Axiomatizing Minkowski Spacetime in First-Order Temporal Logic

Preliminary draft of a chapter of the PhD dissertation

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Contents

1 Classical language and models

1.1 Logical abbreviations

Notation 1 (Vector-notation, projections)**.** If $\vec{x} = \langle x_1, \dots, x_n \rangle$, then we denote the ith member of $\vec{x}$ by $\vec{x}_i$ or $(\vec{x})_i$.

If f is a function with a codomain of some set of n-tuples, then for any $1 \leq k \leq n$,

$$f_k(\vec{x}) \stackrel{\text{def}}{=} (f(\vec{x}))_k$$

We will use the following abbreviations as well: If $i \leq j \leq n$, then for any n-tuple $\vec{x}$,

$$f_{i-j}(\vec{x}) \stackrel{\text{def}}{=} \langle v_i(\vec{x}), v_{i+1}(\vec{x}), \dots, v_j(\vec{x}) \rangle$$

$$f_{i_1,i_2,\dots,i_n}(\vec{x}) \stackrel{\text{def}}{=} \langle v_{i_1}(\vec{x}), v_{i_2}(\vec{x}), \dots, v_{i_n}(\vec{x}) \rangle$$

We also use the vector-notation in syntax; if P is an n-ary predicate then

$$P(\langle x_1, \dots, x_n \rangle) \stackrel{\text{def}}{=} P(x_1, \dots, x_n)$$

Notation 2 (Bounded quantifications)**.** We use the $\in$ symbol and binary relations to bound quantification:

$$\begin{aligned}
(\forall v_1, v_2, \dots, v_n \in \varphi)\psi &\stackrel{\text{def}}{\Leftrightarrow} \forall v_1, \dots, v_n((\varphi(v_1) \wedge \varphi(v_2) \wedge \dots \wedge \varphi(v_n)) \to \psi) \\
(\forall \langle v_1, v_2, \dots, v_n \rangle \in \varphi)\psi &\stackrel{\text{def}}{\Leftrightarrow} \forall v_1, \dots, v_n(\varphi(v_1, v_2, \dots, v_n) \to \psi) \\
(\forall v_2 \varphi v_1)\psi &\stackrel{\text{def}}{\Leftrightarrow} \forall v_2(\varphi(v_1, v_2) \to \psi)
\end{aligned}$$

In Chapter 2, we will frequently define functions in the object language, but most of the time these functions will be partial. The following notational conventions will make the life easier there.

Notation 3 (Functions, partial functions)**.** Let v an arbitrary variable, and $\vec{v}$ is an n-tuple of arbitrary variables. A formula $F(\vec{v}, v')$ is a *function* in the system Γ, iff

$$\Gamma \vdash \exists y(F(\vec{v}, v_1) \wedge \forall z(F(\vec{v}, v_2) \to v_1 = v_2)),$$

We call $\mathrm{F}(\vec{w}, \vec{a}, \vec{x}, y)$ a *partial function* in Γ, if

$$\Gamma \vdash \forall y, z(\mathrm{F}(\vec{w}, \vec{a}, \vec{x}, y) \wedge \mathrm{F}(\vec{w}, \vec{a}, \vec{x}, z) \to y = z).$$

We refer to the only v' which satisfy $\varphi(\vec{v}, v')$ with the lower case, one-argumentless $f(\vec{v})$. Formally:

$$\varphi(f(\vec{v})) \stackrel{\text{def}}{\Leftrightarrow} \exists y(F(\vec{v}, v') \wedge \varphi(v'))$$

So if $F(\vec{w}, \vec{a}, \vec{x}, y)$ is only a partial function, then the truth of $\varphi(\mathrm{f}(\vec{w}, \vec{a}, \vec{x}))$ implies that $f(\vec{w}, \vec{a}, \vec{x})$ *is defined*, and has the property φ. Roughly speaking, using this notation, we will never have to excuse ourselves using partial functions.

1.2 Classical language of clocks

We are going to use the following classical first-order 3-sorted language:

- Symbols:
  - Pointer variables: $a, b, c, \ldots$ $\qquad ClVar \stackrel{\text{def}}{=} \{a_i : i \in \omega\}$
  - Mathematical variables: $x, y, z, \ldots$ $\qquad MVar \stackrel{\text{def}}{=} \{x_i : i \in \omega\}$
  - Event variables: $e, e', e'', \ldots$ $\qquad NVar \stackrel{\text{def}}{=} \{e_i : i \in \omega\}$
  - Mathematical function and relation symbols: $+$, $\cdot$, $\leq$
  - Event predicate: $\prec$
  - Clock predicate: In (Optional)
  - Intersort predicate: P
  - Logical symbols: $\neg, \wedge, =, \exists$
- Terms:

$$\tau ::= x \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$$

- Formulas:

$$\varphi ::= a = b \mid \tau = \tau' \mid \tau \leq \tau' \mid e = e' \mid e \prec e' \mid \mathrm{In}(a) \mid \mathrm{P}(e, a, \tau) \mid \\ \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \mid \exists a\varphi \mid \exists e\varphi$$

On rare occasions we will denote event variables with symbols different from e, e', e_1, ... In these cases, the context always clarifies that the used symbols refer to event variables.

Remark 1. In the light of Theorem **??**, we could explicitly define inertial observers as the geodetic observers. We do not choose this way, by the following reasons: A tétel még készülőben

- We are able to construct our axiomatizations without using the very special geodetic property, and using the more geometrical 'line-like' properties of inertials/geodetics.
- We think that the equivalence of inertiality and geodeticity in Minkowski spacetimes should be on the "theoremhood" rather than the "assumption" side of an axiomatic approach to relativity theories.

1.2.1 Abbreviations

$$\begin{array}{rcl@{\qquad}rcl}
a(e) = \tau & \stackrel{\text{def}}{\Leftrightarrow} & \mathrm{P}(a, e, \tau) & e \ll e' & \stackrel{\text{def}}{\Leftrightarrow} & e \prec e' \wedge \exists a(e\mathcal{E}a \wedge e'\mathcal{E}a) \\
e\mathcal{E}a & \stackrel{\text{def}}{\Leftrightarrow} & \exists x\, \mathrm{P}(a, e, x) & e \underline{\ll} e' & \stackrel{\text{def}}{\Leftrightarrow} & e \ll e' \vee e = e' \\
\mathrm{wline}_a & \stackrel{\text{def}}{=} & \{e : \exists x\, \mathrm{P}(a, e, x)\} & e \rightsquigarrow e' & \stackrel{\text{def}}{\Leftrightarrow} & e \prec e' \wedge \neg\exists a(e\mathcal{E}a \wedge e'\mathcal{E}a) \\
\mathrm{D}_e & \stackrel{\text{def}}{=} & \{a : \exists x\, \mathrm{P}(a, e, x)\} & e \underset{=}{\rightsquigarrow} e' & \stackrel{\text{def}}{\Leftrightarrow} & e \rightsquigarrow e' \vee e = e' \\
a \approx b & \stackrel{\text{def}}{\Leftrightarrow} & \forall e(e\mathcal{E}a \leftrightarrow e\mathcal{E}b) & \overrightarrow{e_1 e_2 e_3} & \stackrel{\text{def}}{\Leftrightarrow} & e_1 \rightsquigarrow e_2 \wedge e_2 \rightsquigarrow e_3 \wedge e_1 \rightsquigarrow e_3
\end{array}$$

D_e is the *domain* of event e, the relation $a \approx b$ is referred as the *cohabitation* of clocks a and b, and $\overrightarrow{e_1 e_2 e_3}$ is the *directed lightlike betweenness* predicate.

1.3 Intended classical clock models

$$\mathfrak{M}^c = \left(\mathbb{R}^4, \mathbb{C}, \mathbb{R}, \prec^{\mathfrak{M}^c}, \mathrm{In}^{\mathfrak{M}^c} +, \cdot, \leq, \mathrm{P}^{\mathfrak{M}^c}\right)$$

where

- $\mathbb{C}$ is the set of those $\alpha : \mathbb{R}^4 \to \mathbb{R} \cup \{\Theta\}$, for which α^{-1}-s are timelike curves that follows the measure system of $\mathbb{R}^4$, i.e.,
  - α^{-1} is differentiable function w.r.t. Euclidean metric:
  $$(\forall x \in U)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in U) \quad |x - y| \leq \delta \Rightarrow \frac{|\alpha^{-1}(x) - \alpha^{-1}(y)|}{|x - y|} \leq \varepsilon,$$
  - $(\alpha^{-1})'$ is continuous:
  $$(\forall x \in U)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in U) \quad |x - y| < \delta \Rightarrow \left|(\alpha^{-1})'(x) - (\alpha^{-1})'(y)\right| < \varepsilon$$
  - $(\alpha^{-1})'$ is timelike: $\mu \circ \left(\alpha^{-1}\right)'(x) > 0$ for all $x \in \mathbb{R}$.
  - Measure system of $\mathbb{R}^4$: $\mu(\alpha^{-1}(x), \alpha^{-1}(x + y)) = y$.
- $\vec{x} \prec^{\mathfrak{M}^c} \vec{y} \overset{\text{def}}{\Leftrightarrow} \mu(\vec{x}, \vec{y}) \geq 0$ and $x_1 < y_1$,
- $\mathrm{In}^{\mathfrak{M}^c} \overset{\text{def}}{=} \left\{ \alpha \in \mathbb{C} \;:\; \begin{array}{r} (\exists x, y \in U)(\forall z \in U)(\exists \lambda \in U) \\ \alpha^{-1}(z) = \alpha^{-1}(x) + \lambda \cdot (\alpha^{-1}(x) - \alpha^{-1}(y)) \end{array} \right\}$
- $\mathrm{P}^{\mathfrak{M}^c} = \{\langle \vec{x}, \alpha, y\rangle \in \mathbb{R}^4 \times \mathbb{C}_I \times \mathbb{R} \;:\; \alpha(\vec{x}) = y\}$,

The non-accelerating intended model $\mathfrak{M}^c_{\mathrm{I}}$ is the largest submodel of $\mathfrak{M}^c$ in which the domain of clocks is $\mathrm{In}^{\mathfrak{M}^c}$.

