say that w_0 sees itself, and can access itself. In this frame, w_0 is also accessible by w_1. We say that w_1 sees w_0 or that w_0 is seen by w_1. We can easily represent frames with graphs, and it is sometimes useful to visualize them in such a way. The graph of the above mentioned frame looks like this:



The idea behind interpreting modal languages on frames is that we say in which worlds, each $p \in \Phi$ is true, with the help of an *assignment function*. Then, we will be able to inductively define truth for all the formulas of the language.

Definition 1.4 (Kripke models). Given a τ-frame $\mathfrak{F}$, a Kripke τ-model is the tuple $\mathfrak{M} = (\mathfrak{F}, V)$ where $V : \Phi \longrightarrow \wp(W)$ is a function from the set of propositional letters Φ to the power set of W. The idea is to associate to each propositional letter a collection of states from W. Thus, a Kripke model consists of:

(i) A non empty set W

(ii) A set of relations on W, such that for each n-ary $\triangle \in \tau$ there is a $(n+1)$-ary relation $R_\triangle$

(iii) A function $V : \Phi \longrightarrow \wp(W)$

$\dashv$

The function V is called a *valuation* (or *assignment*). It decides in which points $w \in W$ each $p \in \Phi$ is *true* in the model. The subset $V(p)$ of W is the set of points where p is true in the model. If we have $w \in V(p)$, then the atomic fact p is true at w. We are now ready to define the notion of truth for complex formulas.

Definition 1.5 (Satisfaction). The following inductive definition tells us when a τ-formula φ is *true* (or *satisfied*) at a point w in a model $\mathfrak{M}$:

$$\mathfrak{M}, w \Vdash p \text{ iff } w \in V(p) \tag{1.1}$$

$$\mathfrak{M}, w \Vdash \bot \quad \text{is never the case} \tag{1.2}$$

$$\mathfrak{M}, w \Vdash \neg\varphi \text{ iff } \text{ it's not the case that } \mathfrak{M}, w \Vdash \varphi \tag{1.3}$$

$$\mathfrak{M}, w \Vdash \varphi \vee \psi \text{ iff } \mathfrak{M}, w \Vdash \varphi \text{ or } \mathfrak{M}, w \Vdash \psi \tag{1.4}$$

$$\begin{aligned}\mathfrak{M}, w \Vdash \triangle(\varphi_1, ..., \varphi_n) \text{ iff } & \text{ for some } v_1, ..., v_n \in W \text{ such that } R_\triangle w v_1 ... v_n, \\ & \text{we have } \mathfrak{M}, v_i \Vdash \varphi_i \text{ for all } i \text{ such that } 1 \leq i \leq n\end{aligned} \tag{1.5}$$

A formula φ is *globally true* if it is satisfied in every state of a model (ie. if we have $\mathfrak{M}, w \Vdash \varphi$ for all $w \in W$ of a model $\mathfrak{M}$), we write $\mathfrak{M} \Vdash \varphi$. We say that φ is *satisfiable* in a model when φ is satisfied in at least one point of the model (ie. if $\exists x \in W$ such that $\mathfrak{M}, x \Vdash \varphi$). A formula φ is *refutable* in some model if its negation is satisfiable. When φ is not true at w in a model $\mathfrak{M}$, we write $\mathfrak{M}, w \nVdash \varphi$, and say that φ is *false* at w. $\dashv$

(1.1) corresponds to our definition of models: an atomic formula p is true in a world when that world is in the valuation of p. The cases (1.2), (1.3) and (1.4) are the standard propositional cases, giving the usual meaning to $\bot, \neg, \vee$. It follows that the meaning of the abbreviations $\rightarrow, \wedge, \top$ are also the usual propositional ones. $\bot$ is the constant *falsum* meaning “false”, and is never satisfied anywhere (ie. at any point) in a model. If a $\neg\varphi$ is true (in a world), then it is not the case that φ is true (in that world). Whenever we have that $\varphi \vee \psi$ holds in some world, then either φ holds in that world, or ψ holds in that world, or both.

The last case (1.5) is the most interesting for us, since it is the one that deals specifically with modalities and the reason models are based on frames. When is a formula containing an n-ary modality true in a world ? If the formula is of the form $\triangle(\varphi_1, ..., \varphi_n)$, then for it to be satisfied at some world w of W, there must be an $(n+1)$-ary relation on W such that w is the first member of the relation, and at each world v_i in the $(i+1)$-th position of the relation, the formula φ_i must be true there (for $0 < i \leq n$).

According to the satisfaction definition we get:

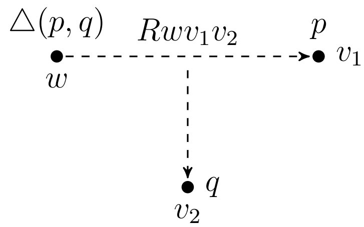
$$\begin{gathered}\mathfrak{M}, w \Vdash \bigtriangledown(\varphi_1, ..., \varphi_n) \\ \text{iff} \\ \forall v_1, ..., v_n \in W \text{ such that } Rwv_1...v_n, \text{ there is an } i \text{ such that } \mathfrak{M}, v_i \Vdash \varphi_i\end{gathered} \tag{1.6}$$

Proof. We verify (1.6). $\mathfrak{M}, w \Vdash \bigtriangledown(\varphi_1, ..., \varphi_n)$ iff $\mathfrak{M}, w \Vdash \neg\triangle(\neg\varphi_1, ..., \neg\varphi_n)$, by the definition of $\bigtriangledown$. Then for any $(n+1)$-ary relation where w is in the first position such as $Rwv_1...v_n$, we have that it is not the case that $\mathfrak{M}, v_i \Vdash \neg\varphi_i$ (for all i such that $1 \leq i \leq n$). So, if we have $Rwv_1...v_n$, then it's not the case that for all i, $\mathfrak{M}, v_i \nVdash \varphi_i$, which means that there is an i such that φ_i is true at v_i. Thus if we have $\mathfrak{M}, w \Vdash \bigtriangledown(\varphi_1, ..., \varphi_n)$ and some v_i's such that $Rwv_1...v_n$, then we have $\mathfrak{M}, v_i \Vdash \varphi_i$ for each i such that $1 \leq i \leq n$, since it cannot be the case that φ_i is false at v_i. $\square$

A set of formulas Σ is true at a state w when all its members are satisfied in w: $\mathfrak{M}, w \Vdash \Sigma$ iff $\forall \varphi \in \Sigma$, $\mathfrak{M}, w \Vdash \varphi$. Similarly, we have that a set of formulas Σ is globally true in a model $\mathfrak{M}$ iff $\forall \varphi \in \Sigma, \forall w \in W$, $\mathfrak{M}, w \Vdash \varphi$, which we write $\mathfrak{M} \Vdash \Sigma$.

Example 1.6. As an example let us assume that we have a binary modality $\triangle \in \tau$ and a τ-model $\mathfrak{M} = (W = \{w, v_1, v_2\}, R = \{(w, v_1, v_2)\}, V)$ such that

$V(p) = \{v_1\}$ and $V(q) = \{v_2\}$. We want to check if the formula $\triangle(p, q)$ is true at w, ie. if $\mathfrak{M}, w \Vdash \triangle(p, q)$. According to valuation V and equivalence (1.1), we have that $\mathfrak{M}, v_1 \Vdash p$ and $\mathfrak{M}, v_2 \Vdash q$. For $\triangle(p, q)$ to be true at w, our model needs to have a ternary relation where w is in the first place, and some $v_i \in V(p)$ and $v_j \in V(q)$ in the second and third place of the relation, respectively. We have only one relation Rwv_1v_2 in our model, so we need to check if $v_1 \in V(p)$ and $v_2 \in V(q)$. Since this is the case, we have that $\mathfrak{M}, w \Vdash \triangle(p, q)$. We can represent this model as a diagram, like so[7]:



$\dashv$

Since the truth of formulas is evaluated on elements of W, we can extend V to tell us at which points of W, each formula $\varphi \in Fml_\tau$ of the language is true. We define the function $\tilde{V} : Fml_\tau \longrightarrow \wp(W)$ such that:

$$\tilde{V}(p) = V(p) \tag{1.7}$$

$$\tilde{V}(\bot) = \emptyset \tag{1.8}$$

$$\tilde{V}(\neg\varphi) = W - \tilde{V}(\varphi) \tag{1.9}$$

$$\tilde{V}(\varphi \vee \psi) = \tilde{V}(\varphi) \cup \tilde{V}(\psi) \tag{1.10}$$

$$\begin{aligned}\tilde{V}(\triangle(\varphi_1, ..., \varphi_n)) = \{w \in W : \exists v_1, ..., v_n \in W \text{ such that } Rwv_1...v_n \\ \text{and } v_i \in \tilde{V}(\varphi_i) \text{ for all } i \leq n\}\end{aligned} \tag{1.11}$$

We show in the next proposition that we get $\tilde{V}(\varphi) = \{w \in W : \mathfrak{M}, w \Vdash \varphi\}$, for all $\varphi \in Fml_\tau$.

[7]We use dashed arrows to represent n-ary relations when $n > 2$

Proposition 1.7. *Given a model* $\mathfrak{M} = (\mathfrak{F}, V)$ *we have that*

$$\mathfrak{M}, w \Vdash \varphi \text{ iff } w \in \tilde{V}(\varphi)$$

Proof. We prove by induction on the complexity of φ. If φ is a propositional letter p:

$$\begin{aligned} \mathfrak{M}, w \Vdash p \text{ iff } & w \in V(p) && \text{(by the satisfaction definition)} \\ \text{iff } & w \in \tilde{V}(p) && \text{(by definition of } \tilde{V}) \end{aligned}$$

If φ is $\perp$. By the satisfaction definition, $\mathfrak{M}, w \Vdash \perp$ is never the case, hence $\tilde{V}(\perp)$ must be empty, which corresponds to the defintion we gave of $\tilde{V}$.

If φ is of the form $\neg\psi$:

$$\begin{aligned} \mathfrak{M}, w \Vdash \neg\psi \text{ iff } & \mathfrak{M}, w \nVdash \psi && \text{(by the satisfaction def.)} \\ \text{iff } & w \notin \tilde{V}(\psi) && \text{(inductive hypothesis)} \\ \text{iff } & w \in W - \tilde{V}(\psi) && \\ \text{iff } & w \in \tilde{V}(\neg\psi) && \text{(by def. of } \tilde{V}) \end{aligned}$$

If φ is of the form $\psi \vee \chi$:

$$\begin{aligned} \mathfrak{M}, w \Vdash \psi \vee \chi \text{ iff } & \mathfrak{M}, w \Vdash \psi \text{ or } \mathfrak{M}, w \Vdash \chi && \text{(satisfaction def.)} \\ \text{iff } & w \in \tilde{V}(\psi) \text{ or } w \in \tilde{V}(\chi) && \text{(inductive hypothesis)} \\ \text{iff } & w \in \tilde{V}(\psi) \cup \tilde{V}(\chi) && \\ \text{iff } & w \in \tilde{V}(\psi \vee \chi) && \text{(by def. of } \tilde{V}) \end{aligned}$$

If φ is of the form $\triangle(\varphi_1, ..., \varphi_n)$:

$$\mathfrak{M}, w \Vdash \triangle(\varphi_1, ..., \varphi_n)$$

$$\text{iff} \qquad\qquad \text{(satisfaction def.)}$$

$$\exists v_1, ..., v_n \text{ and } Rwv_1...v_n \text{ such that } \forall i \leq n,\ \mathfrak{M}, v_i \Vdash \varphi_i$$

$$\text{iff} \qquad\qquad \text{(induction hyp.)}$$

$$\exists v_1, ..., v_n \text{ and } Rwv_1...v_n \text{ such that } \forall i \leq n,\ v_i \in \tilde{V}(\varphi_i)$$

$$\text{iff}$$

$$w \in \{x \in W : \exists v_1, ..., v_n \in W, Rxv_1...v_n \text{ and } v_i \in \tilde{V}(\varphi_i) \text{ for all } i \leq n\}$$

$$\text{iff} \qquad\qquad \text{(by def. of } \tilde{V})$$

$$w \in \tilde{V}(\triangle(\varphi_1, ..., \varphi_n))$$

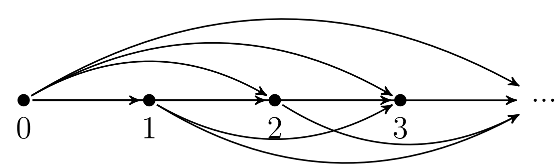
□

Definition 1.8 (Validity). A formula φ is *valid*, when it is satisfied at all worlds, in all models, of all frames (we write: $\Vdash \varphi$). A formula is *valid in a frame* $\mathfrak{F}$, when it is satisfied in all models based on $\mathfrak{F}$ (ie. for all valuations on $\mathfrak{F}$), at all points in the frame (notation $\mathfrak{F} \Vdash \varphi$). A formula is *valid at a state* w of $\mathfrak{F}$: ($\mathfrak{F}, w \Vdash \varphi$), if for any valuation on $\mathfrak{F}$, the formula is satisfied at w.

A formula φ is *valid on a class of frames* F, if it is valid in all frames $\mathfrak{F} \in \mathsf{F}$. We write $\mathsf{F} \Vdash \varphi$.

⊣

Example 1.9. In this example we work with the basic modal similarity type $\tau = \{\Diamond\}$. Let us assume we have a frame $\mathfrak{F} = (\mathbb{N}, <)$, where $\mathbb{N}$ is the natural numbers and $<$ the lesser than relation. This frame looks like this:



First, how do we check if a formula φ is not valid on this frame ? If we can find any valuation that doesn't make the formula φ true, then φ is not valid (on $\mathfrak{F}$).

For instance let us check if $\mathfrak{F} \Vdash p \to \Diamond p$. We assume that this is the case. We choose a valuation V such that $V(p) = \{1\}$, thus we have that $\mathfrak{F}, V, 1 \Vdash p$. By assumption we assumed that $p \to \Diamond p$ is true everywhere, so we must also have that $\mathfrak{F}, V, 1 \Vdash p \to \Diamond p$. According to the definition of $\to$ and the satisfaction criterion, we get that $\mathfrak{F}, V, 1 \Vdash \Diamond p$. For this to be the case, we must have some n related to 1 where p is true. So, we need to find any n such that $1 < n$ and $\mathfrak{F}, V, n \Vdash p$. But we know that there is no such n because $n \notin V(p)$ such that $1 < n$, since $V(p) = \{1\}$. This means that $\mathfrak{F}$, $V, 1 \nVdash p \to \Diamond p$, which contradicts our assumption. Thus, $(\mathbb{N}, <) \nVdash p \to \Diamond p$. The formula $p \to \Diamond p$ is not valid on the frame $(\mathbb{N}, <)$.

How do we verify that a formula is valid on a frame ? We check if the formula is satisfied at any state, for any valuation, on that frame.

As an example, we keep the same frame $\mathfrak{F} = (\mathbb{N}, <)$ and we check if the formula $\Diamond\Diamond p \to \Diamond p$ is valid on it. We have to check that in all worlds, for any valuation whenever $\Diamond\Diamond p$ is true, then $\Diamond p$ must also be true. We pick an arbitrary state $n \in \mathbb{N}$ and an arbitrary valuation V. If $\mathfrak{F}, V, n \Vdash \Diamond\Diamond p$, then we must find that $\mathfrak{F}, V, n \Vdash \Diamond p$. We assume $\mathfrak{F}, V, n \Vdash \Diamond\Diamond p$. Then, there must be a $m \in \mathbb{N}$ such that $n < m$ and $\mathfrak{F}, V, m \Vdash \Diamond p$, according to (1.5). Similarly, there must be a $l \in \mathbb{N}$ such that $m < l$ and $\mathfrak{F}, V, l \Vdash p$. Since it is always the case that for any $n, m, l \in \mathbb{N}$, if $n < m$ and $m < l$, then $n < l$ (by the definition of $<$), then it's always true that $\mathfrak{F}, V, n \Vdash \Diamond p$, since there is an $l \in \mathbb{N}$ such that $n < l$ and $\mathfrak{F}, V, l \Vdash p$. Thus $(\mathbb{N}, <) \Vdash \Diamond\Diamond p \to \Diamond p$. The formula $\Diamond\Diamond p \to \Diamond p$ is valid on the frame $(\mathbb{N}, <)$.

Definition 1.10 (Consequence relation). When can we say that a formula is a *consequence* of some other formulas ? As is usual in logic, we consider a consequence relation $\Sigma \Vdash \varphi$ to be one where the truth of a set of formulas Σ, guarantees the truth of a formula φ. Since we defined truth on states of a frame, we also define the consequence relation on states. We say that φ is a *consequence* of Σ, written $\Sigma \Vdash \varphi$, when for every model $\mathfrak{M}$ and every $w \in W$ if we have $\mathfrak{M}, w \Vdash \Sigma$, then $\mathfrak{M}, w \Vdash \varphi$. This consequence relation is *local* because it considers consequence at the level of states.