1.4 Goals

- Construct coordinate systems for inertial clocks.
- Construct coordinate systems for accelerating clocks.
- Find axiomatic base SClTh for these coordinate construction procedures.
- Extend SClTh into a complete axiomatization of $\mathrm{Th}(\mathfrak{M}^c_I)$.
- Extend SClTh into a complete axiomatization of $\mathrm{Th}(\mathfrak{M}^c)$ or show that it cannot be axiomatized.
- Compare $\mathrm{Th}(\mathfrak{M}^c_I)$ to SpecRel in terms of definitional equivalences.
- Compare $\mathrm{Th}(\mathfrak{M}^c)$ to AccRel in terms of definitional equivalences.

2 Coordinatization

In this section we work in $\mathrm{Th}(\mathfrak{M}^c)$.

2.1 How to build a coordinate system?

During this section keep in mind that we use mostly partial functions, so recall the remarks on Notation 3.

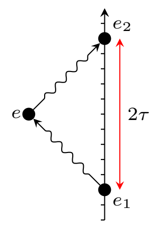
To define coordinatization we have to create the notions of *space and time relative to observers.* To define notions related to time is not a hard job anymore since we can use the structure of $\mathbb{R}$. To construct observer-relative space and the Coordinatization predicate, we follow ideas similar to the paper of Andréka and Németi [2014]. This idea can be summarized in the following steps:

1. **Space:** We define the (spatial) points of clocks. The space of a clock will be the set of its *inertial synchronized co-movers* (or shortly, iscm-s).[1]
2. **Geometry:** We define the *betweenness* and *equidistance* relations, the two primitive relation of Tarski and Givant [1999]. This makes us able to talk about the geometrical structure of the space of any clock.
3. **Coordinate Systems:** We define orthogonality to identify coordinate systems as a 4-tuple of iscm-s, representing the origin and the three direction of the three axes.
4. **Coordinatization:** We use the distances from the axes and a sign-function to build coordinates for every events.
5. **Check:** We check that this coordinatization predicate is good indeed. In Theorem 25 we prove that it is a bijection between W and U^4 for any coordinate system, and in Section 3.5 we show that we can use it to interpret the worldview predicate W of SpecRel. We will check this in an axiomatic environment.

2.2 Space

Definition 4 (Distances)**.** The distance of an *inertial observer from an event* is defined via signalling, see Fig 1.

Figure 1: $\delta^i(a,e)=\tau$



$$\delta^i(a,e)=\tau \overset{\text{def}}{\Leftrightarrow} \mathrm{In}(a)\wedge(\exists e_1,e_2\in \mathrm{wline}_a)\big(e_1 \mathrel{\underset{=}{\rightsquigarrow}} e \mathrel{\underset{=}{\rightsquigarrow}} e_2 \wedge a(e_1)-a(e_2)=2\cdot\tau\big)$$

The *distance of inertials* can be defined in the following way:

$$\delta^i(a,a')=\tau \overset{\text{def}}{\Leftrightarrow} (\forall w\in \mathrm{wline}_{a'})\delta^i(a,w)=\tau$$

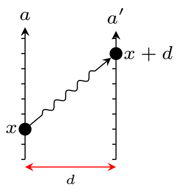
Comovement is defined by having a distance:

$$a\overset{\text{i}}{\uparrow\uparrow}a' \overset{\text{def}}{\Leftrightarrow} \exists x\delta^i(a,a')=x$$

According to the theories of intended models, $\delta^i(a,e)=\tau$ is a total function and $\delta^i(a,a')=\tau$ is a partial, but not a total function. We will prove this in Proposition 8 later, when we will have the final axiom system to work with.

[1]Here we note that the notion of space can be given more generally: simple inertial comovers are enough, but the more special synchronized subset simplifies the coordinatization process.

Figure 2: $a\overset{\text{syn}}{\uparrow\uparrow}a'$



Definition 5 (Inertial synchronized co-movers)**.** Clocks a and a' are *inertial synchronised co-movers* iff a' shows $x + \delta^i(a, a')$ whenever a' sees that a shows x. See Fig 2.

$$a \overset{\text{syn}}{\uparrow\uparrow} a' \overset{\text{def}}{\Leftrightarrow} (\forall w \in \mathrm{D}_a)(\forall w' \in \mathrm{D}'_a) \left(w \underset{=}{\rightsquigarrow} w' \to a'(w') = a(w) + \delta^i(a, a') \right)$$

(Note that comoving is ensured here by the pseudo-term $\delta^i(a, a')$!)

Now we are able to find representatives for points in spatial geometry for a clock a:

Definition 6 (Space)**.** Inertial synchronized comovers of a clock a will be called a *point* of a, and the set of all points of a will be called the *space* of a:

$$a' \in \mathrm{Space}_a \overset{\text{def}}{\Leftrightarrow} a \overset{\text{syn}}{\uparrow\uparrow} a'$$

2.3 Geometry

Now we define the basic primitives of [Tarski and Givant 1999] (The axioms can be found here in Table 1 on p.18 as well):

Definition 7 (Betweenness, Equidistance, Collinearity)**.** We say that a_2 is *between* a_1 and a_3 iff the shortest route from a_1 to a_3 leads through a_2:

$$\mathrm{B}(a_1, a_2, a_3) \overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) + \delta^i(a_2, a_3) = \delta^i(a_1, a_3)$$

Equidistance stands for equal distances:

$$a_1 a_2 \equiv a_3 a_4 \overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) = \delta^i(a_3, a_4)$$

Collinearity is the permutational closure of betweenness:

$$\mathrm{C}(a_1, a_2, a_3) \overset{\text{def}}{\Leftrightarrow} \mathrm{B}(a_1, a_2, a_3) \lor \mathrm{B}(a_3, a_1, a_2) \lor \mathrm{B}(a_2, a_3, a_1)$$

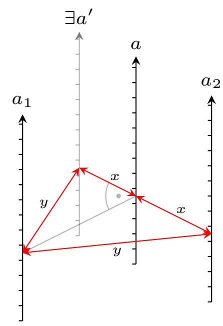
Remark 2. Recall that since δ^i is a partial function, all these relations implies the inertiality and the co-movement of all of its arguments.

2.4 Coordinate systems

Definition 8 (Orthogonality)**.** Distinct lines determined by points a-a_1 and a-a_2 are *orthogonal* iff there is an a' such that a', a_1 and a_2 forms an isoscele triangle and a is in the middle of the segment a' and a_2, see Fig. 3:

$$\begin{aligned}\mathrm{Ort}(a, a_1, a_2) \overset{\text{def}}{\Leftrightarrow}\ & \delta^i(a, a_1) > 0 \land \delta^i(a_1, a_2) > 0 \land \delta^i(a, a_2) > 0 \\ & \land \exists a' \left(\mathrm{B}(a_2, a, a') \land \delta^i(a, a_2) = \delta^i(a, a') \land \delta^i(a_1, a_2) = \delta^i(a_1, a') \right)\end{aligned}$$

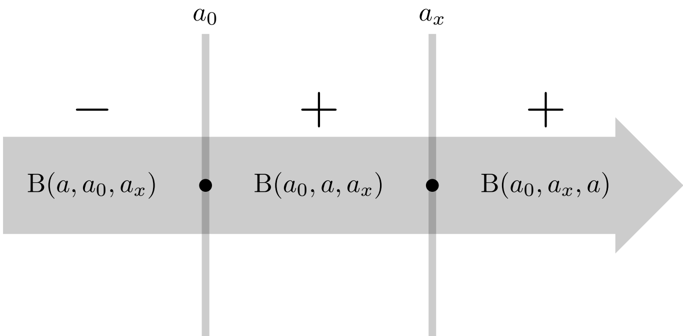
Figure 3: Right angle



Definition 9 (Distances from lines)**.** The distance of a clock a and a line given by the points (a_1, a_2) is τ iff the distance of a and its orthogonal projection on the line (a_1, a_2) is τ.

$$\delta^i(a, (a_1, a_2)) = \tau \overset{\text{def}}{\Leftrightarrow} \exists a' (\mathrm{Ort}(a', a, a_1) \land \mathrm{Ort}(a', a, a_2) \land \delta^i(a, a') = \tau)$$

Figure 4: Definition of the direction function



Definition 10 (Coordinate systems)**.**

$$\mathrm{CoordSys}(a, a_x, a_y, a_z) \overset{\text{def}}{\Leftrightarrow} \mathrm{Ort}(a, a_x, a_y) \wedge \mathrm{Ort}(a, a_y, a_z) \wedge \mathrm{Ort}(a, a_x, a_z)$$

Definition 11 (Directed lines)**.** If a line is given by the points (a_0, a_x), then a point a of that line is in *negative direction* if a_0 is between a and a_x, is in *null-direction* if $a = a_0$, and is in *positive direction* otherwise, see Fig. 4:

$$\begin{aligned}\mathrm{sign}^{-}_{a_0,a_x}(a) = \tau \overset{\text{def}}{\Leftrightarrow} \ & (a \neq a_0 \wedge \mathrm{B}(a, a_0, a_x) \wedge \tau = -1) \vee (a = a_0 \wedge \tau = 0) \vee \\ & \big(a \neq a_0 \wedge (\mathrm{B}(a_0, a, a_x) \vee \mathrm{B}(a_0, a_x, a)) \wedge \tau = 1\big)\end{aligned}$$

If a is not on the line given by (a_0, a_x), then we say that it is in the negative/null/positive direction iff its orthogonal projection on that line is in the negative/null/positive direction, respectively:

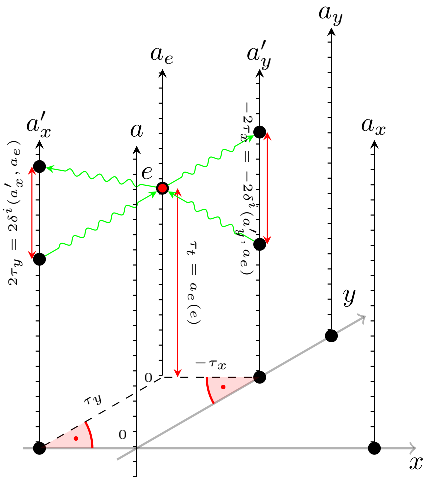
$$\mathrm{sign}_{a_0,a_x}(a) = \tau \overset{\text{def}}{\Leftrightarrow} \exists a'(\mathrm{Ort}(a', a, a_0) \wedge \mathrm{Ort}(a', a, a_x) \wedge \mathrm{sign}^{-}_{a_0,a_x}(a') = \tau)$$

2.5 Coordinatization

Definition 12 (Coordinatization)**.** See Fig. 5. The event e will be coordinatized on the spatiotemporal position $\langle \tau_t, \tau_x, \tau_y, \tau_z \rangle$ by the coordinate system $\langle a, a_x, a_y, a_z \rangle$ iff there is a synchronized co-mover a_e of a that shows the time τ_t in e and $\tau_d = \mathrm{sign}_{a,a_d}(a_e) \cdot \delta^i(a_e, a, a_d)$ for $d \in \{x, y, z\}$.

$$\begin{aligned}&\mathrm{Coord}_{a,a_x,a_y,a_z}(e) = (\tau_t, \tau_x, \tau_y, \tau_z) \overset{\text{def}}{\Leftrightarrow} \\ &\quad (\exists a_e \in \mathrm{Space}_a)\Big(\mathrm{CoordSys}(a, a_x, a_y, a_z) \wedge \mathrm{P}(e, a_e, \tau_t) \wedge \\ &\quad\quad \mathrm{sign}_{a,a_x}(a_e) \cdot \delta^i(a_e, (a, a_x)) = \tau_x \wedge \\ &\quad\quad \mathrm{sign}_{a,a_y}(a_e) \cdot \delta^i(a_e, (a, a_y)) = \tau_y \wedge \\ &\quad\quad \mathrm{sign}_{a,a_z}(a_e) \cdot \delta^i(a_e, (a, a_z)) = \tau_z\Big)\end{aligned}$$

Figure 5: A 2D illustration of the coordinatization process



3 Axiom system SClTh

AxReals The mathematical sort forms a real closed field, see [?]. mit hivatkozzunk?

$$
\begin{array}{llll}
 & (x+y)+z = x+(y+z) & & (x\cdot y)\cdot z = x\cdot(y\cdot z) \\
\exists 0 & x+0 = x & \exists 1 & x\cdot 1 = x \\
\exists(-x) & x+(-x) = 0 & x \neq 0 \rightarrow \exists x^{-1} & x\cdot x^{-1} = 1 \\
 & x+y = y+x & & x\cdot y = y\cdot x
\end{array}
$$

$$x\cdot(x+y) = (x\cdot y)+(x\cdot z)$$

$$
\begin{array}{rr}
a \leq b \wedge b \leq a \rightarrow a = b & \\
a \leq b \wedge b \leq c \rightarrow a \leq c & a \leq b \rightarrow a+c \leq b+c \\
\neg a \leq b \rightarrow b \leq a & a \leq b \wedge 0 \leq c \rightarrow a\cdot c \leq b\cdot c
\end{array}
$$

$$\exists x(\forall y \in \varphi)x \leq y \rightarrow \exists i(\forall y \in \varphi)(i \leq y \wedge \forall i'((\forall y \in \varphi)(i' \leq y \rightarrow i' \leq i)) \tag{AxReals}$$

AxFull Every number occurs as a state of any clock in an event.

$$\forall a \forall x \exists e \quad \mathrm{P}(e,a,x) \tag{AxFull}$$

AxExt We do not distinguish between (1) indistinguishable clocks, (2) states of a particular clock in an event and (3) two events where a clock shows the same time.

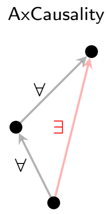
$$
\begin{array}{llrl}
(1) & \forall a, a' & \Big(\forall e \forall x(\mathrm{P}(e,a,x) \leftrightarrow \mathrm{P}(e,a',x))\Big) & \rightarrow\ a = a' \\
(2) & \forall e \forall a \forall x, y & \big(\mathrm{P}(e,a,x) \wedge \mathrm{P}(e,a,y)\big) & \rightarrow\ x = y \\
(3) & \forall e, e' \forall a \forall x & \big(\mathrm{P}(e,a,x) \wedge \mathrm{P}(e',a,x)\big) & \rightarrow\ e = e'
\end{array}
\tag{AxExt}
$$

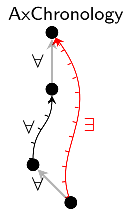
AxForward Clocks are ticking forward.

$$\forall a(\forall e, e' \in \mathrm{wline}_a) \quad \big(e \prec e' \leftrightarrow a(e) < a(e')\big) \tag{AxForward}$$

AxSynchron All clocks occupying the same worldline (i.e., cohabitants) use the same measure system, and for every clock, and delay, there is a cohabitant clock with that delay.

$$\begin{aligned}&\forall a(\forall b\approx a)\exists x(\forall e\in \mathrm{wline}_a)\quad a(e)=b(e)+x\\&\forall a\forall x(\exists b\approx a)(\forall e\in \mathrm{wline}_a)\quad a(e)=b(e)+x\end{aligned}\qquad \text{(AxSynchron)}$$



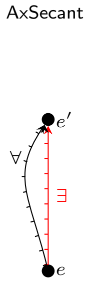


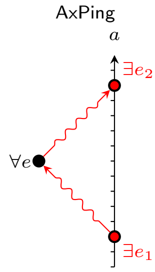
AxCausality Causality is transitive.

$$(e_1 \prec e_2 \wedge e_2 \prec e_3)\to e_1\prec e_3 \qquad \text{(AxCausality)}$$

AxChronology Interiors of two-way lightcones are filled with clocks crossing through the vertex.

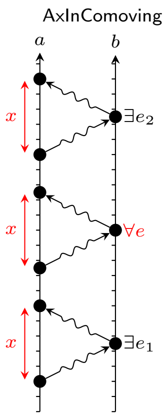
$$(e_1 \preceq e_2 \wedge e_2 \ll e_3 \wedge e_3 \preceq e_4)\to e_1 \ll e_4 \qquad \text{(AxChronology)}$$





AxSecant Any two events that share a clock share an inertial clock as well.

$$e\ll e' \to (\exists a\in \mathrm{In})(e\mathcal{E}a \wedge e'\mathcal{E}a)) \qquad \text{(AxSecant)}$$

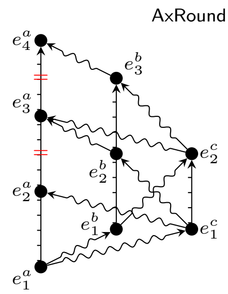


AxInComoving If an inertial clock measures an other inertial clock with the same distance twice, then they are comoving.

$$\big(a,b\in \mathrm{In}\wedge(\exists e_1,e_2\in \mathrm{wline}_b)(e_1\neq e_2 \wedge \delta^i(a,e_1)=\delta^i(a,e_2))\big)\to a\overset{\mathrm{i}}{\uparrow\uparrow} b \qquad \text{(AxInComoving)}$$

AxPing Every inertial clock can send and receive a signal to any event.

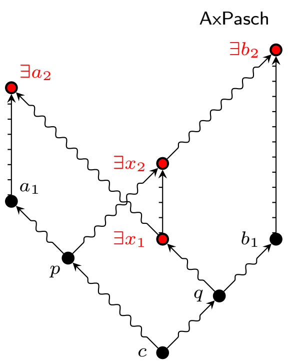
$$(\forall a\in \mathrm{In})\forall e(\exists e_1,e_2\in \mathrm{wline}_a)\qquad e_1 \underset{=}{\nearrow} e \underset{=}{\nearrow} e_2 \qquad \text{(AxPing)}$$



AxRound Given comoving observers a, b and c, the travelling time of simultaneously sent signals on the route $\langle a,b,c,a\rangle$ and $\langle a,c,b,a\rangle$ are (the same, namely,) the average of the travelling time of the $\langle a,c,a\rangle$ and $\langle a,b,c,b,a\rangle$.

$$b\overset{\mathrm{i}}{\uparrow\uparrow} a\overset{\mathrm{i}}{\uparrow\uparrow} c\wedge\begin{pmatrix} e_1^a,e_2^a,e_3^a,e_3^{a\prime},e_4^a\in \mathrm{wline}_a\\ e_1^b,e_2^b,e_3^b\in \mathrm{wline}_b\\ e_1^c,e_2^c\in \mathrm{wline}_c\end{pmatrix}\wedge\begin{pmatrix} e_1^a\nearrow e_1^b\nearrow e_2^c\nearrow e_3^a\\ e_1^a\nearrow e_1^c\nearrow e_2^b\nearrow e_3^{a\prime}\\ e_2^c\nearrow e_3^b\nearrow e_4^a\\ e_1^c\nearrow e_2^a\end{pmatrix}\to$$

$$\to\left(a(e_3^a)=a(e_3^{a\prime})=\frac{a(e_2^a)+a(e_4^a)}{2}\right)\qquad \text{(AxRound)}$$