We can also restrict the consequence relation to a class of frames or models. If we denote this class of frames or models by C, we denote the restricted consequence relation $\Vdash_{\mathsf{C}}$. In such a case, the consequence relation stays the same, except that it should hold not for all models, but for all models from C or all models based on all frames from C. $\dashv$

All the above semantic considerations were given for arbitrary modal languages (ie. arbitrary τ). Since the basic modal language is extremely common, we give its semantics more precisely (which is a particular case of the more general one given above).

Example 1.11 (Semantics for the basic modal language). As defined in example 1.2, the basic modal language is one where $\tau = \{\Diamond\}$. Thus a *frame* for the basic modal language is a structure $\mathfrak{F} = (W, R)$ where R is a binary relation, since $\Diamond$ is unary. Models are based on such frames.

Satisfaction for formulas with unary $\triangle$ such as $\Diamond\varphi$ is a special case of (1.5) such that:

$$\mathfrak{M}, w \Vdash \Diamond\varphi \text{ iff } \exists v \in W \text{ and } Rwv \text{ such that } \mathfrak{M}, v \Vdash \varphi \tag{1.12}$$

We say that $\Diamond\varphi$ is true at w if w *sees* a world v (through the accessibility relation) such that φ is true at v.

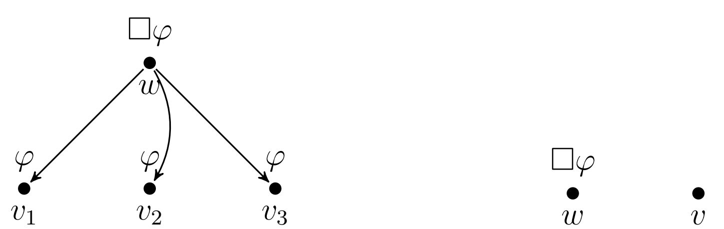
$$\begin{tikzcd}
\overset{\Diamond\varphi}{\underset{w}{\bullet}} \arrow[r, bend left] & \overset{\varphi}{\underset{v}{\bullet}}
\end{tikzcd}$$

We also get:

$$\mathfrak{M}, w \Vdash \Box\varphi \text{ iff } \forall v \in W \text{ such that } Rwv \text{ we have } \mathfrak{M}, v \Vdash \varphi \tag{1.13}$$

This is straightforward from (1.6) or (1.5) but we can nevertheless verify. By the definition of $\Box$ we have $\mathfrak{M}, w \Vdash \Box\varphi$ iff $\mathfrak{M}, w \Vdash \neg\Diamond\neg\varphi$ iff there is no v such that Rwv and $\mathfrak{M}, v \Vdash \neg\varphi$. Then, $\mathfrak{M}, w \Vdash \Box\varphi$ iff whenever it is the case that Rwv for any v, we must have that $\mathfrak{M}, v \nVdash \neg\varphi$. By the satisfaction definition this latter statement is equivalent to $\mathfrak{M}, v \Vdash \neg\neg\varphi$, which is equivalent to $\mathfrak{M}, v \Vdash \varphi$. We get that $\mathfrak{M}, w \Vdash \Box\varphi$ iff whenever w sees any world, then φ must be true in that world. This is in conformity with our definition. We do note however, that if w does not see any other world (ie. there is no v such that Rwv), then w still satifies $\Box\varphi$ vacuously (the condition is that *if* w sees worlds, then φ must be true at those worlds, therefore if w does not see any world, the condition is vacuously true).

We give two models as examples to illustrate:



The clause for $\tilde{V}$ corresponding to (1.11) for $\Diamond$ is as follows:

$$\tilde{V}(\Diamond\varphi) = \{w \in W : \exists v \in W \text{ such that } Rwv \text{ and } v \in \tilde{V}(\varphi)\} \tag{1.14}$$

Since $\Box$ is not primitive in our language, it is not necessary for us to specify the case for boxed formulas for the function $\tilde{V}$. It can nevertheless be helpful and clarifying, so we give it (and it is pretty straightforward anyway).

$$\tilde{V}(\Box\varphi) = \{w \in W : \forall v \in W \text{ such that } Rwv \text{ we have } v \in \tilde{V}(\varphi)\} \tag{1.15}$$

The fact the $\Box\varphi$ is true whenever there is no accessible world makes sense according to the semantics: it can be understood as there being no alternative. And we say that something is necessary when there is no alternative that this something can't be the case. If there is an alternative option, then this something must be true. But if if there is no alternative then, it cannot not be the case in an alternative.

Remark 1.12 (Logic). There are two ways to look at what a logic is: semantic and syntactic. When considered semantically, we associate a logic with a set of *valid* formulas. When considered syntactically, a logic is a set of *theorems*. Whichever approach we take, we get that a logic is a subset of formulas of a language $L \subseteq Fml_\tau$.[8]

A very nice and sought after property of a logic L is when the two approaches coincide, or more precisely, when the semantics and syntax define exactly the same set $L \subseteq Fml_\tau$. Then, we say that L is *sound* and *complete*.

Logics need not be sound and complete. Some logics are defined purely syntactically, and other purely semantically. The search for completeness and soudness for such logics is often an important task.

$\dashv$

Definition 1.13 (Semantic definition of a logic). Given a similarity type τ, a logic can be defined semantically as the set of valid formulas $L \subseteq Fml_\tau$ over a class of frames F.

$$L_{\mathsf{F}} = \{\varphi \in Fml_\tau : \mathsf{F} \Vdash \varphi\}$$

$\dashv$

What does the semantic approach tell us ? Very crudely, semantics give us interpretational tools. For example, in our satisfaction definition we said that if $\neg\varphi$ is the case, then φ is not the case. This is the *meaning* of $\neg$ in our languages. This meaning for instance, lets us interpret $\neg$ as a negation. Let's say our language talks about *truth* as is often the case for formal logical

[8]It is customary for some authors define (modal) logics this way (for example in [9] and [8]), but there are other ways, e.g. [1], [2].

languages. Then clearly this meaning we assigned to $\neg$ seems adequate for the interpretion, since we could interpret[9] $w \Vdash \neg\varphi$ as $\neg\varphi$ is true, which gives us that φ is not true. And if "not φ" is true (ie. φ is false), then it's not true that φ is true (ie. φ is false). And it works both ways, since we get $\neg\neg\varphi$ iff φ. This corresponds to some understanding of truth where "not false" has the same meaning as "true", which seems to validate this interpretation of $\neg$ given the meaning we gave to it in the satisfaction definition.

The main idea is that the meaning we give to symbols (such as $\neg$), lets us interpret those symbols. An important point to note however is that the meaning given to $\neg$ is not limited to interpreting our language as talking about truth (or even, not limited to some intended interpretation for which the meaning has been specifically given). Essentially, *meaning* is just a defined property. And a meaning, once specified, lets itself be interpreted with whichever interpretation. Quite often, a meaning is chosen with a specific *intended interpretation* in mind.

One last remark concerning negation. It may seem that the interpretation of a concept such as negation is straightforward, and hence, that the defined meaning for a negation symbol is always going to be more or less the same. This is not the case, since there are many logics, where negation is not given the same semantical meaning that we gave to it. This happens in some intuitionistic and many-valued logics for example. It is easy to see that in a 3-valued logic for instance, a concept of negation must behave differently.

All the above remarks hold for any symbol given an explicit meaning in a satisfaction definition. Different meanings can and have been given to what we call disjunction (inclusive or exclusive), and implication (materical implication, strict implication), to name just a few others. The fact that we call different defined properties (ie. meanings) by the same name : negation,

[9] We sometimes write $w \Vdash \varphi$ for $\mathfrak{M}, w \Vdash \varphi$ when the choice of model or frame is obvious or not important.

disjunction, implication, comes from how we have interpreted those meanings (as negation, disjunction and implication for instance).

What about modalities ? We defined them as properties on frames. These properties can then be interpreted however we like. Let us take the case of the basic modal languages, and assume that we have a simple model $\mathfrak{M} = (W, R, V)$ where $W = \{w, v, u\}$, $R = \{(w, v), (w, u)\}$ and $V(p) = \{v\}$. According to the satisfaction definition, since Rwv and $v \Vdash p$, we have $w \Vdash \Diamond p$. Also, since Rwu but $u \nVdash p$, we also have $w \Vdash \neg\Box p$. How do these meanings correspond to some of the ways $\Diamond$ and $\Box$ have been and are interpreted ? Let us start with the *alethic* interpretation, where $\Diamond$ is understood as "possibility" and $\Box$ as "necessity". Points $w \in W$ can be interpreted as states of the world (this is just one possible interpretation). Propositional letters can be understood as facts about the world (such as "it rains"). Let's assume that p in our model is a fact such as "it rains". Then $\Diamond p$ is interpreted as "it is possible that it rains". Given that w is a state of the world, R is understood as telling us what are the other possible states of the world, given a valuation (which facts are the case and which are not). We have established that $\Diamond p$ is true means that "it is possible that it rains", thus there is a possible state where p is true, so there must be a state related to w where p is true. This is precisely how we defined $\Diamond$. The idea is that frames have let us defined the notion of possibility in such a way. What about necessity ? If something is necessary then it can't be otherwise. In our model, we have u related to w, so u represents a possible state of the world. Since $u \notin V(p)$, it represents the possibility that "it doesn't rain". Thus at w (a certain state of the world, that is a collection of facts), it is not necessary that it rains (so $w \Vdash \neg\Box p$), because it's possible that it rains (so $w \Vdash \Diamond p$), but also that it doesn't rain ($w \Vdash \Diamond\neg p$). Had we chosen a model where $V(p) = \{v, u\}$, then, since all possible states accesible from w would make p true, we would have that at w, it is necessary that p.

The use of frames in Kripke semantics is a fairly simple yet powerful tool for interpreting modalities.

1.3 Syntax

We want to be able to talk about all instances of certain formulas. For example we might want to talk about all instances of a formula such as $\varphi \vee \neg\varphi$. For example, $(\psi \wedge \Box\chi) \vee \neg(\psi \wedge \Box\chi)$ is a substitution instance of $\varphi \vee \neg\varphi$. Why ? We might for example want to say that all instances of a certain formula, is valid (or not) on a certain frame or class of frames. We need a syntactic tool for that.

Definition 1.14 (Substitution). We first define a substitution function from the set of propositional letters to formulas, and we extend it such that we get a substitution function from formulas to formulas. Given a modal language $ML(\tau, \Phi)$, we define $s : \Phi \longrightarrow Fml_\tau$ and $\tilde{s} : Fml_\tau \longrightarrow Fml_\tau$.

We can substitute any formula of the language for propositional letters, so s is any function with domain Φ and range Fml_τ. We define $\tilde{s}$ like so (for all $p \in \Phi$ and all n-ary $\triangle \in \tau$) :

$$\begin{aligned}
\tilde{s}(p) &= s(p)\\
\tilde{s}(\bot) &= \bot\\
\tilde{s}(\neg\varphi) &= \neg\tilde{s}(\varphi)\\
\tilde{s}(\varphi \vee \psi) &= \tilde{s}(\varphi) \vee \tilde{s}(\psi)\\
\tilde{s}(\triangle(\varphi_1, ..., \varphi_n)) &= \triangle(\tilde{s}(\varphi_1), ..., \tilde{s}(\varphi_n))
\end{aligned}$$

$\dashv$

We work with Hilbert-style axiom systems.

Definition 1.15 (Syntactic definition of a modal logic). Logics viewed syntactically are sets of formulas that satisfy certain closure conditions[10]. A modal logic for a similarity type τ is a set $L \subseteq Fml_\tau$, such that:

(i) L contains all instances of propositional tautologies

(ii) L is closed under the *proof rules* :

  (a) Modus Ponens: if $\varphi \in L$ and $\varphi \to \psi \in L$, then $\psi \in L$.

  (b) Uniform Substitution: if $\varphi \in L$, then all the substitution instances of φ are in L.

When a formula $\varphi \in L$ we say that φ is a *theorem*, and we write it $\vdash_L \varphi$. Thus

$$L = \{\varphi \in Fml_\tau : \ \vdash_L \varphi\}$$

$\dashv$

Definition 1.16 (Axioms). Axioms are sets of formulas. Logics can be generated by adding the condition that they must be the smallest logic containing a certain set of formulas. We may call such sets of formulas axioms. $\dashv$

Definition 1.17 (Normal Modal Logics). A modal logic of similarity type τ is called *normal*[11] when it contains for every $\bigtriangledown$ of arity $ar(\bigtriangledown)$:

(i) Axioms $\mathrm{K}^i_\bigtriangledown$ for all i such that $1 \leq i \leq ar(\bigtriangledown)$:

$$(\mathrm{K}^i_\bigtriangledown) \quad \bigtriangledown(r_1, ..., (p \to q)_i, ..., r_{ar(\bigtriangledown)}) \to$$
$$\to (\bigtriangledown(r_1, ..., p_i, ..., r_{ar(\bigtriangledown)}) \to \bigtriangledown(r_1, ..., q_i, ..., r_{ar(\bigtriangledown)}))$$

(ii) Axiom $\mathrm{Dual}_\bigtriangledown$:

$$(\mathrm{Dual}_\bigtriangledown) \quad \bigtriangleup(r_1, ..., r_{ar(\bigtriangledown)}) \leftrightarrow \neg \bigtriangledown(\neg r_1, ..., \neg r_{ar(\bigtriangledown)})$$

[10]For a different approach see e.g. [1], [2].

[11]Cf. Def 4.13 in [8] and we refer to Chapter 2 in [9]

(iii) $ar(\triangledown)$-many Generalization (proof) rules (for each i such that $1 \leq i \leq ar(\triangledown)$) :

$$\text{if } \vdash_L \varphi, \text{ then } \vdash_L \triangledown(\bot_1, ..., \varphi_i, ..., \bot_{ar(\triangledown)})$$

We need Dual axioms because we use $\triangle$ as primitives in our languages. They are simply the syntactic counterpart of how we defined the $\triangledown$ (semantically) previously. We need them because our axioms are expressed in terms of $\triangledown$.

We note that this is a common way to axiomatize normal modal logics, but not the only one.

Definition 1.18 (Logic $\mathbf{K}_\tau$). The logic $\mathbf{K}_\tau$ is the smallest normal modal logic. That is, according to definitions 1.15 and 1.17, it contains all tautologies, axioms $\mathrm{K}^i_\triangledown$, and $\mathrm{Dual}_\triangledown$ (for all $\triangledown$) and is closed under Modus Ponens, Uniform Substitution, and Generalization Rules.

If we add the set of axioms Σ to the logic $\mathbf{K}_\tau$, we call this new logic $\mathbf{K}_\tau\mathbf{\Sigma}$.

Definition 1.19 (Proofs). A proof for a logic L is a finite sequence of formulas such that each formula is either a tautology, an axiom, or follows from previous formulas in the sequence by the application of a proof rule.

A formula φ is provable in a logic L when it is the last element of a proof sequence. In such a case, we have $\varphi \in L$. We also say that φ is a *theorem* of L, and we write $\vdash_L \varphi$. If not, ie. if $\varphi \notin L$, we write $\nvdash_L \varphi$

$\dashv$

Definition 1.20 (Deduction). A formula φ is *deducible* from a set of formulas Γ in a logic L, written $\Gamma \vdash_L \varphi$, if $\vdash_L \varphi$ or if there are formulas $\psi_1, ..., \psi_n \in \Gamma$ such that $\vdash_L (\psi_1 \wedge ... \wedge \psi_n) \to \varphi$.

If φ is not L-deducible from Γ, ie. if $\Gamma \vdash_L \varphi$ is not the case, we write $\Gamma \nvdash_L \varphi$.

Definition 1.21 (Consistency and Inconsistency). A set of formulas Γ is *L-consistent* if $\Gamma \nvdash_L \bot$, it is *inconsistent* otherwise. A formula φ is L-consistent if $\{\varphi\} \nvdash_L \bot$, inconsistent otherwise.

Example 1.22 (Normal modal logics for the basic similarity type). Many quite common modal logics are normal modal logics of the basic similarity type. The corresponding axioms and rule of 1.17 for the basic similarity type are the following:

(i) Axiom $K_\Box$: $\quad \Box(p \to q) \to (\Box p \to \Box q)$

(ii) $\text{Dual}_\Box$: $\quad \Diamond p \leftrightarrow \neg\Box\neg p$

(iii) Generalization Rule: if $\vdash_L \varphi$, then $\vdash_L \Box\varphi$

Axiom (i) is usually written (K) and is known as the Kripke scheme. Axiom (ii) is introduced as a syntactic equivalent for the definition of $\Box$. The smallest normal logic for the basic similarity type is called **K** and contains all tautologies, (K), (Dual) and is closed under Modus Ponens, Uniform Substitution, and Generalization.