AxPasch Pasch axiom for light signals, See Fig. 6.

$$\big(a\overset{\mathrm{i}}{\uparrow\uparrow} b\wedge(\exists a_1\in \mathrm{wline}_a)(\exists b_1\in \mathrm{wline}_b)(\overrightarrow{cpa_1}\wedge\overrightarrow{cqb_1})\big)\to$$

$$\to(\exists x\overset{\mathrm{i}}{\uparrow\uparrow} a)(\exists x_1,x_2\in \mathrm{wline}_x)(\exists a_2\in \mathrm{wline}_a)(\exists b_2\in \mathrm{wline}_b)(\overrightarrow{px_2b_2}\wedge\overrightarrow{qx_1a_2})\qquad \text{(AxPasch)}$$

Figure 6: Tarski's Inner Pasch axiom

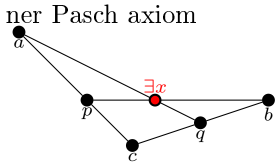
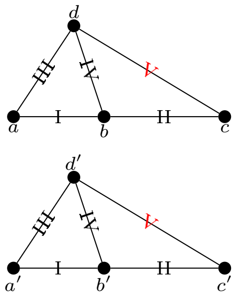
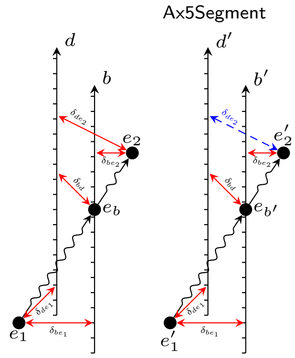


Figure 7: Tarski's Five-segment axiom

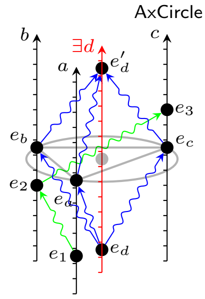


Ax5Segment If there are two pairs of observers b,d and b',d' such that two light signals $e_1 \rightsquigarrow e_2$ and $e'_1 \rightsquigarrow e'_2$ crosses the worldlines of b and b', respectively, then b and b' agree on the distance of e_2 and e_2, respectively, whenever they agree on the distance of e_1 and d, e'_1 and d', respectively. Compare that axiom with Tarski's Five-segment axiom on Fig. 7

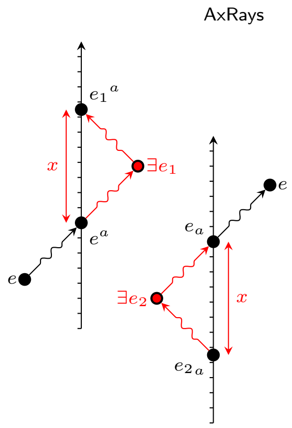


$$
\begin{aligned}
&d \overset{i}{\uparrow\uparrow} d' \wedge b \overset{i}{\uparrow\uparrow} b' \wedge e_b \mathcal{E} b \wedge e'_b \mathcal{E} b' \wedge \overrightarrow{e_1 e_b e_2} \wedge \overrightarrow{e'_1 e'_b e'_2} \wedge \\
&\qquad \wedge \delta^i(b, e_1) = \delta^i(b', e'_1) \wedge \delta^i(b, e_2) = \delta^i(b', e'_2) \wedge \\
&\qquad \wedge \delta^i(d, e_1) = \delta^i(d', e'_1) \wedge \delta^i(b, d) = \delta^i(b', d')) \rightarrow \\
&\qquad\qquad \rightarrow \delta^i(d, e_2) = \delta^i(d', e'_2) \qquad \text{(Ax5Segment)}
\end{aligned}
$$

AxCircle For every three non-collinear inertial observer there is a fourth one that measures them with the same distance.



$$
\begin{aligned}
&(\forall a, b, c \in \mathrm{In})\big((a \overset{i}{\uparrow\uparrow} b \overset{i}{\uparrow\uparrow} c \wedge \\
&\qquad \wedge \exists e_1, e_2, e_3 (e_1 \mathcal{E} a \wedge e_2 \mathcal{E} b \wedge e_3 \mathcal{E} c \wedge e_1 \rightsquigarrow e_2 \rightsquigarrow e_3 \wedge \neg e_1 \rightsquigarrow e_3)) \rightarrow \\
&\qquad \rightarrow \exists d \exists e_a, e_b, e_c, e_d, e'_d (e_a \mathcal{E} a \wedge e_b \mathcal{E} b \wedge e_c \mathcal{E} c \wedge e_d \mathcal{E} d \wedge e'_d \mathcal{E} d \wedge \\
&\qquad\qquad \wedge e_d \rightsquigarrow e_a \rightsquigarrow e'_d \wedge e_d \rightsquigarrow e_b \rightsquigarrow e'_d \wedge e_d \rightsquigarrow e_c \rightsquigarrow e'_d)\big) \qquad \text{(AxCircle)}
\end{aligned}
$$



AxRays For every observer, for any positive x and every direction (given by a light signal) there are lightlike separated events in the past and the future whose distances are exactly x.

$$
\begin{aligned}
&(\forall x > 0) \forall a \forall e \exists e_1 \exists e_2 (\exists e^a, e_a \in \mathrm{wline}_a) \\
&\qquad \overrightarrow{e_2 e_a e} \wedge \delta^i(a, e_2) = x \wedge \overrightarrow{e e^a e_1} \wedge \delta^i(a, e_1) = x \qquad \text{(AxRays)}
\end{aligned}
$$

AxDim$\geq n$ The dimension of the spacetime is at least n. The formula says that $n-1$ lightcones never intersect in only one event.

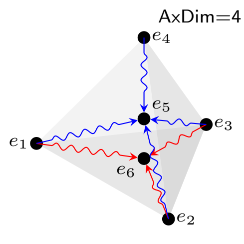
$$
\forall e_1, \ldots, e_n \left(\bigwedge_{i \leq n-1} e_i \rightsquigarrow e_n \rightarrow \exists e_{n+1} \left(\bigwedge_{i \leq n-1} e_i \rightsquigarrow e_n \wedge e_n \neq e_{n+1} \right) \right) \qquad \text{(AxDim}\geq n\text{)}
$$

AxDim$\leq n$ The dimension of the spacetime is at most n. The formula says that there are n lightcones that intersect at most in one event.

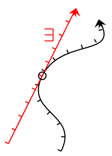
$$
\exists e_1, \ldots, e_{n+1} \left(\bigwedge_{i \leq n} e_i \rightsquigarrow e_{n+1} \wedge \forall e_{n+2} \left(\bigwedge_{i \leq n} e_i \rightsquigarrow e_{n+2} \rightarrow e_{n+1} = e_{n+2} \right) \right) \qquad \text{(AxDim}\leq n\text{)}
$$

AxDim=4 The dimension of the spacetime is exactly 4; 3 lightcones never intersect in only one event and there are 4 lightcones intersect in at most one event.

$$
AxDim_{\leq 4} \wedge AxDim_{\geq 4} \qquad \text{(AxDim=4)}
$$



AxTangent



AxTangent For every event e of every clock a there is an inertial clock b that occurs in e and its velocity is the same as the local instantaneous velocity of a according to any inertial observer.

(AxTangent)

AxNoAcceleration Every clock is inertial.

$$\forall a \quad \mathrm{In}(a) \tag{AxNoAcceleration}$$

AxAcceleration For every coordinate system $\langle a, a_x, a_y, a_z\rangle$ and every definable timelike curve φ there is a clock having that wordline according to $\langle a, a_x, a_y, a_z\rangle$.

(AxAcceleration)

Definition 13 (Axiom systems)**.** We (re)define SClTh to be the following sets of axioms.

$$\mathrm{SClTh} \stackrel{\mathrm{def}}{=} \left\{ \begin{array}{llll} \mathsf{AxFull} & \mathsf{AxCausality} & \mathsf{AxRays} & \mathsf{Ax5Segment} \\ \mathsf{AxExt} & \mathsf{AxChronology} & \mathsf{AxPing} & \mathsf{AxCircle} \\ \mathsf{AxForward} & \mathsf{AxSecant} & \mathsf{AxRound} & \mathsf{AxDim{=}4} \\ \mathsf{AxSynchron} & \mathsf{AxInComoving} & \mathsf{AxPasch} & \mathsf{AxTangent} \end{array} \right\}$$

$$\mathrm{SClTh}^{\mathrm{NoAcc}} \stackrel{\mathrm{def}}{=} (\mathrm{SClTh} - \{\mathsf{AxSecant},\ \mathsf{AxTangent}\}) \cup \{\mathsf{AxNoAcceleration}\}$$
$$\mathrm{SClTh}^{\mathrm{Acc}} \stackrel{\mathrm{def}}{=} \mathrm{SClTh} \cup \{\mathsf{AxAcceleration}\}$$

3.1 Theorems

Our plan is the following:

1. Kronheimer-Penrose axioms: We are working with causal spaces.
2. Signalling (radar-distance) is unique.
3. AxLocExp:For every observer, there is a point (local iscm) in every event. That is equiderivable with AxInComoving.
4. There is a clock in every event (Immediate)
5. Straight signals arrive sooner.
6. $\overset{\mathrm{syn}}{\uparrow\uparrow}$ is an equivalence relation and δ^i is a metric on $\overset{\mathrm{syn}}{\uparrow\uparrow}$-related clocks.
7. There are no two iscms/points in an event.
8. 'Equivalence' of $\overrightarrow{e_a e_b e_c}$ and $\mathrm{B}(a, b, c)$.
9. Tarski's axioms.
10. Coordinatization is a bijection between W and Q^4.
11. Radar-based spatial distance and elapsed time defines the same quantities as coordinate based definition. (Simplifying the coordinate-system based SpecRel axioms)

12. Proving 'Simple-SpecRel'.

Assumptions:
AxForward
AxCausality
AxChronology

Proposition 3. *$\langle W, \preceq, \ll, \underset{=}{\leadsto} \rangle$ is a causal space (see [Kronheimer and Penrose 1967]), i.e., the following statements are all true:*

$$
\begin{aligned}
& e \preceq e \\
& (e_1 \preceq e_2 \wedge e_2 \preceq e_3) \to e_1 \preceq e_3 \\
& (e_1 \preceq e_2 \wedge e_2 \preceq e_1) \to e_1 = e_2 \\
& \neg e \ll e \\
& e_1 \ll e_2 \to e_1 \preceq e_2 \\
& (e_1 \preceq e_2 \wedge e_2 \ll e_3) \to e_1 \ll e_3 \\
& (e_1 \ll e_2 \wedge e_2 \preceq e_3) \to e_1 \ll e_3 \\
& e_1 \underset{=}{\leadsto} e_2 \leftrightarrow (e_1 \preceq e_2 \wedge \neg e_1 \ll e_2)
\end{aligned}
$$

And the following statements are also hold:

$$\neg e \prec e \tag{1}$$

$$(e_1 \prec e_2 \wedge e_2 \ll e_3) \to e_1 \ll e_3 \tag{2}$$

$$(e_1 \ll e_2 \wedge e_2 \prec e_3) \to e_1 \ll e_3 \tag{3}$$

Proof. All the defining properties of the causal spaces are straightforward consequences of AxCausality, (1), (2) and (3) or true simply by the definitions of $\preceq$, $\ll$ and $\underset{=}{\leadsto}$.

- (1) comes from AxForward; $e \prec e$ would lead to $a(e) < a(e)$.
- (2): is AxChronology where $e_1 \neq e_2$ and $e_3 = e_4$.
- (3): is AxChronology where $e_1 = e_2$ and $e_3 \neq e_4$.

■

Assumptions:
AxChronology

Proposition 4. *Signalling is unique:*

$$
\begin{aligned}
& \forall e \forall a (\forall e_a, e'_a \in \mathrm{wline}_a)(e_a \leadsto e \wedge e'_a \leadsto e) \to e_a = e'_a \\
& \forall e \forall a (\forall e^a, e^{a\prime} \in \mathrm{wline}_a)(e \leadsto e^a \wedge e \leadsto e^{a\prime}) \to e^a = e^{a\prime}
\end{aligned}
$$

Proof. Suppose that $e_a \neq e'_a$ but $e_a \leadsto e \wedge e'_a \leadsto e$ and $e_a, e'_a \in \mathrm{wline}_a$. Then by definition $e_a \ll e'_a$ or $e_a \gg e'_a$. By AxChronology, $e_a \ll e$ or $e'_a \ll e$ which contradicts to the assumption. The proof is similar for the symmetrical formula as well. ■

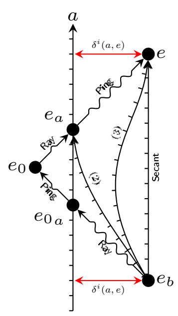
The following theorem is equiderivable with AxInComoving above SClTh.

Proposition 5 (AxLocExp)**.** *For every inertial observer, there is a synchronized inertial observer (i.e., a point) in any event.*

Assumptions:
Proposition 3
AxPing
AxRays
AxSecant

$$(\forall a \in \mathrm{In}) \forall e \exists b \qquad e \mathcal{E} b \wedge a \overset{\mathrm{syn}}{\uparrow\uparrow} b \tag{AxLocExp}$$

Proof. Let $a \in \mathrm{In}$ and e be arbitrary. If $e \in \mathrm{wline}_a$ then we are ready. Suppose now that $e \notin \mathrm{wline}_a$. By AxPing, there are $e_a, e^a \in \mathrm{wline}_a$ s.t. $e_a \leadsto e \leadsto e^a$. Let $x \overset{\mathrm{def}}{=} a(e^a) - a(e_a)$. Note that $\delta^i(a, e) = x$ is true. By AxCausality and AxForward and by the assumption that $e \notin \mathrm{wline}_a$, this x is strictly positive. By AxRays,



there is an e_0 s.t. e_0 is 1 distance away from a and $\overrightarrow{e_0 e_a e}$. By AxPing, there is an $e_{0a} \in \mathrm{wline}_a$ s.t. $e_{0a} \nearrow e_0$. By AxRays again, there is an event e_b s.t. $\overrightarrow{e_b e_{0a} e_0}$ and $\delta^i(a, e_b) = x$. Since $e_b \nearrow e_{0a} \ll e_a \nearrow e$, by AxChronology we have $e_b \ll e$. By AxSecant, there is an inertial clock b through e_b and e. Now since both a and b are inertial and $\delta^i(a, e_b) = x$ and $\delta^i(a, e) = x$, by AxInComoving, $a \overset{i}{\uparrow\uparrow} b$, and by AxSynchron again, there is an a-synchronized b' cohabitant of b here as well; that is the clock having delay x. ∎

Proposition 6. *There is a clock in every event.*

Assumptions:
AxFull
AxSecant
Proposition 5

$$\forall e \exists c \quad e\mathcal{E}c$$

Proof. Let e be an arbitrary event. There is a clock a in some event e_0 by AxFull (and by the tautology $\exists a\, a = a$). By AxSecant, there is an inertial clock at e_0 as well. By Proposition 5, there is an inertial comover of a at e. ∎

Corollary 7. *The pointing relation* P *is a surjective function* $\mathrm{P} : \mathbb{C} \times U \to W$.

Assumptions:
AxFull
AxSecant
Proposition 5
AxExt

Proof. It is a function by Proposition AxExt, and is surjective by 6. ∎

Proposition 8. $\delta^i(a, e) = \tau$ *is a total function.*

Assumptions:
AxPing

Proof. This is true by AxPing: Since every observer can ping an event, it is always defined, and functionality comes from Proposition 4. ∎

Proposition 9. $\delta^i(a, a') = \tau$ *is a partial function.*

Proof. It is a partial function *by definition*. ∎

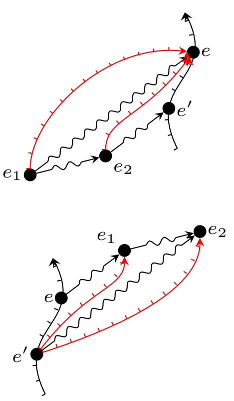
Proposition 10. *Straight signals arrive sooner:*

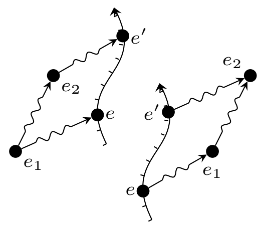
Assumptions:
AxPing
AxCausality
AxChronology
AxForward

$$\forall a \forall e_1, e_2, e, e' (e\mathcal{E}a \wedge e'\mathcal{E}a \wedge e_1 \underset{=}{\nearrow} e \wedge e_1 \underset{=}{\nearrow} e_2 \underset{=}{\nearrow} e') \to a(e) \leq a(e') \tag{4}$$

$$\forall a \forall e_1, e_2, e, e' (e\mathcal{E}a \wedge e'\mathcal{E}a \wedge e' \underset{=}{\nearrow} e_2 \wedge e \underset{=}{\nearrow} e_1 \underset{=}{\nearrow} e_2) \to a(e) \leq a(e') \tag{5}$$

Indirect proof





Proof.

- For (4) suppose indirectly that $a(e) > a(e')$. Then by AxForward, $e \succ e'$, and since they share the clock a, $e' \ll e$. If $e_2 = e_1$ or $e_2 = e'$ then by AxPing, $e = e'$ or $e = e'$ which contradicts to $e \succ e'$. So we have the chain

$$e_1 \nearrow e_2 \nearrow e' \ll e$$

This implies $e_2 \preceq e'$, and by AxCausality, $e_1 \preceq e'$. From (2) we have that (1) $e_2 \ll e$ and then (2) $e_1 \ll e$ which contradicts to $e_1 \nearrow e$.

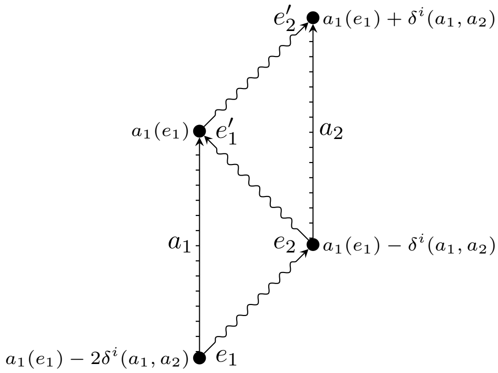
- For (5) suppose indirectly that $a(e') < a(e)$. Then by AxForward, $e' \prec e$, and since they share the clock a, $e' \ll e$. If $e_1 = e$ or $e_1 = e_2$ then by AxPing, $e = e'$ or $e = e'$ which contradicts to $e' \prec e$. So we have the chain

$$e' \ll e \nearrow e_1 \nearrow e_2$$

This implies $e \prec e_1$, and by AxCausality, $e' \succ e_1$. From (3) we have that (1) $e' \ll e_1$, and then (2) $e' \ll e_2$, which contradicts to $e' \nearrow e_2$.

∎

Figure 8: Proof of (8) and (12).