Why are normal modal logics so important? As we will see later, the minimal modal logic **K** is *sound* for the class of all frames. This means that the theorems of **K** are valid in all frames. If $\vdash_{\mathbf{K}} \varphi$, then $\Vdash \varphi$. If we use Kripke semantics, that is, we use frames to interpret modal languages, then **K** is the minimal set of axioms and rules we need, if we want to have completeness and soundness results with Kripke semantics. If we have Kripke semantics, as we do, then the corresponding logics defined syntactically, are *normal* modal logics.

For any normal modal logic L we have $\mathbf{K} \subseteq L$. Thus we can say that **K** is the logic of the class of all frames.

Common examples of modal logics such as **S5**, **S4**, **T**, etc. can be found in e.g. Chapter 2 in [9], and in [8], [10].

Chapter 2

Completeness results for Kripke semantics

In this chapter we discuss completeness results for Kripke semantics, via building canonical models. Then we turn our discussion to the incompleteness limitations inherent to this frame based semantics.

2.1 Completeness for Kripke semantics

Definition 2.1 (Soundness). A logic L is *sound* with respect to a class of structures F when:

$$\vdash_L \varphi \text{ implies } \mathsf{F} \Vdash \varphi$$

In other words, if φ is a theorem of L, then φ is valid on all structures $\mathfrak{F} \in \mathsf{F}$.

$\dashv$

We claimed at earlier that the logic $\mathbf{K}$, the smallest normal modal logic, is important because its theorems are valid on all frames. This means that $\vdash_{\mathbf{K}} \varphi$ implies $\Vdash \varphi$.

Thus, any normal modal logic contains **K**. Any normal modal logic is either **K** or an extension of it.

Definition 2.2 (Completeness). A logic L is *strongly complete* with respect to F (for any set of formulas $\Gamma \cup \{\varphi\}$) if:

$$\Gamma \Vdash_{\mathsf{F}} \varphi \text{ implies } \Gamma \vdash_L \varphi$$

A logic L is *weakly complete* with respect to F (for any formula φ) if:

$$\mathsf{F} \Vdash \varphi \text{ implies } \vdash_L \varphi$$

We note that F is often a class of frames, but it can also be a class of models, or general frames.

When we talk about *completeness* we mean strong completeness.

We prove strong completeness of a logic L with respect to a class of frames by building canonical models. A canonical model for a normal modal logic L is a model that satisfies exactly the theorems of L. The main idea behind canonical models is that we use a set of maximal consistent sets as the underlying set of the frame of the canonical model.

We introduce the two following lemmas that will be useful for later proofs.

Lemma 2.3. *For any normal modal logic L we have*

$$\begin{aligned} \Sigma \vdash_L \varphi \quad &\text{iff} \quad \Sigma \cup \{\neg\varphi\} \text{ is inconsistent} \\ \Sigma \vdash_L \neg\varphi \quad &\text{iff} \quad \Sigma \cup \{\varphi\} \text{ is inconsistent} \end{aligned}$$

Proof. ($\Rightarrow$) If $\Sigma \vdash_L \varphi$, then $\Sigma \cup \{\neg\varphi\} \vdash_L \varphi \wedge \neg\varphi$, thus $\Sigma \cup \{\neg\varphi\} \vdash_L \bot$, so $\Sigma \cup \{\neg\varphi\}$ is inconsistent.

($\Leftarrow$) If $\Sigma \cup \{\neg\varphi\}$ is inconsistent, there is a ψ such that $\Sigma \cup \{\neg\varphi\} \vdash_L \psi$

and $\Sigma \cup \{\neg\varphi\} \vdash_L \neg\psi$:

1.	$\Sigma \cup \{\neg\varphi\} \vdash_L \psi$	
2.	$\Sigma \cup \{\neg\varphi\} \vdash_L \neg\psi$	
3.	$\Sigma \vdash_L \neg\varphi \to \psi$	Deduction, 1
4.	$\Sigma \vdash_L \neg\varphi \to \neg\psi$	Ded., 2
5.	$\Sigma \vdash_L (\neg\varphi \to \neg\psi) \to ((\neg\varphi \to \psi) \to \varphi)$	Tautology
6.	$\Sigma \vdash_L (\neg\varphi \to \psi) \to \varphi$	MP,4,5
7.	$\Sigma \vdash_L \varphi$	MP,3,6

Thus if $\Sigma \cup \{\neg\varphi\}$ is inconsistent, then $\Sigma \vdash_L \varphi$. By uniform substitution and the tautology $\varphi \leftrightarrow \neg\neg\varphi$, we obtain that $\Sigma \vdash_L \neg\varphi$ iff $\Sigma \cup \{\varphi\}$ is inconsistent. □

Corollary 2.4. *For any normal modal logic L we have*

$$\begin{aligned} \Sigma \nvdash_L \varphi &\quad \text{iff} \quad \Sigma \cup \{\neg\varphi\} \text{ is consistent} \\ \Sigma \nvdash_L \neg\varphi &\quad \text{iff} \quad \Sigma \cup \{\varphi\} \text{ is consistent} \end{aligned}$$

Proof. Direct from Lemma 2.3. □

Lemma 2.5. *If Σ is consistent and $\Sigma \vdash_L \varphi$, then $\Sigma \cup \{\varphi\}$ is consistent.*

Proof. If Σ is consistent and $\Sigma \vdash_L \varphi$, then $\Sigma \nvdash_L \neg\varphi$ (otherwise Σ would be inconsistent). By Lemma 2.3, we have $\Sigma \nvdash_L \neg\varphi$ iff $\Sigma \cup \{\varphi\}$ is consistent. □

Theorem 2.6. *A logic L is strongly complete with respect to a class of structures F iff every L-consistent set of formulas Σ is satisfiable on some $\mathfrak{A} \in \mathsf{F}$, ie. $\exists \mathfrak{A} \in \mathsf{F}, \mathfrak{A} \Vdash \Sigma$ for all L-consistent Σ.*

Proof. ($\Rightarrow$) We show that if L is strongly complete with respect to F, then for every L-consistent Σ there exists an $\mathfrak{A} \in \mathsf{F}$ such that $\mathfrak{A} \Vdash \Sigma$. By contraposition, this means that if there exist a consistent Σ such that $\forall \mathfrak{A} \in \mathsf{F}, \mathfrak{A} \nVdash \Sigma$, then L is not strongly complete. But then we would have $\Sigma \Vdash_F \varphi$ for any φ since the definition of $\Sigma \Vdash_F \varphi$ is that if any $\mathfrak{A} \in \mathsf{F}$ makes Σ true, then $\mathfrak{A}$ makes φ true. But since there is no such $\mathfrak{A}$, it is vacuously true. But then L is not strongly complete, since if it were, it would imply that $\Sigma \vdash_L \varphi$ for any φ, for example $\bot$. But this is not possible since Σ is consistent.

($\Leftarrow$) We prove by contraposition, so we assume that L is not strongly complete with respect to F, and want to show that $\exists \Sigma$ that can't be satisfied in any structure $\mathfrak{A} \in \mathsf{F}$. Since L is not strongly complete with respect to F, then there is a set $\Gamma \cup \{\varphi\}$ such that $\Gamma \Vdash_F \varphi$ and $\Gamma \nvdash_L \varphi$ (by the definition of strong completeness). Since $\Gamma \nvdash_L \varphi$, by Corollary 2.4, $\Gamma \cup \{\neg\varphi\}$ is consistent. But $\Gamma \cup \{\neg\varphi\}$ is not satisfiable on any $\mathfrak{A} \in \mathsf{F}$. Because $\Gamma \Vdash_F \varphi$ means that $\forall \mathfrak{A} \in \mathsf{F}$ if $\mathfrak{A} \Vdash \Gamma$ then $\mathfrak{A} \Vdash \varphi$. But then clearly $\mathfrak{A} \nVdash \Gamma \cup \{\neg\varphi\}$. $\square$

Definition 2.7 (Maximal Consistent Sets). A set of formulas Σ is maximal L-consistent if Σ is L-consistent ($\Sigma \nvdash_L \bot$) and for any Γ such that $\Sigma \subsetneq \Gamma$, then Γ is inconsistent ($\Gamma \vdash_L \bot$). We write that Σ is L-MCS, if it is maximal consistent.

Lemma 2.8 (Properties of MCS). *For all L-MCS Σ, the following statements hold:*

(i) *$\Sigma = Ded(\Sigma)$, where $Ded(\Sigma)$ is the set of all φ such that $\Sigma \vdash_L \varphi$. Or, equivalently if Σ is a L-MCS, then $\Sigma \vdash_L \varphi$ iff $\varphi \in \Sigma$.*

Proof. By Corollary 2.4, if $\Sigma \vdash_L \varphi$ then $\Sigma \cup \{\varphi\}$ is consistent. But then $\varphi \in \Sigma$. $\square$

(ii) *If $\varphi \in \Sigma$ and $\varphi \to \psi \in \Sigma$, then $\psi \in \Sigma$*

Proof. We assume that $\varphi \in \Sigma$ and $\varphi \to \psi \in \Sigma$. By Lemma 2.8-(i) we have $\Sigma \vdash_L \varphi$ and $\Sigma \vdash_L \varphi \to \psi$. By Modus Ponens this yields $\Sigma \vdash_L \psi$. And by Lemma 2.8-(i) $\psi \in \Sigma$. $\square$

(iii) For all φ, either $\varphi \in \Sigma$ or $\neg\varphi \in \Sigma$

Proof. Suppose that $\varphi \notin \Sigma$. Then since Σ is maximal, $\Sigma \cup \{\varphi\}$ is inconsistent (by definition of MCS). By Lemma 2.3 if $\Sigma \cup \{\varphi\}$ is inconsistent, then we have $\Sigma \vdash_L \neg\varphi$. By Lemma 2.8-(i) $\Sigma \vdash_L \neg\varphi$ yields $\neg\varphi \in \Sigma$. Thus if $\varphi \notin \Sigma$, then $\neg\varphi \in \Sigma$. The argument is the similar for $\neg\varphi \notin \Sigma$ (we substitute φ with $\neg\varphi$ in the argument). □

(iv) For all φ, ψ we have $\varphi \vee \psi \in \Sigma$ iff $\varphi \in \Sigma$ or $\psi \in \Sigma$

Proof. ($\Rightarrow$) We have $\varphi \vee \psi \in \Sigma$, thus by Lemma 2.8-(i) we get $\Sigma \vdash_L \varphi \vee \psi$. Hence by maximality of Σ either $\Sigma \vdash_L \varphi$ or $\Sigma \vdash_L \psi$. Then, by Lemma 2.8-(i), $\varphi \in \Sigma$ or $\psi \in \Sigma$.

($\Leftarrow$) The argument is similar to the left to right direction. $\varphi \in \Sigma$ or $\psi \in \Sigma$ iff $\Sigma \vdash_L \varphi$ or $\Sigma \vdash_L \psi$ iff $\Sigma \vdash_L \varphi \vee \psi$ iff $\varphi \vee \psi \in \Sigma$. □

(v) For all φ, ψ we have $\varphi \wedge \psi \in \Sigma$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$.

Proof. $\varphi \wedge \psi \in \Sigma$ iff (by Lemma 2.8-(i)) $\Sigma \vdash_L \varphi \wedge \psi$ iff $\Sigma \vdash_L \varphi$ and $\Sigma \vdash_L \psi$ iff $\varphi \in \Sigma$ and $\psi \in \Sigma$. □

Lemma 2.9 (Lindenbaum's Lemma). *If Σ is a L-consistent set of formulas, then there is a L-MCS Σ^+ such that $\Sigma \subseteq \Sigma^+$.*

Proof. Let $\varphi_0, \varphi_1, ...$ be an enumeration of all the formulas of the language of L. We define Σ^+ :

$$\Sigma_0 = \Sigma$$

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{\varphi_n\} & \text{if it is } L\text{-consistent} \\ \Sigma_n \cup \{\neg\varphi_n\} & \text{otherwise} \end{cases}$$

$$\Sigma^+ = \bigcup \Sigma_n$$

We need to show that (i) for each n, the set Σ_n is consistent. Furthermore, we have to prove that (ii) Σ^+ is a maximal consistent set.

(i) We assumed that Σ is consistent, so Σ_0 is obviously consistent. To prove that Σ_{n+1} is consistent, we have to show that $\Sigma_n \cup \{\varphi_n\}$ is consistent or if it isn't, that $\Sigma_n \cup \{\neg\varphi_n\}$ is consistent. If $\Sigma_n \cup \{\varphi_n\}$ is consistent, then it is obviously consistent. If $\Sigma_n \cup \{\varphi_n\}$ is inconsistent, then by Lemma 2.3, we have $\Sigma_n \vdash_L \neg\varphi_n$. Since Σ_n is consistent and $\Sigma_n \vdash_L \neg\varphi_n$, by Lemma 2.5 we obtain that $\Sigma_n \cup \{\neg\varphi_n\}$ is consistent. Thus Σ_{n+1} is always consistent.

(ii) We show that Σ^+ is consistent and maximal. Towards a contradiction, we assume that Σ^+ is inconsistent, ie. $\Sigma^+ \vdash_L \bot$. This means that there are $\psi_1, ..., \psi_n \in \Sigma^+$ such that (by the Deduction theorem) $\vdash_L (\psi_1 \wedge ... \wedge \psi_n) \to \bot$. But then there must be a k such that $\psi_1, ...\psi_n \in \Sigma_k \subseteq \Sigma^+$. Such a $\Sigma_k \vdash_L \bot$, ie. would be inconsistent. Which contradicts our assumption.

Finally, we need to prove that Σ^+ is maximal. Towards a contradiction, we suppose that it is not a maximal consistent set. Then, there is a φ such that $\varphi \notin \Sigma^+$ and $\neg\varphi \notin \Sigma^+$ (by Lemma 2.8-(iii)). Since all formulas of the language are enumerated, φ must be one of φ_n. But then according to the definition of Σ_{n+1}, we must either have $\varphi_n \in \Sigma_{n+1}$ or $\neg\varphi_n \in \Sigma_{n+1}$. Since $\Sigma_{n+1} \subseteq \Sigma^+$, then $\varphi_n \in \Sigma^+$ or $\neg\varphi_n \in \Sigma^+$. But this contradicts our assumption. Therefore, Σ^+ must be a maximal consistent set. □

We first start by giving a general completeness result for the normal modal logics of the basic similarity type, then we show such a result for arbitrary similarity type. Each contains three important steps (i) definition of canonical models, (ii) proof of Existence lemma and (iii) Truth lemma. Then we are able to state the Canonical Model Theorem, the completeness result we wanted to achieve.

Definition 2.10 (Canonical model (for the basic similarity type)). The canonical model for a normal modal logic L (with the basic similarity type) is the model $\mathfrak{M}^L = (W^L, R^L, V^L)$ such that :

(i) W^L is the set of all L-MCS

(ii) R^L is a binary relation on W^L defined by $R^L wv$ if $\varphi \in v$ implies $\Diamond\varphi \in w$ (for all φ).

(iii) V^L is a function $V^L : \Phi \longrightarrow \wp(W^L)$ such that for all $p \in \Phi$, $V^L(p) = \{w \in W^L : p \in w\}$

For a normal modal logic L, its canonical model is a Kripke model which satisfies exactly the theorems of L.

Lemma 2.11. *$\big(\Box\varphi \in w$ implies $\varphi \in v\big)$ iff $R^L wv$.*

Proof. ($\Rightarrow$) We prove that if $\Box\varphi \in w$ implies $\varphi \in v$, then $R^L wv$. Equivalently by taking the contrapositive of the antecedant we can prove that if $\varphi \notin v$ implies $\Box\varphi \notin w$, then $R^L wv$. Since v is a MCS and $\varphi \notin v$, is equivalent to $\neg\varphi \in v$. Similarly $\Box\varphi \notin w$ is equivalent with $\neg\Box\varphi \in w$. So, we need to show that if $\neg\varphi \in v$ implies $\neg\Box\varphi \in w$, then $R^L wv$. By the Dual axiom, from $\neg\Box\varphi \in w$ we get $\neg\neg\Diamond\neg\varphi \in w$, and thus $\Diamond\neg\varphi \in w$. But if $\neg\varphi \in v$ implies $\Diamond\neg\varphi \in w$, then, by clause (ii) of the canonical model definition 2.10, we have $R^L wv$.

($\Leftarrow$) We prove that if $R^L wv$, then $\Box\varphi \in w$ implies $\varphi \in v$. We prove by contraposition: we assume $R^L wv$ and $\varphi \notin v$ and show that $\Box\varphi \notin w$. Since v is maximal consistent and $\varphi \notin v$, we have $\neg\varphi \in v$. By clause (ii) of the canonical model definition 2.10, since we have $R^L wv$ and $\neg\varphi \in v$, then we also have $\Diamond\neg\varphi \in w$. w is a MCS, so because it is consistent $\neg\Diamond\neg \notin w$. But then we have $\Box\varphi \notin w$, the result we wanted. $\Box$

For the next three statements (Lemmas 2.12, 2.13 and Theorem 2.14) we refer to Lemmas 4.20, 4.21 and Theorem 4.22 in [8]. Here we give full and more detailed proofs.

Lemma 2.12 (Existence Lemma). *For any normal modal logic L (with the basic modal similarity type) and for any $w \in W^L$:*

$$\Diamond\varphi \in w \text{ implies } \exists v \in W^L \text{ such that } R^L wv \text{ and } \varphi \in v$$

Proof. We suppose that $\Diamond\varphi \in w$. We want to show that there is v such that $R^L wv$ and $\varphi \in v$. Let u be $\{\varphi\} \cup \{\psi : \Box\psi \in w\}$. We will construct $v \supseteq u$ such that it extends u. φ should clearly be in u, because we want it to be in v. We also want all ψ such that $\Box\psi \in w$ to be in u, and in v, to ensure that we have the relation $R^L wv$, since according to the previous Lemma 2.11, if for all ψ we have that $\Box\psi \in w$ such that $\psi \in v$, then $R^L wv$. If all the mentioned conditions are met, all that is left to be shown is that v is a L-MCS. By the Lindenbaum Lemma 2.9, any L-consistent set can be extended to an L-MCS. If we can show that u is L-consistent, then we simply extend it to a L-MCS v and we are done.

We show that u is consistent, and prove it by contradiction. We assume that u is inconsistent. Since u is inconsistent and $\varphi \in u$, by the Deduction theorem there are $\psi_1, ..., \psi_n \in u$ such that $\vdash_L (\psi_1 \wedge ... \wedge \psi_n) \to \neg\varphi$. By Generalization we get $\vdash_L \Box((\psi_1 \wedge ... \wedge \psi_n) \to \neg\varphi)$. By the previous formula and the instance of (K) $\vdash_L \Box((\psi_1 \wedge ... \wedge \psi_n) \to \neg\varphi) \to (\Box(\psi_1 \wedge ... \wedge \psi_n) \to \Box\neg\varphi)$, we have $\vdash_L \Box(\psi_1 \wedge ... \wedge \psi_n) \to \Box\neg\varphi$.

For the next step we need to show that $(\Box\psi_1 \wedge ... \wedge \Box\psi_n) \to \Box(\psi_1 \wedge ... \wedge \psi_n)$

is a theorem of every normal modal logic L:

1.	$\vdash_L \psi_1 \to (\psi_2 \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...)$	Tautology
2.	$\vdash_L \Box(\psi_1 \to (\psi_2 \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...)$	Gen, 1
3.	$\vdash_L \Box(\psi_1 \to (\psi_2 \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...) \to$	
	$\to (\Box\psi_1 \to \Box(\psi_2 \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...)$	inst. (K)
4.	$\vdash_L \Box\psi_1 \to \Box(\psi_2 \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...)$	MP, 2, 3
5.	$\vdash_L \Box(\psi_2 \to (\psi_3 \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...) \to$	
	$\to (\Box\psi_2 \to \Box(\psi_3 \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...)$	inst. (K)
5^*.	$\vdash_L \Box(\psi_i \to (\psi_{i+1} \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...) \to$	inst. (K)
	$\to (\Box\psi_i \to \Box(\psi_{i+1} \to ... \to (\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)...)$	for i, $0 < i < n$
6.	$\vdash_L \Box(\psi_n \to (\psi_1 \wedge ... \wedge \psi_n)) \to (\Box\psi_n \to \Box(\psi_1 \wedge ... \wedge \psi_n))$	inst. (K)
7.	$\vdash_L \Box\psi_1 \to (\Box\psi_2 \to ... \to (\Box\psi_n \to \Box(\psi_1 \wedge ... \wedge \psi_n)...)$	Ded. 4,5,5^*, 6
8.	$\vdash_L (\Box\psi_1 \wedge ... \wedge \Box\psi_n) \to \Box(\psi_1 \wedge ... \wedge \psi_n)$	Ded., 7

We had $\vdash_L \Box(\psi_1 \wedge ... \wedge \psi_n) \to \Box\neg\varphi$, and since we just showed that $\vdash_L (\Box\psi_1 \wedge ... \wedge \Box\psi_n) \to \Box(\psi_1 \wedge ... \wedge \psi_n)$, we get $\vdash_L (\Box\psi_1 \wedge ... \wedge \Box\psi_n) \to \Box\neg\varphi$. We defined u such that $\Box\psi_1, ..., \Box\psi_n \in w$ and since w is a L-MCS, it is closed under conjunction, thus $\Box\psi_1 \wedge ... \wedge \Box\psi_n \in w$. Since we just showed that $(\Box\psi_1 \wedge ... \wedge \Box\psi_n) \to \Box\neg\varphi$ is a theorem and $\Box\psi_1 \wedge ... \wedge \Box\psi_n \in w$, then we must have $\Box\neg\varphi \in w$. By the Dual axiom we have $\neg\Diamond\varphi \in w$. But this contradicts our assumption that $\Diamond\varphi \in w$, since w is consistent. Hence we reach a contradiction, and get that u is consistent, which is what we needed to finish the proof. $\Box$

Lemma 2.13 (Truth Lemma). *For all normal modal logics L (in the basic modal similarity type) and all φ:*

$$\mathfrak{M}^L, w \Vdash \varphi \text{ iff } \varphi \in w$$

Proof. We prove by induction on the complexity of φ. If φ is a propositional letter p:

$$\begin{aligned} \mathfrak{M}^L, w \Vdash p \text{ iff } & w \in V^L(p) && \text{(satisfaction def.)} \\ \text{iff } & p \in w && \text{(def. of } V^L\text{)} \end{aligned}$$

If φ is $\bot$. $\mathfrak{M}^L, w \Vdash \bot$ is never the case by definition, which corresponds to w being consistent (ie. w never contains $\bot$).

If φ is of the form $\neg\psi$:

$$\begin{aligned} \mathfrak{M}^L, w \Vdash \neg\psi \text{ iff } & \mathfrak{M}^L, w \nVdash \psi && \text{(satisfaction def.)} \\ \text{iff } & \psi \notin w && \text{(inductive hypothesis)} \\ \text{iff } & \neg\psi \in w && \text{(by maximality of } w\text{)} \end{aligned}$$

If φ is of the form $\psi \vee \chi$:

$$\begin{aligned} \mathfrak{M}^L, w \Vdash \psi \vee \chi \text{ iff } & \mathfrak{M}^L, w \Vdash \psi \text{ or } \mathfrak{M}^L, w \Vdash \chi \\ \text{iff } & \psi \in w \text{ or } \chi \in w && \text{(inductive hypothesis)} \\ \text{iff } & \psi \vee \chi \in w && \text{(by maximality of } w\text{)} \end{aligned}$$

If φ is of the form $\Diamond\psi$:

$$\begin{aligned} \mathfrak{M}^L, w \Vdash \Diamond\psi \text{ iff } & \exists v \text{ such that } R^L wv \text{ and } \mathfrak{M}^L, v \Vdash \psi \\ \text{iff } & \exists v \text{ such that } R^L wv \text{ and } \psi \in v && \text{(ind. hypothesis)} \end{aligned}$$

$$\begin{aligned} & \exists v \text{ such that } R^L wv \text{ and } \psi \in v \Rightarrow \Diamond\psi \in w && \text{(def. of } R^L\text{)} \\ & \exists v \text{ such that } R^L wv \text{ and } \psi \in v \Leftarrow \Diamond\psi \in w && \text{(Existence Lemma 2.12)} \end{aligned}$$

Hence,

$$\mathfrak{M}^L, w \Vdash \Diamond\psi \text{ iff } \Diamond\psi \in w$$

$\square$

Theorem 2.14 (Canonical Model Theorem). *Every normal modal logic L in the basic similarity type is strongly complete with respect to its canonical model.*

Proof. For any consistent set Σ of a normal modal logic L, by the Lindenbaum Lemma, there is a maximal consistent set Σ^+ such that $\Sigma \subseteq \Sigma^+$. By the Truth Lemma (Lemma 2.13), $\mathfrak{M}^L, \Sigma^+ \Vdash \Sigma$. $\square$

Theorem 2.15. *The logic* $\mathbf{K}$ *of the basic similarity type is strongly complete with respect to the class of all frames.*

Proof. By Theorem 2.6 we need to find a model $\mathfrak{M}$ such that for every $\mathbf{K}$-consistent Σ there is a w in $\mathfrak{M}$ that satisfies Σ, ie. we need to find a model $\mathfrak{M}$ such that $\mathfrak{M}, w \Vdash \Sigma$, for every consistent Σ. There is such a model, namely the canonical model for $\mathbf{K}$: $\mathfrak{M}^{\mathbf{K}}$. For w, we choose any $\mathbf{K}$-MCS Σ^+ such that $\Sigma \subseteq \Sigma^+$. By the Lindenbaum Lemma, we can always find such a Σ^+. By the Truth Lemma, we obtain $\mathfrak{M}^{\mathbf{K}}, \Sigma^+ \Vdash \Sigma$. $\square$

So far we only dealt with the basic similarity type. We now do the same for an arbitrary similarity type, to get a general result.

Definition 2.16 (Canonical model (for arbitrary similarity type)). We let τ be any modal similarity type and L a normal τ-modal logic. The canonical model $\mathfrak{M}^L = (W^L, R^L_\triangle, V^L)_{\triangle \in \tau}$ for the logic L is defined such that:

(i) W^L is the set of all L-MCS

(ii) There is a relation $R^L_\triangle \subseteq (W^L)^{n+1}$ for each n-ary $\triangle \in \tau$, defined by $R^L_\triangle w v_1 ... v_n$ if for all formulas $\varphi_1 \in v_1, ..., \varphi_n \in v_n$ we have $\triangle(\varphi_1, ..., \varphi_n) \in w$.

(iii) V^L is a function $V^L : \Phi \longrightarrow \wp(W^L)$ such that for all $p \in \Phi$, $V^L(p) = \{w \in W^L : p \in w\}$

Proposition 2.17. *For any normal modal logic L,* $\big(\bigtriangledown(\varphi_1, ..., \varphi_n) \in w$ *implies* $\exists i\ \varphi_i \in v_i\big)$ *iff* $R^L_\triangle wv_1...v_n$.

Proof. ($\Rightarrow$) We prove that if $\bigtriangledown(\varphi_1, ..., \varphi_n) \in w$ implies $\exists i\ \varphi_i \in v_i$, then $R^L_\triangle wv_1...v_n$. By taking the contrapositive of the antecedant, we can show that if $\varphi_i \notin v_i$ (for all $1 \leq i \leq n$) implies $\bigtriangledown(\varphi_1, ..., \varphi_n) \notin w$, then $R^L_\triangle wv_1...v_n$. By Lemma 2.8, since v_i are L-MCS and $\varphi_i \notin v_i$, we have $\neg\varphi_i \in v_i$. Similarly if $\bigtriangledown(\varphi_1, ..., \varphi_n) \notin w$, then $\neg \bigtriangledown(\varphi_1, ..., \varphi_n) \in w$. It's enough to show that if $\forall i\ \neg\varphi_i \in v_i$ implies $\neg \bigtriangledown(\varphi_1, ..., \varphi_n)$, then $R^L_\triangle wv_1...v_n$. By the axiom Dual$_\triangle$ from $\neg \bigtriangledown(\varphi_1, ..., \varphi_n) \in w$ we get $\neg\neg\triangle(\neg\varphi_1, ..., \neg\varphi_n) \in w$, and thus $\triangle(\neg\varphi_1, ..., \neg\varphi_n) \in w$. But then if for all i such that $1 \leq i \leq n$, $\neg\varphi_i \in v_i$ implies $\triangle(\neg\varphi_1, ..., \neg\varphi_n) \in w$, then by clause (ii) of the Canonical model definition, we obtain that $R^L_\triangle wv_1...v_n$.

($\Leftarrow$) We prove that if $R^L_\triangle wv_1...v_n$, then $\bigtriangledown(\varphi_1, ..., \varphi_n) \in w$ implies $\exists i\ \varphi_i \in v_i$. By taking the contrapositive of the consequent, if we assume that $R^L_\triangle wv_1...v_n$ and $\forall i\ \varphi_i \notin v_i$, then we need to show that $\bigtriangledown(\varphi_1, ..., \varphi_n) \notin w$. By Lemma 2.8, since all v_i are maximal consistent sets and $\varphi_i \notin v_i$, we have that $\neg\varphi_i \in v_i$. By the clause defining $R^L_\triangle wv_1...v_n$ in the Canonical model definition, since we have $R^L_\triangle wv_1...v_n$ and $\forall i\ \neg\varphi_i \in v_i$, then we also have $\triangle(\neg\varphi_1, ..., \neg\varphi_n) \in w$. Since w is a maximal consistent set, by consistency, $\neg\triangle(\neg\varphi_1, ..., \neg\varphi_n) \notin w$. By Dual$_\triangle$, this gives us $\bigtriangledown(\varphi_1, ..., \varphi_n) \notin w$, the result we needed. $\square$

Next we state and prove, for the arbitrary similarity type, the analogs of Lemmas 2.12, 2.13 and Theorem 2.14. For these statements (Lemmas 2.18, 2.19, and Theorem 2.20), we give complete and detailed proofs, which are missing in [8].

Lemma 2.18 (Existence Lemma for the arbitrary modal similarity type). *For all normal modal logics L we have*

$$\triangle(\varphi_1, ..., \varphi_n) \in w \text{ implies } \exists v_1, ...v_n \in W^L$$
$$\text{such that } \varphi_1 \in v_1, ..., \varphi_n \in v_n \text{ and } R^L w v_1 ... v_n$$

Proof. Suppose $\triangle(\varphi_1, ..., \varphi_n) \in w$. Let $\psi_0, \psi_1, ...$ enumerate all the formulas of our language. We construct by induction n sequences of sets of formulas

$$\begin{array}{ll} \{\varphi_1\} & = \Sigma_0^{\varphi_1} \subseteq \Sigma_1^{\varphi_1} \subseteq \dots \\ \{\varphi_2\} & = \Sigma_0^{\varphi_2} \subseteq \Sigma_1^{\varphi_2} \subseteq \dots \\ \vdots & \qquad \vdots \\ \{\varphi_n\} & = \Sigma_0^{\varphi_n} \subseteq \Sigma_1^{\varphi_n} \subseteq \dots \end{array}$$

such that for $1 \leq j \leq n$, all $\Sigma_i^{\varphi_j}$ are finite and consistent, and

$$(\star) \quad \Sigma_{i+1}^{\varphi_j} \text{ is either } \Sigma_i^{\varphi_j} \cup \{\psi_i\} \text{ or } \Sigma_i^{\varphi_j} \cup \{\neg\psi_i\}$$

and

$$(\star\star) \quad \triangle(\sigma_i^{\varphi_1}, \dots, \sigma_i^{\varphi_n}) \in w, \text{ where } \sigma_i^{\varphi_j} := \bigwedge \Sigma_i^{\varphi_j}.$$

Suppose the sets $\Sigma_i^{\varphi_j}$ have already been defined for $i \leq k$, for all $1 \leq j \leq n$. We have to construct $\Sigma_{k+1}^{\varphi_j}$.