Proposition 11. *$\overset{\text{syn}}{\upuparrows}$ is an equivalence relation and δ^i is a(n U-relative) metric on $\overset{\text{syn}}{\upuparrows}$ related clocks, i.e.,*

Assumptions:
AxPing
AxExt
AxRound
AxForward
AxCausality
AxChronology
Proposition 5

$$\tag{6}$$

$$a \overset{\text{syn}}{\upuparrows} a \tag{7}$$

$$a_1 \overset{\text{syn}}{\upuparrows} a_2 \Rightarrow a_2 \overset{\text{syn}}{\upuparrows} a_1 \tag{8}$$

$$a_1 \overset{\text{syn}}{\upuparrows} a_2 \wedge a_2 \overset{\text{syn}}{\upuparrows} a_3 \Rightarrow a_1 \overset{\text{syn}}{\upuparrows} a_3 \tag{9}$$

$$\delta^i(a, a) = 0 \tag{10}$$

$$\delta^i(a_1, a_2) = 0 \Rightarrow a_1 = a_2 \tag{11}$$

$$\delta^i(a_1, a_2) = \delta^i(a_2, a_1) \tag{12}$$

$$\delta^i(a_1, a_2) + \delta^i(a_2, a_3) \geq \delta^i(a_1, a_3) \tag{13}$$

Proof.

- **Self-distance**, proof of (10): By $e \underset{=}{\rightsquigarrow} e \underset{=}{\rightsquigarrow} e$ we have $\delta^i(a, e) = a(e) - a(e) = 0$. The truth of $\delta^i(a, a) = 0$ is trivially implied by that fact.

- **Reflexivity of $\overset{\text{syn}}{\upuparrows}$**, proof of (7): By (10) we have $a(e') = a(e) + 0$ whenever $e \underset{=}{\rightsquigarrow} e'$, so $\overset{\text{syn}}{\upuparrows}$ is reflexive.

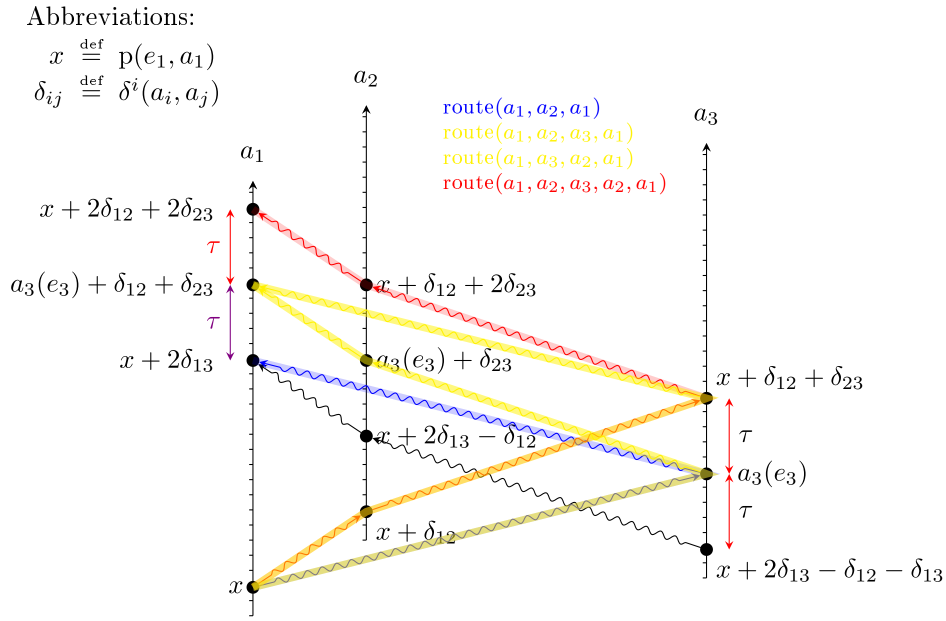
- **Symmetry of $\overset{\text{syn}}{\upuparrows}$ and δ^i**, proofs of (8) and (12), see Fig. 8. Suppose that $a_1 \overset{\text{syn}}{\upuparrows} a_2$, i.e.,

$$a_2(e_2) = a_1(e_1) + \delta^i(a_1, a_2) \text{ whenever } e_1 \underset{=}{\rightsquigarrow} e_2. \tag{14}$$

Take an arbitrary event $e_1' \mathcal{E} a_1$ s.t $e_2 \mathcal{E} a_2$ and $e_2 \underset{=}{\rightsquigarrow} e_1'$. We have to show that $a_1(e_1') = a_2(e_2) + \delta(a_2, a_1)$. By AxPing, there is an $e_2' \mathcal{E} a_2$ such that $e_2 \underset{=}{\rightsquigarrow} e_1' \underset{=}{\rightsquigarrow} e_2'$. By (14), we have that $a_2(e_2') = a_1(e_1') + \delta^i(a_1, a_2)$. Also from AxPing we know that there is $e_1 \mathcal{E} a_1$ s.t. $e_1 \underset{=}{\rightsquigarrow} e_2 \rightsquigarrow e_1'$. Here by definition of δ^i, $a_1(e_1) = a_1(e_1') - 2\delta^i(a_1, a_2)$, therefore, by (14) again we have that $a_2(e_2) = a_1(e_1') - \delta^i(a_1, a_2)$. Therefore we showed

$$a_2(e_2) + \delta^i(a_1, a_2) = a_1(e_1').$$

Figure 9: Transitivity of $\overset{\text{syn}}{\upuparrows}$.



Note that here $\delta^i(a_1, a_2) = \delta^i(a_2, a_1)$ since $\delta^i(a_2, a_1) = a_2(e_2') - a_2(e_2) = \delta^i(a_1, a_2)$, so we are ready with *both* (8) and (12).

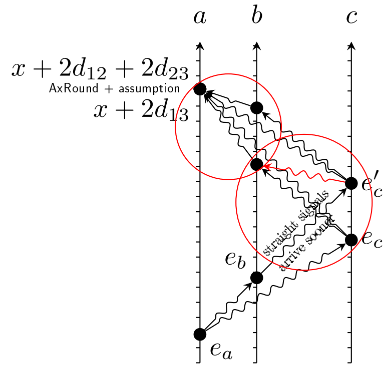
- **Identity of indiscernibles**, proof of 11. Take arbitrary iscm's a_1 and a_2 for which $\delta^i(a_1, a_2) = 0$, i.e.,

$$(\forall e \in \mathrm{wline}_{a_2})(\exists w_1, w_2 \in \mathrm{wline}_{a_1}) w_1 \underset{=}{\leadsto} e \underset{=}{\leadsto} w_2 \wedge a_1(w_2) - a_1(w_2) = 0$$

but that means that $a_1(w_1) = a_1(w_2)$, and by AxExt, $w_1 = w_2$. It cannot be the case that $w_1 \leadsto e$ and $e \leadsto w_2 = w_1$, because by AxCausality we would have $w_1 \prec w_1$ which contradicts to the irreflexivity of $\prec$ (Prop. 3). It cannot be the case either that $w_1 = e \leadsto w_2$ or $w_2 = e \leadsto w_1$, since we know that w_1 and w_2 share the clock a. So the only possiblity is that $w_1 = e = w_2$. Since this is true for all $e \in \mathrm{wline}_{a_2}$, we have that $\mathrm{wline}_{a_2} \subseteq \mathrm{wline}_{a_1}$. Using (12) we have that $\mathrm{wline}_{a_2} = \mathrm{wline}_{a_1}$. Now since a_1 and a_2 are iscms, they show the same numbers in the same events, therefore $a_1 = a_2$ by AxExt.

- **Transitivity of $\overset{\text{syn}}{\upuparrows}$**, proof of (9): We start to circuit signals between a_1, a_2 and a_3 and track the time tags, see Fig. 9 Following the abbreviation of Fig. 9, we have to show that $a_3(e_3) = x + d_{13}$. To show that, it is enough to show that $a_3(e_3)$ is the average of $x + d_{12} + d_{23}$ and $x + 2d_{13} - d_{12} - d_{23}$, i.e., to show that a_3 measures the same elapsed time between them. Since we can project these distances along a_2 to a_1 by our assumption that $a_1 \overset{\text{syn}}{\upuparrows} a_2 \overset{\text{syn}}{\upuparrows} a_3$, it is enough to show that $a_3(e_3) + d_{12} + d_{23}$ is the average of $x + 2d_{12} + 2d_{23}$ and $x + 2d_{13}$. But this is true by AxRound.

Figure 10: ‘Equivalence’ of lightlike betweenness and triangle equality



- **Triangle inequality**, proof of (13): By AxPing, we can take $e_1 \in \text{wline}_{a_1}$, $e_2 \in \text{wline}_{a_2}$, $e_3, e_3^* \in \text{wline}_{a_3}$ s.t. $e_1 \nearrow_{=} e_2 \nearrow_{=} e_3$ and $e_1 \nearrow e_3^*$. Since all clocks are iscm's of each other by (7)-(8)-(9), we have that

$$a_3(e_3) = a_1(e_1) + \delta^i(a_1, a_2) + \delta^i(a_2, a_3)$$
$$a_3(e_3^*) = a_1(e_1) + \delta^i(a_1, a_3)$$

Proposition 10 says that $a(e_3^*) \leq a(e_3)$, so

$$a_1(e_1) + \delta^i(a_1, a_3) \leq a_1(e_1) + \delta^i(a_1, a_2) + \delta^i(a_2, a_3)$$

which can be simplified to (13).