By the inductive hypothesis $(\star\star)$ we have $\triangle(\sigma_k^{\varphi_1}, \dots, \sigma_k^{\varphi_n}) \in w$, hence it follows that

$$\triangle(\sigma_k^{\varphi_1} \wedge (\psi_k \vee \neg\psi_k), \dots, \sigma_k^{\varphi_n} \wedge (\psi_k \vee \neg\psi_k)) \in w,$$

and hence

$$\triangle((\sigma_k^{\varphi_1} \wedge \psi_k) \vee (\sigma_k^{\varphi_1} \wedge \neg\psi_k), \dots, (\sigma_k^{\varphi_n} \wedge \psi_k) \vee (\sigma_k^{\varphi_n} \wedge \neg\psi_k)) \in w.$$

As $\triangle$ distributes over $\vee$ [1], one of the formulas $\triangle(\sigma_k^{\varphi_1} \wedge \psi_k^{\varepsilon_1}, \dots, \sigma_k^{\varphi_n} \wedge \psi_k^{\varepsilon_n})$ should belong to w, where ε_i is either 0 or 1, and $\psi^1 = \psi$ and $\psi^0 = \neg\psi$. Fix

[1] $\triangle(\varphi \vee \psi) \leftrightarrow \triangle\varphi \vee \triangle\psi$

the sequence of those ε_i's for which the previous formula is in w and, for all $1 \leq j \leq n$, let

$$\Sigma_{k+1}^{\varphi_j} = \begin{cases} \Sigma_k^{\varphi_j} \cup \{\psi_k\} & \text{if } \varepsilon_j = 1 \\ \Sigma_k^{\varphi_j} \cup \{\neg\psi_k\} & \text{otherwise.} \end{cases}$$

It's easy to check that each $\Sigma_{k+1}^{\varphi_j}$ satisfies our inductive hypotheses: each is finite, $(\star)$ and $(\star\star)$ hold. To see that $\Sigma_{k+1}^{\varphi_j}$ is consistent, observe that $\sigma_{k+1}^{\varphi_j} = \bigwedge \Sigma_{k+1}^{\varphi_j} \not\leftrightarrow \bot$. This is because $\triangle(\sigma_{k+1}^{\varphi_1}, \ldots, \sigma_{k+1}^{\varphi_n}) \in w$ and w is consistent (because $w \in W^L$ is L-MCS) and $\triangle(\cdots, \bot, \cdots) \leftrightarrow \bot$.

Finally, put $v_j = \bigcup_i \Sigma_i^{\varphi_j}$ for $1 \leq j \leq n$. Thus all $v_1, \ldots, v_n$ are L-MCS, and we have $\varphi_1 \in v_1, \ldots, \varphi_n \in v_n$ and $R_\triangle^L w v_1 \ldots v_n$ as we wanted. $\square$

Lemma 2.19 (Truth Lemma for arbitrary similarity type). *For all normal modal logics L, and all φ we have*

$$\mathfrak{M}^L, w \Vdash \varphi \text{ iff } \varphi \in w$$

Proof. We prove by induction on the complexity of φ. The cases when φ is of the form $p, \neg\psi, \psi \vee \chi$ are the same as in Lemma 2.13. Thus we only need to show the modal case for arbitrary modalities, ie. to prove that $\mathfrak{M}^L$, $w \Vdash \triangle(\psi_1, ..., \psi_n)$ iff $\triangle(\psi_1, ..., \psi_n) \in w$, for all n-ary modal operators.

$$\mathfrak{M}^L, w \Vdash \triangle(\psi_1, ..., \psi_n)$$

$$\text{iff} \qquad \text{(satisfaction def.)}$$

$$\exists v_1, ...v_n \in W^L \text{ and } R_\triangle^L w v_1 ... v_n \text{ such that } \mathfrak{M}^L, v_i \Vdash \psi_i \ (\forall i\ 1 \leq i \leq n)$$

$$\text{iff} \qquad \text{(induction hyp.)}$$

$$(\star\star\star) \quad \exists v_1, ...v_n \in W^L \text{ and } R_\triangle^L w v_1 ... v_n \text{ such that } \psi_i \in v_i \ (\forall i\ 1 \leq i \leq n)$$

$$(\star\star\star) \Rightarrow \triangle(\psi_1, ..., \psi_n) \in w \qquad \text{(def 2.16-(ii) of } R_\triangle^L w v_1 ... v_n)$$

$$(\star\star\star) \Leftarrow \triangle(\psi_1, ..., \psi_n) \in w \qquad \text{(Existence Lemma 2.18)}$$

Hence,

$$\mathfrak{M}^L, w \Vdash \triangle(\psi_1, ..., \psi_n) \text{ iff } \triangle(\psi_1, ..., \psi_n) \in w$$

$\square$

Theorem 2.20 (Canonical Model Theorem, for arbitrary modal similarity type). *For any τ, any normal modal logic L is strongly complete with respect to its canonical model.*

Proof. Same as proof of Theorem 2.10. For any consistent set Σ of of the normal modal logic L, by Lindenbaum's Lemma there is a L-MCS Σ^+ extending Σ. By the Truth Lemma (Lemma 2.19) we have $\mathfrak{M}^L, \Sigma^+ \Vdash \Sigma$. $\square$

2.2 An incomplete but consistent normal modal logic

We show that there exist consistent normal modal logics that are incomplete with respect to Kripke semantics. Thomason in [21], 1974, was the first to show that such a logic existed.[2] An (incomplete) version of this proof can be found in e.g. [8], here we work out and discuss the proof in full details.

To show this we introduce what is usually called the basic temporal language, with a similarity type containing two unary diamond modalities. They are usually denoted by F and P. We stick with this standard notation. Thus the similarity type for this language is $\tau = \{F, P\}$.

To construct models for this language we need two binary relations (since we have two unary modalities) R_F and R_P. We define R_P to be the converse of R_F, that is $R_P := R_F^{-1}$. The duals of F and P are G and H respectively. That is, $G := \neg F \neg$ and $H := \neg P \neg$.

[2]A similar result was obtained independently by Fine [12] and published in the same issue as Thomason [21]. Thomason mentions this fact in a footnote of his article.

The Normal Axioms and proof rules are given for both F and P. For instance we will have as the (K) axioms: $G(p \to q) \to (Gp \to Gq)$ and $H(p \to q) \to (Hp \to Hq)$. Similarly for Dual and Generalization rule.

The smallest normal modal logic in this language, corresponding to the frame where R_F and R_P are converses of each other (as we defined them) is the smallest normal modal logic augmented with axioms $p \to GPp$ and $p \to HFp$. This logic is named $\mathbf{K}_t$, and can be shown to be complete with respect to the class of frames with two converse binary relations (cf. [8, Section 4.3]).

We will show that there is a logic, namely $\mathbf{K}_t\mathbf{ThoM}$ that can be proven to be consistent, but incomplete, ie. consistent but without any non-empty class of frames validating its axioms.

We introduce the logic $\mathbf{K}_t\mathbf{Tho}$ which is the logic $\mathbf{K}_t$ extended with the following three axioms:

$$
\begin{array}{ll}
(.3_F) & Fp \wedge Fq \to (F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)) \\
(D_F) & Gp \to Fp \\
(L_P) & H(Hp \to p) \to Hp
\end{array}
$$

$(.3_F)$ is valid on frames that are non-branching to the right. So if a frame $\mathfrak{F}$ validates $(.3_F)$, then $\mathfrak{F}$ is non-branching to the right. We show this by contraposition: if $\mathfrak{F}$ is branching to the right, then $(.3_F)$ is not valid. Thus for every $\mathfrak{F}$ branching to the right, we need to find a model over $\mathfrak{F}$ such that the negation of $(.3_F)$ is satisfied. Pick an arbitrary $\mathfrak{F}$. As it branches to the right there are distinct w, v_1, y_1 such that $\{(w, v_1), (w, y_1)\} \subseteq R_F$. Then we choose V such that $V(p) = \{v_1\}$ and $V(q) = \{y_1\}$.

Such a model satisfies $\neg(Fp \wedge Fq \to (F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)))$

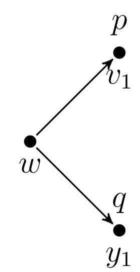


Figure 2.1: $\{(w, v_1), (w, y_1)\} \subseteq R_F$ under V

at w.

$$w \Vdash \neg(Fp \wedge Fq \to (F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)))$$
$$w \Vdash \neg(\neg(Fp \wedge Fq) \vee (F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)))$$
$$w \Vdash (Fp \wedge Fq) \wedge \neg(F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q))$$

The first part $Fp \wedge Fq$ of the conjunction is true at w because wRv_1 and p is satisfied at v_1, and wRy_1 with q satisfied at y_1. The second part of the conjunction is true because:

- it's not the case that there is a point seen by w such that p is true there and that point sees a point where q is true. w sees v_1 and p is true there, but v_1 does not see any point where q is true.

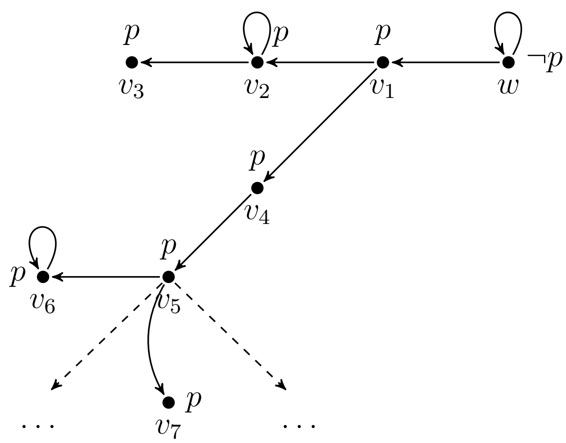
- it's not the case either that there is a point seen by w where both p and q are true, since $v_1 \neq y_1$.

- lastly, it's neither the case that there is a point seen by w where q is true such that it sees a point where p is true.

We show that (D_F) is valid on right-unbounded frames. We pick an arbitrary model and point. We have $w \Vdash Gp \to Fp$. Thus $w \Vdash \neg Gp \vee Fp$, then $w \Vdash F\neg p \vee Fp$. This means that at every state, there is R_F-successor state. Thus any frame validating (D_F) must be right-unbounded.

(L_P) is the Gödel-Löb axiom for P. Forgetting our particular frame for an instant, it can be shown that the Gödel-Löb axiom is one that guarantees converse well-foundedness (see [8, Example 3.9]). In other words, given a relation R, it prevents infinite R-paths. Thus it prevents infinite ascending chains, loops, and reflexivity. Coming back to our case, what we are interested in, for reasons that will become obvious later, is to prevent our frame from having reflexive points. But we also want our frame to have right-unboundedness, which is why we included axiom (D_F). Thus the axioms (D) and (L) in the basic modal similarity type, respectively $\Box p \to \Diamond p$ and $\Box(\Box p \to p) \to \Box p)$ can't be valid on the same frame. But since what we are interested in is to force irreflexivity, we choose Gödel-Löb for the converse of the relation R (or R_F as we called it). Luckily, we can easily pick modalities and frames that can help us with this. As we said before, such a frame can be defined with two modalities, having as corresponding relations, two relations, converses of each other. Hence, if we give the Gödel-Löb axiom for the modality P, we have converse well-foundedness for R_P and not R_F, thus we can have (D_F), and right-unboundedness for R_F. Since R_P and R_F are converses of each other, (L_P) actually forces well-foundedness for R_F, which is fine and is not incompatible with right-unboundedness. However, since what we were interested in is the irreflexivity property that the Gödel-Löb axiom provides, and that irreflexivity defined for R_P still holds for R_F (and vice versa), then adding (L_P) is satisfactory for our purpose. We also note that (L_P) forces R_P to be transitive. However this is not crucial for us.

We show that (L_P) implies irreflexivity on our frame. We show by contraposition that if a frame $\mathfrak{F}$ is not irreflexive, then $\mathfrak{F} \nVdash H(Hp \to p) \to Hp$. For every non-irreflexive frame $\mathfrak{F}$ we need to find a model where some point satisfies $\neg(H(Hp \to p) \to Hp)$. Pick an arbitrary $\mathfrak{F}$. Since the frame is non-irreflexive it must contain a point w that is reflexive. We choose V such that $V(p) = W - \{w\}$. We show that $\neg(H(Hp \to p) \to Hp)$ is satisfied in w. Since $w \Vdash \neg(H(Hp \to p) \to Hp) \Leftrightarrow w \Vdash H(Hp \to p) \wedge \neg Hp$, we must show that (i) $w \Vdash \neg Hp$ and (ii) $w \Vdash H(Hp \to p)$. (i) holds since w sees itself where $\neg p$ is true, hence it cannot be the case that p is true in all the points related to w. (ii) $w \Vdash H(Hp \to p)$ implies that in all points v_i seen by w, we have $v_i \Vdash Hp \to p$. Since this last statement is equivalent to $v_i \Vdash \neg Hp \vee p$, all that remains to be shown is that for all v_i, either $v_i \Vdash \neg Hp$ or $v_i \Vdash p$. If $v_i \neq w$ then by definition of $V(p)$ we have $v_i \Vdash p$ so it holds. If $v_i = w$ then $\neg Hp$ holds as we've seen in (i). Hence if $\mathfrak{F}$ is not irreflexive then $\mathfrak{F} \nVdash H(Hp \to p) \to Hp$. Thus (L_P) implies irreflexivity.



We show that $\mathbf{K}_t\mathbf{Tho}$ is consistent by giving a frame that validates it. $(\mathbb{N}, <, >)$ is such a frame, where $\mathbb{N}$ is the natural numbers and $<$ the lesser than relation, and $>$ the greater than relation. R_F needs to be non-branching to the right $(.3_F)$, right-unbounded (D_F), and irreflexive, well-founded and transitive (because of (L_P)). We also need R_P to be R_F's converse. Since $<$ and $>$ are each other's converses and $<$ has the properties of R_F, we can understand R_F as $<$ and R_P as $>$.

We check that the axioms are valid on this frame. The K axioms are valid on any frame, but we check nonetheless. For any $n \in \mathbb{N}$ we should have $n \Vdash H(p \to q) \to (Hp \to Hq)$. If $n \Vdash H(p \to q)$, then $\forall m < n, m \Vdash p \to q$.

Thus if $\forall m < n, m \Vdash p$ then $m \Vdash q$. But then if $n \Vdash Hp$, we have $\forall m < n$, $m \Vdash p$, thus for all $m < n, m \Vdash q$, so we get $n \Vdash Hq$. The argument is similar for $G(p \to q) \to (Gp \to Gq)$.

We check the validity of axioms $p \to GPp$ and $p \to HFp$. Interpreted on our frame, the first axiom says that if p is true at n, then for every m larger than n, there exists a l smaller than m where p is true. This is obviously always true, for instance if $n = l$. The argument for the second axiom is similar.

For axiom $(.3_F)$, we need to check that for any $n \in \mathbb{N}$, if $n \Vdash Fp \wedge Fq$ then $n \Vdash F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)$. If $n \Vdash Fp \wedge Fq$ then $\exists m > n$ such that $m \Vdash p$ and $\exists l > n$ such that $l \Vdash q$. But then this makes $n \Vdash F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)$ true because it's the case that either $n < m < l$ and then $F(p \wedge Fq)$ is true, or $n < m = l$ in which case $F(p \wedge q)$ is true, or $n < l < m$ and then $F(Fp \wedge q)$ is true.

The validity of (D_F) is very easy to check. For any n, we have to check that if $n \Vdash Gp$ then $n \Vdash Fp$. Assuming that $n \Vdash Gp$ is the case, this means that $\forall m > n, m \Vdash p$. Since we can always find a $l > n$, then there exists a $l > n$ such that $l \Vdash p$, and thus, $n \Vdash Fp$ is true.

Finally, we check (L_P). We have to show that for any $n \in \mathbb{N}$, if $n \Vdash H(Hp \to p)$ is true, then $n \Vdash Hp$ should be true, that is, $\forall m < n, m \Vdash p$. We assume $n \Vdash H(Hp \to p)$. This is true if $\forall m < n, m \Vdash Hp \to p$. This in turn is true if for any $l < m$, $l \Vdash p$. If there are some $l < m$, then each l satisfies p. But if this is the case then $m \Vdash Hp$ is true, and hence $m \Vdash p$. If there is no such $l < m$, then $m \Vdash Hp$ is vacuously true, hence we also have $m \Vdash p$. Thus $\forall m < n, m \Vdash p$ and hence $n \Vdash Hp$.

We've shown that all the axioms of $\mathbf{K}_t\mathbf{Tho}$ are valid on $(\mathbb{N}, <, >)$. Thus any model based on this frame satisfies these axioms, and $\mathbf{K}_t\mathbf{Tho}$ is consistent.

Lemma 2.21. *No frame validates* $\boldsymbol{K_t}$ ***ThoM****.*

Proof. $\mathbf{K}_t$**ThoM** is the logic $\mathbf{K}_t$**Tho** with an additional axiom (M), the McKinsey axiom, given in the basic temporal language: $GFp \to FGp$.

Any frame validating $\mathbf{K}_t$**ThoM** should also validate $\mathbf{K}_t$**Tho**, since a frame validating $\mathbf{K}_t$**ThoM** should validate the axioms of $\mathbf{K}_t$**Tho** and (M).

Towards a contradiction we suppose that there exists a frame $\mathfrak{F}$ such that $\mathfrak{F} \Vdash \mathbf{K}_t\mathbf{ThoM}$. If this is the case then we should also have $\mathfrak{F} \Vdash \mathbf{K}_t\mathbf{Tho}$. We show that we can find a valuation such that $\mathfrak{F}, V \Vdash \neg(M)$, ie. we can find a model $(\mathfrak{F}, V)$ satisfies the negation of the McKinsey axiom : $\mathfrak{F}, V \Vdash \neg(GFp \to FGp)$, which would contradict our assumption that there exists such a $\mathfrak{F}$.