■

Proposition 12. *For any three distinct inertial comovers a, b and c, the clock b is between a and c iff a can send a light signal to c through b.*

Assumptions:
AxExt
??
AxChronology
Proposition 10
Proposition 4

$$\forall a_0 (\forall a, b, c \in \text{Space}_{a_0})$$
$$a \neq b \neq c \wedge \text{B}(a, b, c) \leftrightarrow \exists e_a, e_b, e_c (e_a \mathcal{E} a \wedge e_b \mathcal{E} b \wedge e_c \mathcal{E} c \wedge \overrightarrow{e_a e_b e_c})$$

Proof. $\Leftarrow$: Since we have iscm observers, and of course by AxExt, we have

$$\begin{aligned} c(e_c) &= a(e_a) + \delta^i(a, b) + \delta^i(a, c) && \text{by } e_a \nearrow e_b \nearrow e_c \\ &= a(e_a) + \delta^i(a, c) && \text{by } e_a \nearrow e_c \end{aligned}$$

therefore $\delta^i(a, b) + \delta^i(b, c) = \delta^i(a, c)$.

The $\Rightarrow$ comes from the idea of the unique signalling Thm. 4; the assumption that there is no $\overrightarrow{e_a e_b e_c}$ while $\delta^i(a, b) + \delta^i(b, c) = \delta^i(a, c)$ leads to forbidden triangles as it is depicted on Fig. 10

■

Assumptions:
(10)
(11)
(9)

Proposition 13. *No clock has two different inertial synchronized comovers at the same event.*

$$(\forall a \in \text{In}) \forall e (\forall a_1, a_2 \in \text{D}_e) \quad a_1 \overset{\text{syn}}{\uparrow\uparrow} a \overset{\text{syn}}{\uparrow\uparrow} a_2 \Rightarrow a_1 = a_2 \tag{15}$$

Proof. Let $e \in \mathrm{wline}_{a_1} \cap \mathrm{wline}_{a_2}$ be arbitrary but fixed. Let a_1 and a_2 be inertial comovers of a occurring at e. By (9), $a_1 \overset{\mathrm{syn}}{\uparrow\uparrow} a_2$. By the proof of (10) we know that $\delta^i(a_1, e) = \delta^i(a_2, e) = 0$. Since $a_1 \overset{\mathrm{syn}}{\uparrow\uparrow} a_2$ implies comovement, i.e., constant distance, $\delta^i(a_1, a_2) = 0$. By (11), $a_1 = a_2$. ∎

3.2 Geometry

To treat the sets Space_a as n dimensional Euclidean spaces we have two prove that they satisfy the (first-order) axioms of Euclidean geometry. We will use the axiom system of Tarski and Givant [1999]. Let $\forall\mathsf{EG}^n$ denote the set of the universal closures of the axioms of the n dimensional elementary geometry of Tarski and Givant [1999, p. 190.], i.e., the axioms 1 – 7, 8^n, 9^n and $10_2{}^2$, see Table 1.

Let ξ be variable mapping that maps every variable of the language of $\forall\mathsf{EG}$ to a clock variable other than a_0, and let T_ξ be the following translation of the language of $\forall\mathsf{EG}^n$ to the language of SClTh:

$$
\begin{array}{lcl}
\mathrm{T}_\xi(a = b) & \stackrel{\mathrm{def}}{=} & \xi(a) = \xi(b) \\
\mathrm{T}_\xi(B(abc)) & \stackrel{\mathrm{def}}{=} & \delta^i(\xi(a), \xi(b)) + \delta^i(\xi(b), \xi(c)) = \delta^i(\xi(a), \xi(c)) \\
\mathrm{T}_\xi(ab \equiv cd) & \stackrel{\mathrm{def}}{=} & \delta^i(\xi(a), \xi(b)) = \delta^i(\xi(c), \xi(d)) \\
\mathrm{T}_\xi(\neg\varphi) & \stackrel{\mathrm{def}}{=} & \neg\mathrm{T}_\xi(\varphi) \\
\mathrm{T}_\xi(\varphi \wedge \psi) & \stackrel{\mathrm{def}}{=} & \mathrm{T}_\xi(\varphi) \wedge \mathrm{T}(\psi) \\
\mathrm{T}_\xi(\forall a\varphi) & \stackrel{\mathrm{def}}{=} & (\forall \xi(a) \in \mathrm{Space}_{a_0})\mathrm{T}_\xi(\varphi)
\end{array}
$$

Now under the *Tarski axioms for n-dimensional space of inertial clocks* we understand the following set of statements

(AxGeom) $\quad \{\forall a_0 \mathrm{T}_\xi(\varphi) \,:\, \varphi \in \forall\mathsf{EG}^n\}$

Now we prove (AxGeom) to show that $\mathrm{Space}^{\mathfrak{M}}_\alpha$ is an Euclidean space for all $\alpha \in \mathbb{C}$.

Corollary 14 (Axiom 1.). *Reflexivity axiom for equidistance.* Assumptions: (12)

$$\forall a_0 (\forall a, b \in \mathrm{Space}_{a_0}) \quad ab \equiv ba$$

Proof. That comes from the symmetry of δ^i, i.e., from (12). ∎

Corollary 15 (Axiom 2.). *Transitivity of equidistance.* Assumptions: Proposition 8

$$\forall a_0 (\forall a, b, c, d, e, f \in \mathrm{Space}_{a_0})\ (ab \equiv cd \wedge ab \equiv fe) \rightarrow cd \equiv ef$$

Proof. By Proposition 8 this is just the consequence of the transitivity of $=$. ∎

Corollary 16 (Axiom 3.). *Identity axiom for equidistance.* Assumptions: Proposition 11

$$\forall a_0 (\forall abc \in \mathrm{Space}_{a_0}) \quad (ab \equiv cc \rightarrow a = b)$$

Proof. $\delta^i(a_3, a_3) = 0 = \delta^i(a_1, a_2)$ which implies $a_1 = a_2$ according the identity of indiscernibles provided by Proposition 11. ∎

Table 1: Tarski's 11 axioms of elementary geometry

1. $ab \equiv ba$ (Reflexivity for $\equiv$)

2. $(ab \equiv pq \wedge ab \equiv rs) \rightarrow pq \equiv rs$ (Transitivity for $\equiv$)

3. $ab \equiv cc \rightarrow a = b$ (Identity for $\equiv$)

4. $\exists x(B(qax) \wedge ax \equiv bc)$ (Segment Construction)

5. $(a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge$

$\wedge\, ad \equiv a'd' \wedge bd \equiv b'd') \rightarrow cd \equiv c'd'$ (Five-segment)

6. $B(aba) \rightarrow a = b$ (Identity for B)

7. $(B(apc) \wedge B(bqc)) \rightarrow \exists x(B(pxb) \wedge B(qxa))$ (Pasch)

8^n. $\exists a, b, c, p_1, \ldots p_{n-1} \Bigg(\bigwedge_{i<j<n} p_i \neq p_j \wedge \bigwedge_{1<i<n} (ap_1 \equiv ap_i \wedge bp_1 \equiv bp_i \wedge cp_1 \equiv cp_i) \wedge$

$\wedge \neg(B(abc) \vee B(bca) \vee B(cab)) \Bigg)$ (Lower n-dimension)

9^n. $\Bigg(\bigwedge_{i<j<n} p_i \neq p_j \wedge \bigwedge_{1<i<n} (ap_1 \equiv ap_i \wedge bp_1 \equiv bp_i \wedge cp_1 \equiv cp_i) \Bigg) \rightarrow$

$\rightarrow (B(abc) \vee B(bca) \vee B(cab))$ (Upper n-dimension)

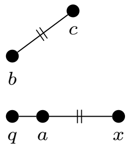
10_2. $B(abc) \vee B(bca) \vee B(cab) \vee \exists x(ax \equiv bx \wedge ax \equiv cx)$ (Circumscribed tr.)

11. $\exists a \forall x, y(\alpha \wedge \beta \rightarrow B(axy)) \rightarrow \exists b \forall x, y(\alpha \wedge \beta \rightarrow B(aby))$ (Continuity scheme)

where α and β are first-order formulas, the first of which does not contain any free occurrences of a, b and y and the second any free occurrences of a, b, x.

Corollary 17 (Axiom 4.). *Axiom of segment construction.*

Assumptions:
AxPing
AxRays
Proposition 5



$$\forall a_0 (\forall a,b,c,q \in \mathrm{Space}_{a_0})(\exists x \in \mathrm{Space}_{a_0})\ (bc \equiv ax \wedge \mathrm{B}(q,a,x))$$

Proof. Let $e_q \mathcal{E} q$ be arbitrary but fixed. By AxPing, there is an $e_a \mathcal{E} a$. s.t. $e_a \rightsquigarrow e_q$. By AxRays there is an e_x for which $\overrightarrow{e_d e_a e_q}$ and $\delta^i(a, e_x) = \delta^i(b,c)$. Proposition 5 then provides the desired iscm x in e_x. ∎

Corollary 18 (Axiom 5.). *Five-segment axiom.*

Assumptions:
AxPing
Proposition 12
Ax5Segment

$$\forall a_0 (\forall a,a',b,b',c,c',d,d' \in \mathrm{Space}_{a_0})$$
$$(a \neq b \wedge \mathrm{B}(a,b,c) \wedge \mathrm{B}(a',b',c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge$$
$$\wedge\, ad \equiv a'd' \wedge bd \equiv b'd') \rightarrow cd \equiv c'd'$$

Proof. Suppose that we have the inertial comovers a,b,c,d,a',b',c',d' with the properties described by the premise. By AxPing, b can ping a and c and b' can ping a' and c' s.t. the receiving event of the ping of a is the same as the sending event of the ping of c. By the equivalence of betweenness' provided by Proposition 12, all these events are on a lightline, therefore they satisfies the conditions of Ax5Segment, which provide the conclusion that $\delta^i(c,d) = \delta^i(c',d')$. ∎