Since $\mathfrak{F} \Vdash \mathbf{K}_t\mathbf{Tho}$, we know that we have two accessbility relations converses of each other $<$ and $>$. We also know that our frame $\mathfrak{F}$ is right-unbounded, non-branching to the right, well-founded, transitive and irreflexive. We let $\mathfrak{F} = (W, <, >)$. We pick any $w \in W$, and we let $X = \{x \in W : w < x\}$. By the mentioned properties of $<$, this makes $(X, <)$ a right-unbounded strict total order. We pick a $S \subseteq X$ such that S and $X - S$ are cofinal in X. Such a subset S exists. Thus $\forall x \in X, \exists s \in S(x < s)$ and $\forall x \in X, \exists t \in X - S(x < t)$. We choose a V such that $V(p) = S$. We show that under this valuation, we can satisfy the negation of the McKinsey formula, ie. that $(\mathfrak{F}, V), w \Vdash \neg(GFp \to FGp)$.

We show that if $(\mathfrak{F}, V), w \Vdash GFp$, then $(\mathfrak{F}, V), w \Vdash \neg FGp$. We assume that $(\mathfrak{F}, V), w \Vdash GFp$. Then, $\forall v > w$ we have $(\mathfrak{F}, V), v \Vdash Fp$. Thus $\forall v > w$, $\exists v' > v, v' \Vdash p$. Since X is the set of all x such that $w < x$, then if GFp is satisfied at w, we have $\forall x \in X, x \Vdash Fp$. But since S is cofinal in X, by cofinality, for any x, we can always find a $s \in S$ such that $x < s$. So $\forall x > w, \exists s > x$, and since $V(p) = S$, then $\forall x > w, \exists s > x, s \Vdash p$. Thus $(\mathfrak{F}, V), w \Vdash GFp$ is true.

We now show that $(\mathfrak{F}, V), w \Vdash \neg FGp$. If that's the case then $w \Vdash G\neg Gp$, and $w \Vdash GF\neg p$. This means that we have to show that $\forall x \in X, x \Vdash F\neg p$, that is we have to show that for each x, $\exists x' > x, x' \Vdash \neg p$. By the cofinality of $X - S$, there is such an x', and since any such $x' \notin S = V(p)$, then $x' \Vdash \neg p$.

We've shown that $(\mathfrak{F}, V), w \nVdash GFp \to FGp$, ie. we can satisfy the negation of (M) in $\mathfrak{F}$. Thus $\mathfrak{F} \nVdash GFp \to FGp$. But our assumption was that $\mathfrak{F}$ validates $\mathbf{K}_t\mathbf{ThoM}$ (hence should validate (M) since it is one of the axioms), thus we reached a contradiction. No frame validating $\mathbf{K}_t\mathbf{Tho}$ can validate $\mathbf{K}_t\mathbf{ThoM}$. Thus, no frame validates $\mathbf{K}_t\mathbf{ThoM}$. □

Lemma 2.22. ***K**$_t$**ThoM*** *is consistent.*

Proof. We let $(\mathbb{N}, <, >)$ be the same structure we mentioned previously. To it, we add a set $A = \{X \subseteq \mathbb{N} : X \text{ is finite or } \mathbb{N} - X \text{ is finite}\}$, ($A$ is the set of subsets X of $\mathbb{N}$ such that X is finite or X is cofinite), and we restrict valuations to this set, ie. $\forall p \in \Phi, V(p) \in A$. We show that $(\mathbb{N}, <, >, A)$[3]. validates $\mathbf{K}_t\mathbf{ThoM}$. We've already shown that $(\mathbb{N}, <, >)$ validates the axioms of $\mathbf{K}_t\mathbf{Tho}$, so these must be true on $(\mathbb{N}, <, >, A)$. It remains to show that the McKinsey axiom $GFp \to FGp$ is valid on this structure.

We show that for any $n \in \mathbb{N}$ if $(\mathbb{N}, <, >, A), V, n \Vdash GFp$ then $n \Vdash FGp$. We pick any $n \in \mathbb{N}$ and assume $n \Vdash GFp$. This means that $\forall m > n, m \Vdash Fp$ and $\forall m > n, \exists k > m$ such that $k \Vdash p$. For any such k we have $k > n$, by the transitivity of $<$ since $n < m < k$. Thus we have $k \Vdash Fp$. But then $\exists k' > k$, $k' \Vdash p$. Since such a $k' > n$ the previous argument can be repeated. $(\mathbb{N}, <, >)$ is right-unbounded, so the argument can be repeated infinitely many times. This means that p is true in infinitely many points of the model, thus $V(p)$ can't be finite, and since we restricted V to A, then $V(p) \in A$ thus if $V(p)$ isn't finite, it must be cofinite. We now have to show that $n \Vdash FGp$. This holds if $\exists n' > n, n' \Vdash Gp$, that is, if $\forall l > n', l \Vdash p$. By the cofiniteness of

[3] This structure is a generalized frame. For more details about generalized frames see [8], eg. Def. 1.32 and Section 5.5

$V(p)$ there is a n' such that p is satisfied for any $l > n'$.

We show that A is closed under the boolean and modal operations, that is, if V is an evaluation such that $\forall p \in \Phi,\ V(p) \in A$, then $\forall \varphi\ \tilde{V}(\varphi) \in A$. The boolean cases are easy. When φ is atomic, this is by definition $A \ni \tilde{V}(p) = \mathbb{N} - \tilde{V}(\neg p) \in A$. For $\neg\varphi$ this is similar since we have $\tilde{V}(\neg\varphi) = \mathbb{N} - \tilde{V}(\varphi)$. For $\vee$, by definition $\tilde{V}(\varphi \vee \psi) = \tilde{V}(\varphi) \vee \tilde{V}(\psi)$, and we have $\tilde{V}(\neg(\varphi \vee \psi)) = \tilde{V}(\neg\varphi \wedge \neg\psi) = \tilde{V}(\neg\varphi) \cap \tilde{V}(\neg\psi) = (\mathbb{N} - \tilde{V}(\varphi)) \cap (\mathbb{N} - \tilde{V}(\psi)) = \mathbb{N} - (\tilde{V}(\varphi) \cup \tilde{V}(\psi))$. The modal case is more interesting. For any $\Diamond \in \{F, P\}$ we have to show that if $\tilde{V}(\varphi) \in A$ then $\tilde{V}(\Diamond\varphi) \in A$. It is easy to see that if $\tilde{V}(\varphi)$ is finite then $\tilde{V}(F\varphi)$ is cofinite and $\tilde{V}(P\varphi)$ is finite, so they are both in A as required. If $\tilde{V}(\varphi)$ is cofinite then $\tilde{V}(F\varphi)$ is finite and $\tilde{V}(P\varphi)$ is cofinite, so they are also both in A.

As A is closed under the boolean and modal operations (in the above sense), $(\mathbb{N}, <, >, A)$ is a general frame, and we've seen that any model based on $(\mathbb{N}, <, >, A)$ satisfies $\mathbf{K}_t\mathbf{ThoM}$, thus $\mathbf{K}_t\mathbf{ThoM}$ is consistent (cf. [8, Thm 4.49]). $\square$

Theorem 2.23. *$\boldsymbol{K}_t\boldsymbol{ThoM}$ is incomplete with respect to any non-empty class of frames.*

Proof. By Lemma 2.22 and Lemma 2.21 we have seen that no class of frames validates the normal modal logic $\mathbf{K}_t\mathbf{ThoM}$, and yet it is not inconsistent. Thus, $\mathbf{K}_t\mathbf{ThoM}$ is incomplete (both strongly and weakly). $\square$

We remark that even though this might seem contradictory with the statement made in the Canonical Model Theorem (Theorem 2.20), this is not the case. There is no contradiction simply because we proved that we can't build a (Kripke) frame for $\mathbf{K}_t\mathbf{ThoM}$ (Theorem 2.23), but $\mathbf{K}_t\mathbf{ThoM}$ is satisfied on certain general frames.

Corollary 2.24. *Not every normal modal logic is complete with respect to a class of frames.*

The incompleteness of $\mathbf{K}_t\mathbf{ThoM}$ with respect to any non empty class of frames is not trivial. Any logic extending it would also be incomplete. Therefore this result is not only stating the incompleteness of a single logic with respect to frames, but states that there exists a large class of normal modal logics which are incomplete with respect to non-empty classes of frames.

As we have seen in our discussion of $\mathbf{K}_t\mathbf{Tho}$ and $\mathbf{K}_t\mathbf{ThoM}$, it is not just the axioms, but some specific properties (of relations on a set) defined by the axioms that lead to this result.

By considering modal logic algebraically, we will be able to show more general completeness results, for instance with algebraic semantics we are able to obtain a completeness result for every normal modal logic.

Chapter 3

Algebraic modal logic

In order to view modal logic algebraically, we *algebraize* modal logic. A universal method for such algebraization for arbitrary logics can be found in [1], [2]. For the first-order case see [15]. For the algebraization of modal logic we refer to [10].

3.1 Modal languages viewed algebraically

We start by showing how any modal language $ML(\tau, \Phi)$ can be algebraized.

Definition 3.1 (Modal algebraic similarity type). Given a modal similarity type τ, the modal algebraic similarity type $\mathcal{S}_\tau$ is the set of symbols $\{\neg, \vee, \bot, \triangle\}_{\triangle\in\tau}$, each with a fixed arity: $\neg$ is unary, $\vee$ is binary, $\bot$ is 0-ary (constant) and each $\triangle \in \tau$ of arity $ar(\triangle) \in \mathbb{N}$.

Notation 3.2 (Terms). The algebraic $\mathcal{S}_\tau$-terms over a set of variables Φ is denoted $Ter_\tau(\Phi)$.

Hence, the set of $\mathcal{S}_\tau$-terms is the same as the set of formulas Fml_τ of a modal language: $Ter_\tau(\Phi) = Fml_\tau$. This is so because we want to consider modal formulas as terms of algebras.

To each modal similarity type τ corresponds a class of boolean algebras with an additional operator for each $\triangle \in \tau$. These algebras are known as *Boolean Algebras with Operators* (or BAO). Given a particular similarity type τ, we call such algebras: Boolean Algebras with τ-Operators (or BAτO).

Definition 3.3 (Boolean Algebras with Operators). Give a similarity type τ, a Boolean Algebra with τ-Operators is an algebra $\mathfrak{A}$ such that:

$$\mathfrak{A} = (A, +, -, 0, f_\triangle)_{\triangle \in \tau}$$

where $(A, +, -, 0)$ is a Boolean Algebra and each $f_\triangle$ is an $ar(\triangle)$-ary operation satisfying *normality* and *additivity*.

Normality. f is normal if (for all i such that $0 < i \leq ar(\triangle)$):

$$f_\triangle(a_1, ..., a_{ar(\triangle)}) = 0 \text{ whenever } a_i = 0$$

Additivity. f is additive if (for all i such that $0 < i \leq ar(\triangle)$):

$$f_\triangle(a_1, ..., a_i + a_i', ..., a_{ar(\triangle)}) = f_\triangle(a_1, ..., a_i, ...a_{ar(\triangle)}) + f_\triangle(a_1, ..., a_i', ..., a_{ar(\triangle)})$$

Definition 3.4 (Operators). Given a Boolean Algebra $\mathfrak{A}$, an operation $f : A^n \to A$ is called an *operator* if it satifies normality and additivity.

Example 3.5 (BAO for the basic modal similarity type). BAO for the basic similarity type is an algebra $\mathfrak{A} = (A, +, -, 0, f_\diamond)$ such that $(A, +, -, 0)$ is a Boolean Algebra, and $f_\diamond$ is unary and satisfies normality and addivity. Hence:

$$f_\diamond(0) = 0$$
$$f_\diamond(a + b) = f_\diamond a + f_\diamond b$$

We denote a BAO for the basic modal similarity type by BAO$_\diamond$

Remark 3.6. Normal τ-languages can be interpreted in Boolean Algebras with τ-Operators. This can be easily noticed by looking at the axiomatization of $\mathbf{K}$, different than the one we gave but nonetheless equivalent, given by the two axioms $\vdash_{\mathbf{K}} \Diamond\bot \leftrightarrow \bot$ and $\vdash_{\mathbf{K}} \Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ and the rule that if $\vdash_{\mathbf{K}} \varphi \to \psi$ then $\vdash_{\mathbf{K}} \Diamond\varphi \to \Diamond\psi$, which gives us a glimpse about the normal and additive behaviour of $\Diamond$. As we will see, this symmetry between boolean operators and modal operators is quite important. We note that this doesn't only hold for unary modalities or operators, however we gave it as an example to illustrates an important idea that we will discuss thoroughly.

3.2 Algebraic Semantics

We introduce Power Set Algebras, and then expand them to obtain Complex Algebras.

Definition 3.7 (Power Set Algebras and Set Algebras). A Power Set Algebra is the structure

$$\mathfrak{P}(A) = (\wp(A), \cup, -, \emptyset)$$

such that $\cup$ denotes the usual operation of union between two sets. $-$ is unary and denotes the operation of complementation relative to A. $\emptyset$ is the empty set. From those we can define the intersection operation $\cap$ between sets, and the distinguished element A.

A *Set Algebra* is a subalgebra of a Power Set Algebra. The universe of a Set Algebra is a subset S of $\wp(A)$ such that $\emptyset \in S$, if $x, y \in S$ then $x \cup y \in S$, if $x \in S$ then $A - x \in S$.

The theory of algebraic propositional logic is very rich and the literature abundant. We refer to [19], [11].

We introduce Full Complex Algebras and Complex Algebras, the modal equivalent of Power Set Algebras and Set Algebras. They will let us interpret the propositional operations similarly to Set Algebras, but provide us with the possibility of adding operations to interpret modalities.

Definition 3.8 (Full Complex Algebras and Complex Algebras). Assuming that $\mathfrak{F} = (W, R_\triangle)_{\triangle \in \tau}$ is a frame, then the Full Complex Algebra $\mathfrak{F}^+$ for this frame is a Power Set Algebra $\mathfrak{P}(W)$ with an additional operations $m_{R_\triangle}$ for each $\triangle \in \tau$:

$$\mathfrak{F}^+ := (\wp(W), \cup, -, \emptyset, m_{R_\triangle})_{\triangle \in \tau}$$

Assuming an $(n+1)$-ary relation R on a set W, we define the n-ary map $m_R : \wp(W)^n \longrightarrow \wp(W)$ such that

$$m_R(X_1, ..., X_n) = \{w \in W : Rwv_1...v_n \text{ for some } v_1 \in X_1, ..., v_n \in X_n\}$$

A Complex Algebra is a subalgebra of a Full Complex Algebra. Complex Algebras are Set Algebras with the $m_{R_\triangle}$ operations added to them. If F is a class of frames, the class of Full Complex Algebras of frames in F is denoted by $\mathbf{Cm}\mathsf{F}$.

What does $m_{R_\triangle}$ correspond to? It is the set of points that can see at least a point v_i in each subset X_i, for i such that $0 < i \leq n$. This is exactly how we defined $\tilde{V}(\triangle(\varphi_1, ..., \varphi_n))$, if we put X_i for $\tilde{V}(\varphi_i)$.

Theorem 3.9. *The Full Complex Algebra $\mathfrak{F}^+$ for a τ-frame $\mathfrak{F} = (W, R_\triangle)_{\triangle \in \tau}$ is a Boolean Algebra with τ-operators.*

Proof. We have to show that the operations $m_{R_\triangle}$ for each $\triangle \in \tau$ are normal and additive.

For normality we have to show that $m_{R_\triangle}(X_1, ..., X_n) = \emptyset$ if any $X_i = \emptyset$ for i such that $0 < i \leq n$. If any $X_i = \emptyset$, then there is no v_i in X_i such that $Rwv_1...v_i...v_n$, thus $m_{R_\triangle}(X_1, ..., X_n)$ is empty.

For additivity, we have to show that $m_{R_\triangle}(X_1, ..., X_i \cup X_i', ..., X_n) = m_{R_\triangle}(X_1, ..., X_i, ..., X_n) \cup m_{R_\triangle}(X_1, ..., X_i', ...X_n)$ which is straightforward by definition. $\square$

Definition 3.10 (Assignment and meaning functions). Given a Boolean Algebra with τ-operators $\mathfrak{A} = (A, +, -, 0, f_\triangle)_{\triangle \in \tau}$ and a set of variables Φ, an assignment is a function $\alpha : \Phi \longrightarrow A$. The meaning function is a unique extension $\tilde{\alpha} : Ter_\tau(\Phi) \longrightarrow A$ of any assignment function such that :

$$\begin{aligned}
\tilde{\alpha}(p) &= \alpha(p), \text{ for all } p \in \Phi \\
\tilde{\alpha}(\perp) &= 0 \\
\tilde{\alpha}(\neg t) &= -\tilde{\alpha}(t) \\
\tilde{\alpha}(t \vee s) &= \tilde{\alpha}(t) + \tilde{\alpha}(s) \\
\tilde{\alpha}(\triangle(t_1, ..., t_n)) &= f_\triangle(\tilde{\alpha}(t_1), ..., \tilde{\alpha}(t_n))
\end{aligned}$$

We note that this meaning function is very similar to the function $\tilde{V}$ that we defined in Chapter 1, which assigned to each formula of the language, the set of worlds where it is true.

Given an assignment on variables, we can determine the meaning of any term.

Definition 3.11 (Equations). We denote equations between two algebraic terms t and s as $t \approx s$. An equation $t \approx s$ is true in an algebra $\mathfrak{A}$ if $\forall \alpha$, $\tilde{\alpha}(t) = \tilde{\alpha}(s)$, or in words, if the meanings of t and s are the same under every assignment. If an algebra $\mathfrak{A}$ makes the equation $t \approx s$ true, it is denoted by $\mathfrak{A} \models t \approx s$.

Example 3.12 (Complex Algebras for the basic similarity type). A Complex Algebra for the basic similarity type $\tau = \{\Diamond\}$, is a Complex Algebra with one operator $m_R : \wp(W) \longrightarrow \wp(W)$ such that

$$m_R(X) = \{w \in W : \exists v \in W \text{ such that } Rwv\}$$

Since Complex Algebras are BAO (Theorem 3.9), m_R is normal and additive:

$$m_R(\emptyset) = \emptyset$$
$$m_R(X \cup Y) = m_R(X) \cup m_R(Y)$$

Theorem 3.13. *Given a similarity type τ, a τ-frame $\mathfrak{F}$, formulas $\varphi, \psi \in Fml_\tau$, an arbitrary state w in $\mathfrak{F}$, and an assignment α, we have :*

$$\mathfrak{F}, \alpha, w \Vdash \varphi \quad \text{iff} \quad w \in \tilde{\alpha}(\varphi) \qquad \text{(i)}$$
$$\mathfrak{F} \Vdash \varphi \quad \text{iff} \quad \mathfrak{F}^+ \vDash \varphi \approx \top \qquad \text{(ii)}$$
$$\mathfrak{F} \Vdash \varphi \leftrightarrow \psi \quad \text{iff} \quad \mathfrak{F}^+ \vDash \varphi \approx \psi \qquad \text{(iii)}$$

Proof. (i) We prove by induction on the complexity of φ. Boolean cases are straightforward. If φ is of the form $\triangle(\varphi_1, ..., \varphi_n)$, then $\mathfrak{F}, \alpha, w \Vdash \triangle(\varphi_1, ..., \varphi_n)$ iff there are v_i such that $Rwv_1...v_n$ and $\mathfrak{F}, \alpha, v_i \Vdash \varphi_i$, for all i such that $0 < i \leq n$ (by the satisfaction definition). By the inductive hypothesis this is the case iff there are v_i such that $Rwv_1...v_n$ and $v_i \in \tilde{\alpha}(\varphi_i)$. This means that $w \in m_{R_\triangle}(\tilde{\alpha}(\varphi_1), ..., \tilde{\alpha}(\varphi_n))$ by the definition of $m_{R_\triangle}$. The definition of the meaning function (def. 3.10) states that $m_{R_\triangle}(\tilde{\alpha}(\varphi_1), ..., \tilde{\alpha}(\varphi_n)) = \tilde{\alpha}(\triangle(\varphi_1, ..., \varphi_n))$, thus we have $w \in \tilde{\alpha}(\triangle(\varphi_1, ..., \varphi_n))$.

(ii) and (iii) follow directly from (i) and definition 3.11. □

Theorem 3.14. *Let* F *be a class of τ-frames, and* $\boldsymbol{Cm}\mathsf{F}$ *the class of Full Complex Algebras of frames in* F*.*

$$\mathsf{F} \Vdash \varphi \quad \text{iff} \quad \boldsymbol{Cm}\mathsf{F} \vDash \varphi \approx \top$$
$$\mathsf{F} \Vdash \varphi \leftrightarrow \psi \quad \text{iff} \quad \boldsymbol{Cm}\mathsf{F} \vDash \varphi \approx \psi$$

Proof. By Theorem 3.13. □

What Theorems 3.13 and 3.14 tell us is that $L_F = \{\varphi : \forall \mathfrak{F} \in \mathsf{F}, \mathfrak{F}^+ \vDash \varphi \approx \top\}$. The logic of a class of frames F is the same as the equational theory of the class of algebras $\mathbf{Cm}\mathsf{F}$.

Without going into the details of the theory of equational classes, we still wished to state this theorem because it is nice to notice that to each class of frames corresponds a class of algebras that validate the same formulas.

3.3 Lindenbaum-Tarski Algebras

Definition 3.15. By V_Σ we mean the class of BAτO in which the set $\Sigma^{\approx} = \{\sigma \approx \top : \sigma \in \Sigma\}$ is valid.

Definition 3.16 (Formula Algebra). Let Φ be a set of propositional variables and let τ be a modal similarity type. The formula algebra of τ over Φ is the structure

$$\mathfrak{Form}(\tau, \Phi) = (Fml(\Phi, \tau), +, -, 0, f_\triangle)_{\triangle \in \tau}$$

where the operations are defined like so :

$$0 := \bot$$
$$-\varphi := \neg\varphi$$
$$\varphi + \psi := \varphi \vee \psi$$
$$f_\triangle(\varphi_1, ..., \varphi_n) := \triangle(\varphi_1, ..., \varphi_n)$$

Definition 3.17 (Equivalence relation modulo L). For any normal modal logic L, we define

$$\varphi \equiv_L \psi \quad \text{iff} \quad \vdash_L \varphi \leftrightarrow \psi$$

Theorem 3.18. *$\equiv_L$ is a congruence relation on $\mathfrak{Form}(\tau, \Phi)$.*

Proof. The boolean cases are straightforward. The modal cases that if $\varphi_i \equiv_L \psi_i$ then $\triangle(\varphi_1, ..., \varphi_n) \equiv_L \triangle(\psi_1, ..., \psi_n)$ follows from the fact that from $\vdash_L \varphi_i \leftrightarrow \psi_i$ we can easily deduce $\vdash_L \triangle(\varphi_1, ..., \varphi_n) \leftrightarrow \triangle(\psi_1, ..., \psi_n)$, for all i such that $0 < i \leq n$. $\square$

Definition 3.19 (Lindenbaum-Tarski Algebras). The Lindenbaum-Tarski Algebra of L over Φ is the structure :

$$\mathfrak{L}_L(\Phi) = (Fml(\tau, \Phi)/\equiv_L, +, -, 0, f_\triangle)_{\triangle \in \tau}$$

where $Fml(\tau, \Phi)/\equiv_L$ is the set of equivalence classes under the relation $\equiv_L$ and such that :

$$0 := [\bot]$$
$$[\varphi] + [\psi] := [\varphi \vee \psi]$$
$$-[\varphi] := [\neg\varphi]$$
$$f_\triangle([\varphi_1], ..., [\varphi_n]) := [\triangle(\varphi_1, ..., \varphi_n)]$$

The Lindenbaum-Tarski Algebra of a normal modal logic L is the quotient algebra of the formula algebra over $\equiv_L$.

Theorem 3.20. *For any normal modal τ-logic L, the* Lindenbaum-Tarski Algebra $\mathfrak{L}_L(\Phi)$ *is a Boolean Algebra with τ-Operators.*

Proof. We show that for each $\triangle \in \tau$, the operation $f_\triangle$ satisfies normality and additivity.

Normality. We have to show that $f_\triangle([\varphi_1], ..., [\varphi_n]) = [\bot]$ whenever any $[\varphi_i] = [\bot]$ for i such that $0 < i \leq n$. If we have $f_\triangle([\varphi_i], ..., [\bot], ..., [\varphi_n])$, by definition this is equivalent to $[\triangle(\varphi_1, ..., \bot, ..., \varphi_n)]$. But since $\bot$ can never be satisfied (by the satisfaction definition), then there is no v_i in $Rwv_1...v_i...v_n$ that satisfies it. Hence $\triangle(\varphi_1, ..., \bot, ..., \varphi_n)$ can never be satisfied, which is the definition of $\bot$. Hence $[\triangle(\varphi_1, ..., \bot, ..., \varphi_n)] = [\bot]$.

Additivity. For an n-ary $f_\triangle$ we show that for each i we have

$$f_\triangle([\varphi_1], \ldots, [\varphi_i] + [\varphi_i'], \ldots, [\varphi_n])$$
$$=$$
$$f_\triangle([\varphi_1], \ldots, [\varphi_i], \ldots, [\varphi_n]) + f_\triangle([\varphi_1], \ldots, [\varphi_i'], \ldots, [\varphi_n])$$

To ease notation we write $f_{\triangle}(\cdots,\psi,\cdots)$ to denote $f_{\triangle}(\varphi_1,\ldots,\psi,\ldots,\varphi_n)$.

We have $f_{\triangle}(\cdots,[\varphi_i]+[\varphi_i'],\cdots) = [\triangle(\cdots,\varphi_i\vee\varphi_i',\cdots)]$, by Definition 3.19.

We have

$$\begin{aligned} f_{\triangle}(\cdots,[\varphi_i],\cdots) + f_{\triangle}(\cdots,[\varphi_i'],\cdots) &= [\triangle(\cdots,\varphi_i,\cdots)] + [\triangle(\cdots,\varphi_i',\cdots)] \\ &= [\triangle(\cdots,\varphi_i,\cdots) \vee \triangle(\cdots,\varphi_i',\cdots)] \end{aligned}$$

by Definition 3.19.

Hence we have to show that

$$[\triangle(\cdots,\varphi_i \vee \varphi_i',\cdots)] = [\triangle(\cdots,\varphi_i,\cdots) \vee \triangle(\cdots,\varphi_i',\cdots)]$$

By Definition 3.17, this amounts to show

$$\vdash_L \triangle(\cdots,\varphi_i \vee \varphi_i',\cdots) \leftrightarrow \big(\triangle(\cdots,\varphi_i,\cdots) \vee \triangle(\cdots,\varphi_i',\cdots)\big).$$

But this is a theorem of every normal modal logic, thus it holds. Actually, this formula is often chosen to axiomatize **K** of arbitrary similarity type, in addition to $\triangle(...,\bot,...) \leftrightarrow \bot$. $\square$

Theorem 3.21.

$$\mathfrak{L}_L(\Phi) \models [\varphi] \approx [\top] \quad \text{iff} \quad \vdash_L \varphi$$

Proof. Direct from Definitions 3.17 and 3.19. $\square$

Theorem 3.22. *For any normal modal τ-logic, $\mathfrak{L}_L(\Phi)$ is a member of the class of BAτO which validates L.*

$$\mathfrak{L}_L(\Phi) \in \mathsf{V}_L$$

Proof. Direct from theorem 3.21. $\square$

3.4 The Jónsson-Tarski Theorem

3.4.1 Motivations

So far we've shown two important points :

- We can build a Boolean Algebra with τ Operators from any frame $(W, R_\triangle)_{\triangle\in\tau}$ (Def. 3.8). The obtained algebra is a Complex Algebra, which is a concrete BAO (Theorem 3.9).
- We've seen that for any normal modal logic, we can build an abstract BAO using Lindenbaum-Tarski algebras (Def 3.19, Theorem 3.20).

If we would be able to build a Complex Algebra from any Lindenbaum-Tarski algebra of a normal modal logic, then we would have a general completeness result for any normal modal logic, since we would have corresponding structures for the provability relation (Lindenbaum-Tarski algebras) of a logic, and for the validity relation (Complex Algebras).

This is precisely what the Jónsson-Tarski theorem will provide us with: a representation theorem stating that every Boolean Algebra with Operators is isomorphic to a Complex Algebra.

Thus by theorems 3.13 and 3.21 the algebraic recipe to obtain a completeness result for any normal modal logic is as follows :

(i) Build the Lindenbaum-Tarski algebra of any normal modal logic (which is a BAO).

(ii) Build the *ultrafilter frame* of the Lindenbaum-Tarski algebra.

(iii) Build the Complex Algebra of the obtained ultrafilter frame: the resulting algebra is the *canonical embedding algebra* of the Lindenbaum-Tarski algebra we started with.

We already know how to perform step (i). The Jónsson-Tarski theorem guarantees that step (ii) and (iii) can always be done. Hence, by proving the Jónsson-Tarski theorem we obtain a general completeness result for every normal modal logic.

3.4.2 The Jónsson-Tarski Theorem and its proof

We show the Jónsson-Tarski theorem in three main steps :

(i) How to build the *ultrafilter frame* $(Uf\mathfrak{A}, R_i)_{i\in I}$ of an arbitrary Boolean Algebra with Operators $\mathfrak{A}$:

  (a) Building the underlying set $Uf\mathfrak{A}$ of ultrafilters of $\mathfrak{A}$

  (b) Building the relations R_i on $Uf\mathfrak{A}$

(ii) Showing that every BAO is embeddable in the Full Complex Algebra of its ultrafilter frame.

(i)-(a) Let $\mathfrak{A} = (A, +, -, 0, f_i)_{i\in I}$ be an arbitrary Boolean Algebra with operators. We would like to build the set $Uf\mathfrak{A}$ of ultrafilters of $\mathfrak{A}$. To build such a set, considering the stricly boolean part $(A, +, -, 0)$ of $\mathfrak{A}$ is enough. The operators f_i will be relevant when we need to build the relations on $Uf\mathfrak{A}$.

We recall some basic definitions and properties of ultrafilters. An *ultrafilter* of a Boolean Algebra $(A, +, -, 0)$ is a subset $U \subseteq A$ such that:[1] (i) $1 \in U$, (ii) if $a, b \in U$ then $a \cdot b \in U$, (iii) if $a \in U$ and $a \leq b$ then $b \in U$, (iv) $0 \notin U$, (v) $\forall a \in A$ either $a \in U$ or $-a \in U$. A subset of A is a *filter* if it satisfies conditions (i) to (iii), a *proper filter* if it additionally satisfies

[1] $x \cdot y$ is the standard boolean operation such that $x \cdot y = -(-x + -y)$. Also $1 = -0$, and $a \leq b$ iff $a + b = b$. Let us recall the *monotonicity* laws for boolean arithmetic as well: if $x \leq x'$ and $y \leq y'$ then $x + y \leq x' + y'$ and $x \cdot y \leq x' \cdot y'$ and $-x' \leq -x$. For more boolean identities and arithmetical laws we refer to [20].

(iv), and an *ultrafilter* if it satisfies all of them. We denote the collection of ultrafilters of $\mathfrak{A}$ as $Uf\mathfrak{A}$.

Proposition 3.23 (Ultrafilter Theorem). *If P is a proper filter of a Boolean Algebra $\mathfrak{A}$ such that for an element a of $\mathfrak{A}$, we have $a \notin P$, then there exists an ultrafilter U extending P such that $a \notin U$.*

Proposition 3.24. *If U is an ultrafilter of $\mathfrak{A}$, then $\forall a, b \in A$, $a + b \in U$ iff $a \in U$ or $b \in U$.*

Lemma 3.25. *We can embed the boolean reduct of $\mathfrak{A}$ into the power set of the collection $Uf\mathfrak{A}$ of ultrafilters of $\mathfrak{A}$.*

Proof. We need to find an injective boolean homomorphism from A to $\wp(Uf\mathfrak{A})$. We let $g : A \longrightarrow \wp(Uf\mathfrak{A})$ be such a map :

$$g(a) = \{U \in Uf\mathfrak{A} : a \in U\}$$

We need to check two things: that g is a homomorphism, and that it is injective. We show that g is a homomorphism. We have $g(0) = \emptyset$ since no ultrafilter contains 0 by definition. We have $g(-a) = \{U \in Uf\mathfrak{A} : -a \in U\} = \{U \in Uf\mathfrak{A} : a \notin U\} = -\{U \in Uf\mathfrak{A} : a \in U\} = -g(a)$.

We show that $g(a + b) = g(a) \cup g(b)$.

By definition $g(a + b) = \{U \in Uf\mathfrak{A} : a + b \in U\}$. By Proposition 3.24 we have $\{U \in Uf\mathfrak{A} : a + b \in U\} = \{U \in Uf\mathfrak{A} : a \in U \text{ or } b \in U\}$. Then we get $\{U \in Uf\mathfrak{A} : a \in U\} \cup \{U \in Uf\mathfrak{A} : b \in U\}$, which yields $g(a) \cup g(b)$.

We have to show that f is injective, that is for any elements a, b of A such that $a \neq b$ we show that $g(a) \neq g(b)$. We assume that $a \neq b$, then $a \nleq b$. Since $a \neq b$, by the ultrafilter properties and Theorem 3.23 this means that $\exists U$ such that $a \in U$, but $b \notin U$. Then $U \in g(a)$ but $U \notin g(b)$, thus $g(a) \neq g(b)$. $\square$

(i)-(b) We define relations on the set of ultrafilters $Uf\mathfrak{A}$ to obtain the *ultrafilter frame* of a Boolean Algebra with Operators.

Definition 3.26. Given a similarity type τ and a BAτO $\mathfrak{A} = (A, +, -, 0, f_\triangle)_{\triangle\in\tau}$, we define an $(n+1)$-ary relation R_f on $Uf\mathfrak{A}$ such that

$$R_f uu_1...u_n$$
$$\text{iff}$$
$$f(a_1, ..., a_n) \in u \text{ for all } a_1 \in u_1, ..., a_n \in u_n$$

Lemma 3.27.

$$f(a_1, ..., a_n) \in u \text{ for all } a_1 \in u_1, ..., a_n \in u_n$$
$$\text{iff}$$
$$-f(-a_1, ..., -a_n) \in u \text{ implies } \exists i \text{ such that } a_i \in u_i$$

Proof. $\Downarrow$ direction: Suppose $R_f uu_1 \dots u_n$ and $-f(-a_1, \dots, -a_n) \in u$. We have to show that there is i such that $a_i \in u_i$. For if not, for all i we have $a_i \notin u_i$ which amounts to $-a_i \in u_i$. Then, by $R_f uu_1 \dots u_n$ we obtain $f(-a_1, \dots, -a_n) \in u$ which contradicts to $-f(-a_1, \dots, -a_n) \in u$.

For the other direction, suppose that whenever $-f(-a_1, \dots, -a_n) \in u$ is the case, we have $a_i \in u_i$ for some i. We have to show for that if $a_i \in u_i$ holds for all i, then $f(a_1, \dots, a_n) \in u$. Pick arbitrary elements $a_i \in u_i$ and, by way of contradiction, suppose $f(a_1, \dots, a_n) \notin u$. Then $-f(a_1, \dots, a_n) \in u$, whence by our assumption it follows that $-a_i \in u_i$ for some i. This is a clear contradiction. $\square$

The frame $(Uf\mathfrak{A}, R_{f_\triangle})_{\triangle\in\tau}$ is the *ultrafilter frame* of the BAO $\mathfrak{A}$. The Complex Algebra $(Uf\mathfrak{A}, R_{f_\triangle})^+_{\triangle\in\tau}$ of an ultrafilter frame is called the *canonical embedding algebra* of $\mathfrak{A}$ and we denote it by $\mathfrak{Em}\mathfrak{A}$

(ii) Now that we know how to build the ultrafilter frame of an arbitrary Boolean Algebra with operators, we are ready to prove the Jónsson-Tarski theorem.

In what follows we prove a version of the celebrated Jónsson-Tarski Theorem [17], [18].

Theorem 3.28 (Jónsson-Tarski Theorem). *Every Boolean Algebra with Operators is embeddable in the Full Complex Algebra of its ultrafilter frame. Given a modal similarity type τ and a Boolean Algebra with τ-Operators $\mathfrak{A} = (A, +, -, 0, f_\triangle)_{\triangle \in \tau}$, we can embed $\mathfrak{A}$ into $\mathfrak{Em}\mathfrak{A}$.*

Proof. By Lemma 3.25 the function $g : A \longrightarrow \wp(Uf\mathfrak{A})$

$$g(a) = \{U \in Uf\mathfrak{A} : a \in U\}$$

was shown to be a boolean embedding of a Boolean Algebra into the Power Set Algebra of $Uf\mathfrak{A}$, ie. it is an injective homomorphism that preserves the boolean operations. We keep the same map, and show that g preserves the modal operations. We have to show for all $f_\triangle$ that

$$g(f_\triangle(a_1, ..., a_n)) = m_{R_{f_\triangle}}(g(a_1), ..., g(a_n))$$

We prove by induction on the arity of f. It is good to keep in mind that :

$$u \in g(a) \text{ iff } a \in u$$

Base case: f is unary (this would be a boolean operator corresponding to a diamond modal operator). We have to show that $g(f(a)) = m_{R_f}(g(a))$. We prove that $m_{R_f}(g(a)) \subseteq g(f(a))$. If $u \in m_{R_f}(g(a))$, then $\exists u_1 \in g(a)$ and $R_f u u_1$, by definition of m_{R_f}. This means that $a \in u_1$ and $f(a) \in u$, thus $u \in g(f(a))$, as we wanted.

We check that $g(f(a)) \subseteq m_{R_f}(g(a))$. If $u \in g(f(a))$, then we must show that $u \in m_{R_f}(g(a))$. To do so, we need to find a u_1 with $R_f u u_1$ and $u_1 \in g(a)$. We want to have in u_1 all the elements that should be in there by $R_f u u_1$. Since $u \in g(f(a))$ is equivalent to $f(a) \in u$, by Lemma 3.27 we have that for any element if $-f(-b) \in u$, then $b \in u_1$ (given the relation $R_f uu_1$). This will help us to put in u_1 all the elements that should be there. We can note that what we are doing is actually mimicking a boxed operator (note the similarity $\Diamond\varphi$, $\neg\Box\neg\varphi$, and $f(a)$,$-f(-a)$). So, we define the set $X = \{b \in A : -f(-b) \in u\}$. We need to show that $\exists u_1$ such that[2] $X \cup \{a\} \subseteq u_1$. By the ultrafilter theorem (Theorem 3.23), it is enough to show that $X \cup \{a\}$ is a proper filter. We show that $X \cup \{a\}$ is closed under $\cdot$ the meet operation. By the definition of $\cdot$ and the additivity axiom we get for any $x, y \in X$, $f(x \cdot y) = -f(-x + -y) = -(-f(x) + -f(y)) = f(x) \cdot f(y)$. We can show that $X \cup \{a\}$ is closed under meet by showing that $a \cdot x \neq 0$ for any $x \in X$. Towards a contradiction we assume that there is an $x \in X$ such that $a \cdot x = 0$. This gives us that $a \leq -x$, then $f(a) \leq f(-x)$. But then $f(-x) \in u$, contradicting $x \in X$, since $x \in X$ if $-f(-x) \in u$ by definition. We show that $a \neq 0$. By normality we have $f(0) = 0$. Since $f(a) \in u$, we can't have $a = 0$ otherwise we would have $f(0) = 0 \in u$, and 0 can't be in u. Thus $X \cup \{a\}$ is a proper filter and by the Ultrafilter Theorem we can extend it to an ultrafilter u_1. Hence $R_f uu_1$ holds, since, by definition of X and of $R_f uu_1$ (Lem. 3.27) we have that if $-f(-x) \in u$ then $x \in X \subseteq u_1$.

General case: We prove for arbitrary cases of arbitrary arity n. By the induction hypothesis, we assume that it holds for n. The case for the direction $m_{R_f}(g(a_1), ..., g(a_{n+1})) \subseteq g(f(a_1, ..., a_{n+1}))$ is the same as the base case.

For the other direction we have to check that $g(f(a_1, ..., a_{n+1})) \subseteq m_{R_f}(g(a_1), ..., g(a_{n+1}))$. For $f(a_1, ..., a_{n+1}) \in u$ we have to find $u_1, ..., u_{n+1}$ such that $R_f uu_1 ... u_{n+1}$ and $a_i \in u_i$ for all i such that $0 < i \leq n+1$. We let $f' : A^n \longrightarrow A$ such that $f'(x_1, ..., x_n) = f(x_1, ..., x_n, a_{n+1})$. By the normality and additivity

[2]For the sake of clarity, but forgetting formal rigour, what has been said could be (incorrectly) rephrased as $X = \{b \in A : \Box b \in u\} \subseteq u_1$, which might help understand why this is guaranteed by the relation Ruu_1, if we have in mind the satisfaction definition for the common basic modal language.

of f', and the induction hypothesis we have $f'(a_1, ..., a_n) \in u$, thus we have $u_1, ..., u_n$ such that $a_i \in u_i$ (for $0 < i \leq n$). Thus by the definition of f' we get that $f(x_1, ..., x_n, a_{n+1}) \in u$ whenever $x_i \in u_i$. We need to find a u_{n+1} such that $R_f uu_1...u_{n+1}$ and $a_{n+1} \in u_{n+1}$. Similarly to the argument for the base case we build a set $X \cup \{a_{n+1}\}$ and show that it can be extended to an ultrafilter u_{n+1}. By Lemma 3.27 we have $R_f uu_1...u_{n+1}$ iff $\forall x_i, y$, if $x_i \in u_i$, then $-f(x_1, ..., x_n, -y) \in u$ implies $y \in u_{n+1}$. Then, similarly to the base case we set $X = \{y \in A : \exists x_1 \in u_1, ..., x_n \in u_n(-f(x_1, ..., x_n, -y) \in u)\}$. All that remains is to show that $X \cup \{a_{n+1}\}$ is a proper filter, which by the Ultrafilter theorem 3.23, would let us extend it to the ultrafilter u_{n+1} that we need. Hence, the last step is to show that $X \cup \{a_{n+1}\}$ is closed under $\cdot$ meet. We start by showing that X is closed under meet, ie. that $-f(x'_1x''_1, ..., x'_nx''_n, -(y'y'')) \in u$ (in accordance with the definition of X). We have $f(x'_1x''_1, ..., x'_nx''_n, -y') \leq f(x'_1, ..., x'_n, -y')$. By monotonicity we get $-f(x'_1, ..., x'_n, -y') \leq -f(x'_1x''_1, ..., x'_nx''_n, -y')$. By the upward closedness of u (since it is a filter) and our assumption, then $-f(x'_1, ..., x'_n, -y') \in u$ yields $-f(x'_1x''_1, ..., x'_nx''_n, -y') \in u$. Similary we get $-f(x'_1x''_1, ..., x'_nx''_n, -y'') \in u$. The following equalities hold

$$f(x'_1x''_1, ..., x'_nx''_n, -(y'y'')) = f(x'_1x''_1, ..., x'_nx''_n, -y' + -y'') \tag{3.1}$$

$$= f(x'_1x''_1, ..., x'_nx''_n, -y') + f(x'_1x''_1, ..., x'_nx''_n, -y'') \tag{3.2}$$

where (3.1) holds by definition of meet, and (3.2) holds by additivity. Hence

$$-f(x'_1x''_1, ..., x'_nx''_n, -(y'y''))$$

$$=$$

$$-(f(x'_1x''_1, ..., x'_nx''_n, -y') + f(x'_1x''_1, ..., x'_nx''_n, -y''))$$

$$= \tag{3.3}$$

$$-f(x'_1x''_1, ..., x'_nx''_n, -y') \cdot -f(x'_1x''_1, ..., x'_nx''_n, -y'') \tag{3.4}$$

where the equality (3.3) holds by De Morgan's laws. (3.4) shows that X is closed under meet, hence $-f(x_1'x_1'', ..., x_n'x_n'', -(y'y'')) \in u$ as we wanted.

It is left for us to show that $X \cup \{a_{n+1}\}$ is closed under meet. Similarly to the base case, this means that we have to show that $a_{n+1} \cdot y \neq 0$ for any $y \in X$. Towards a contradiction we assume that there is a $y \in X$ such that $a_{n+1} \cdot y = 0$. Then we get $a_{n+1} \leq -y$, hence by monotonicity $f(x_1'x_1'', ..., x_n'x_n'', a_{n+1}) \leq f(x_1'x_1'', ...x_n'x_n'', -y)$. Then $-f(x_1'x_1'', ..., x_n'x_n'', -y) \leq -f(x_1'x_1'', ..., x_n'x_n'', a_{n+1})$, which yields by definition of X and upward closedness that $-f(x_1'x_1'', ..., x_n'x_n'', a_{n+1}) \in u$. But this means that $-a_{n+1} \in u_{n+1}$ which is not possible, hence we reach a contradiction. $X \cup \{a_{n+1}\}$ is closed under meet, which means that it can be extended to an ultrafilter u_{n+1}. This concludes the proof of the Jónsson-Tarski theorem. $\square$

A short remark is in order here to show that not every Boolean Algebra with Operators is isomorphic to a Full Power Set Algebra. For example, take $B = \{X \subseteq \mathbb{N} : X \text{ is finite or cofinite}\}$ with the usual operations. Define $\Diamond X = \emptyset$ if X is finite, and $\Diamond X = \mathbb{N}$ if X is cofinite. Then $\Diamond$ is normal and additive, therefore we obtain a BAO. To see that it is not isomorphic to a Full Power Set Algebra, observe that in Full Power Set Algebras every subset has a least upper bound which is not the case with B.

Conclusion

We've seen that although intuitive and very useful, Kripke semantics are inherently limited since they are unable to provide positive completeness results for a large family of normal modal logics. However, as we have seen, this limitation can be healed with the help of algebraic tools. The Jónsson-Tarski Theorem plays the key role in this respect. With it, we have a completeness result for any normal modal logic of arbitrary similarity type. As we've seen, algebraic logic provides us with tools to build the Lindenbaum-Tarski algebra of any normal modal logic. As it turns out, the Lindenbaum-Tarski algebra of a normal modal logic is a Boolean Algebra with operators. What the Jónsson-Tarski Theorem states is that any BAO is isomorphic to a Complex Algebra. In this sense, it is a generalization of Stone's Theorem. What it does more precisely for us is that it shows us how to build a specific frame (the *ultrafilter frame*) from the starting Lindenbaum-Tarski algebra. Then we can turn this ultrafilter frame into a Complex Algebra, isomorphic to the starting Lindenbaum-Tarski algebra. The required syntactic conditions are guaranteed by the starting Lindenbaum-Tarksi algebra, and the structure that lets us interpret the logic semantically, is the corresponding Complex Algebra (or canonical embedding algebra). The isomorphism between the two structures is what guarantees positive completeness and soundness results.

Bibliography

[1] Andréka, H., Németi, I. and Sain, I., Universal algebraic logic. In preparation, 2011.

[2] Andréka, H., Németi, I. and Sain, I., Algebraic Logic. In: Handbook of Philosophical Logic Vol.II, 2nd Edition. Editors: D. M. Gabbay and F. Guenthner. Kluwer Academic Publishers, 2001.

[3] Beal, J.C., Frassen, van B.C., Possibilities and Paradox: An Introduction to modal and many-valued logic. Oxford University Press, 2003.

[4] Benthem, van J., Two simple incomplete modal logics. Theoria. 44:25–37, 1978.

[5] Benthem, van J., Canonical modal logics and ultrafilter extensions. Journal of Symbolic Logic, 44:1–8, 1980.

[6] Boolos, G., The logic of provability. Cambridge University Press, 1995.

[7] Burris, S., Sankappanavar, H.P., A course in Universal Algebra. Graduate Texts in Mathematics. Springer, 1981.

[8] Blackburn, P., Rijke, M. de., Venema, Y., Modal Logic. Cambridge tracts in theoretical computer science 53, 2001.

[9] Blackburn, P., Benthem, van J., Wolter, F., Handbook of Modal Logic. Studies in Logic and Practical Reasoning, Elsevier, 2006.

[10] Chagrov, A., Zakharyaschev, M., Modal Logic, volume 35 of Oxford Logic Guides. Oxford University Press, 1997.

[11] Cori, R., Lascar, D., Mathematical Logic: Part 1. Oxford University Press, 2000.

[12] Fine, K., An incomplete logic containing S4. Theoria 40:23–29, 1974.

[13] Gabbay, D., Horty, J., Parent, X., et al. (eds.) Handbook of Deontic Logic and Normative Systems, London: College Publications, 2013.

[14] Halmos, P.R., Algebraic Logic. Chelsea Publishing Company, 1962.

[15] Henkin, L., Monk, J.D., Tarski, A., Cylindric Algebras. Parts I–II. North-Holland, Amsterdam, 1985.

[16] Jónsson, B., A survey of boolean algebras with operators. In Algebras and Orders, pp. 239–286. Kluwer, 1993.

[17] Jónsson, B., Tarski, A., Boolean algebras with operators, I. Amer. J. Math., 73, pp. 891939, 1951.

[18] Jónsson, B., Tarski, A., Boolean algebras with operators, II. Amer. J. Math., 73, pp. 891939, 1951.

[19] Monk, J.D., Mathematical logic. Graduate Texts in Mathematics, Volume 37, 1976.

[20] Monk, J.D., Koppelberg, S., Bonnet, R., (eds). Handbook of Boolean Algebras, Vol 1–3. North-Holland, 1989.

[21] Thomason, An incompleteness theorem in modal logic. Theoria, 40:150–158, 1974.