Corollary 19 (Axiom 6.). *Identity axiom for betweenness.*

Assumptions:
Proposition 11

$$(\forall a,b \in \mathrm{Space}_{a_0})\ \big(\mathrm{B}(a,b,a) \rightarrow a = b\big)$$

Proof. $\delta^i(a,b) + \delta^i(b,a) = \delta^i(a,a) = 0$ and the fact that $\delta^i(a,b) \geq 0$ implies that $\delta^i(a,b) = 0$. By the identity of indiscernibles provided by Proposition 11, $a = b$. ∎

Corollary 20 (Axiom 7.). *Inner form of the Pasch axiom.*

Assumptions:
AxPing
Proposition 12
AxPasch

$$\forall a_0 (\forall a,b,c,p,q \in \mathrm{Space}_{a_0})\ \big((\mathrm{B}(a,p,c) \wedge \mathrm{B}(b,q,c)) \rightarrow$$
$$\rightarrow (\exists d \in \mathrm{Space}_{a_0})(\mathrm{B}(p,d,b) \wedge \mathrm{B}(q,d,a))\big)$$

Proof. Suppose that a,b,c,p,q satisfies the premise. Then by AxPing a and b can ping an event of c in a way that these light signals will cross the wordline of p and q (the latter is provided by Proposition 12). Now AxPasch provides the existence of the desired clock x. ∎

Corollary 21 (Axiom 8^n.). *Lower n-dimensional axiom:* UNDER CONSTRUCTION

Proof. UNDER CONSTRUCTION ∎

Corollary 22 (Axiom 9^n.). *Upper n-dimensional axiom:* UNDER CONSTRUCTION

Proof. UNDER CONSTRUCTION ∎

Corollary 23 (Axiom 10_2.). *Every triangle can be circumscribed:*

Assumptions:
Proposition 12, AxCircle

$$\mathrm{B}(a,b,c) \vee \mathrm{B}(b,c,a) \vee \mathrm{B}(c,a,b) \vee \exists x (ax \equiv bx \wedge ax \equiv cx)$$

[2] Here we used some results of **?** and **?**: we used axioms 7, 6 and 10_2 instead of 7_1, 15 and 10_2, respectively.

Proof. It is enough to prove that $\exists x(ax \equiv bx \land ax \equiv cx)$ whenever $\mathrm{B}(a,b,c) \lor \mathrm{B}(b,c,a) \lor \mathrm{B}(c,a,b)$ is false. Suppose that this is false. Then by Proposition 12, they a, b and c can not be connected with a lightline. Therefore by AxCircle, there is an x s.t. this x has the same signalling distance from a, b and c, and that is what we needed. ∎

Corollary 24 (Axiom 11.). *Tarski's axiom scheme of continuity (p.185.)*

Assumptions: AxReals

$$\forall a_0(\exists a \in \mathrm{Space}_{a_0})(\forall c,d \in \mathrm{Space}_{a_0})(\varphi(c) \land \psi(d) \to \mathrm{B}(a,c,d)) \to \\ \to (\exists b \in \mathrm{Space}_{a_0})(\forall c,d \in \mathrm{Space}_{a_0})(\varphi(c) \land \psi(d) \to \mathrm{B}(c,b,d))$$

Proof. This comes from the continuity (or infimum-supremum) scheme of the real closed fields: The transition of that scheme to events is granted by AxRays, and the existence of the specific point through the event is granted by Proposition 5. ∎

3.3 Coordinatization

Theorem 25 (Coordinatization)**.** *For arbitrary coordinatesystem, the coordinatization function is a bijection between W and U^4. In other words, given an arbitrary but fixed coordinate system, the following statements are true:*

Totality *Every event is coordinatized with a 4-tuple.*

Surjectivity *Every 4-tuple is a coordinate of an event.*

Functionality *No event has two different coordinates.*

Injectivity *No 4-tuple is a coordinatization of 2 different events.*

Proof. Let a, a_x, a_y, a_z be an arbitrary but fixed coordinate system.

Totality Every event is coordinatized with a 4-tuple. Let e be an arbitrary event. By Proposition 5, we have a synchronized comover a_e of a in e. Then by definition, $a_e(e)$ will be the time coordinate. We can use Tarski's axioms to conclude that there are (unique) a'_x, a'_y and a'_z that are projections of the point a_e to the lines (a, a_x), (a, a_y) and (a, a_z), respectively. By (AxPing), these projections can ping a_e, i.e., they can measure the spatial distance between them and a_e (and e), and thus we will have the spatial coordinates of e as well.

Surjectivity Every 4-tuple is a coordinate of an event. Let (t, x, y, z) be an arbitrary 4-tuple. It follows from Tarski's axioms that there are planes there are inertial comovers a'_x, a'_y and a'_z of a on the axes (a, a_x), (a, a_y) and (a, a_z), respectively, such that $\delta^i(a, a_x) = x$, $\delta^i(a, a_y) = y$ and $\delta^i(a, a_t) = t$. For all $i \in \{x, y, z\}$ Let P_i denote the plane that contains a'_i and is orthogonal to the line (a, a_i). Then by Tarski's axioms, these planes has one (unique) intersection, a_e. By the definition of the Coord, any event of wline_{a_e} are coordinatized on the spatial coordinates (x, y, z). Now we know from (**??**) that there is an event e of wline_{a_e} such that $a(e) = t$.

Functionality No event has two different coordinates. In the proof of Totality, a_e is unique by Proposition 13. After that, as we noted above, Tarki's axioms provided the uniqueness of the projections as well, and this is enough for the uniqueness of the coordinates.

Injectivity No 4-tuple is a coordinate of 2 different events. From the proof of surjectivity we saw that a_e was unique. But for a given t, the e is also unique by (**??**). ∎

3.4 Simplifying SpecRel

Note that a lot of physical quantities can be defined without referring to coordinate systems. Spatial distance, elapsed time and speed are nice examples of that. Here we are going to define these concepts and prove that they are indeed equivalent with the usual spacetime diagram-based definitions. These proofs will allow to identify the monstrous axioms of SpecRel with the lightweight propositions what we will call "Simple-SpecRel" in Section **??**.

Definition 14 (spatial distance)**.** We say that the spatial distance between events e and e' according to an inertial clock a is τ iff

$$\mathrm{sd}_a(e,e') = \tau \overset{\text{def}}{\Leftrightarrow} (\exists a' \in \mathrm{Space}_a)(a \in \mathrm{D}_e \wedge \delta^i(a,e') = \tau)$$

Proposition 26. $\mathrm{sd}_a(,)$ *is a total function for all* a*.*

Proof. There is such a' by Proposition 5, and this a' is unique by Proposition 13. ∎

Proposition 27. *The spacetime diagram-based definition of spatial distance and our definition are the same.*

$$\begin{aligned}\mathrm{sd}_a(e,e') = \tau \iff & (\exists \langle a_x, a_y, a_z\rangle \in \mathrm{CoordSys(a)})\exists \vec{x}\vec{y} \\ & \mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \wedge \mathrm{Coord}_{a,a_x,a_y,a_z}(e') = \vec{y} \wedge \tau = |\vec{x}_{2-4} - \vec{y}_{2-4}|\end{aligned}$$

Proof. By Tarski's axioms of geometry, this is just Pythagoras's theorem:[3]

$$\delta^i(a_e,a_{e'})^2 = \delta^i(a_e,b)^2 + \delta^i(b,a_{e'})^2$$

where $b \in \mathrm{Space_a}$ is a clock with which

$$\mathrm{Ort}(a'_x,a,b) \wedge \mathrm{Ort}(a'_y,a,b) \wedge \mathrm{Ort}(a'_z,a,b)$$

where a'_x, a'_y, a'_z, are the projections of a_e to the axes of the coordinate system (See Fig. 5). ∎

Definition 15 (elapsed time)**.** We say that the elapsed time between events e and e' according to an inertial clock a is τ iff

$$\mathrm{et}_a(e,e') = \tau \overset{\text{def}}{\Leftrightarrow} (\exists b,b' \in \mathrm{Space}_a)\ |b(e) - b'(e')| = \tau$$

Proposition 28. $\mathrm{et}_a(,)$ *is a total function for all* a*.*

Proof. That is true by the same reasons as Proposition 26. ∎

Proposition 29. *The spacetime diagram-based definition of elapsed time and our definition are the same.*

$$\begin{aligned}\mathrm{et}_a(e,e') = \tau \iff & (\exists \langle a_x, a_y, a_z\rangle \in \mathrm{CoordSys(a)})\exists \vec{x},\vec{y} \\ & \mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \wedge \mathrm{Coord}_{a,a_x,a_y,a_z}(e') = \vec{y} \wedge \tau = |\vec{x}_1 - \vec{y}_1|\end{aligned}$$

[3] UNDER CONSTRUCTION

Proof. The clocks that measures the time in the events are the same in both definitions by Proposition 13, so practically, both formula refer to the same measurement. ■

Definition 16 (speed)**.** Speed is defined using the standard $v = \frac{\Delta s}{\Delta t}$ formula:

$$\mathrm{v}_a(e, e') \stackrel{\text{def}}{=} \frac{\mathrm{sd}_a(e, e')}{\mathrm{et}_a(e, e')}$$

3.5 Proving '**Simple-**SpecRel'

The following theorems are important because of their resemblance to the axioms of SpecRelComp.

During the proofs we follow the notation of the definition of coordinatization predicate, e.g., we always refer to the inertial synchronized co-mover a that witness the event e by a_e.

Proposition 30 (Simple-AxSelf)**.**

$$\forall a(\forall e \in \mathrm{wline}_a)(\forall\langle a_x, a_y, a_z\rangle \in \mathrm{CoordSys(a)})\exists t\ \mathrm{Coord}_{a,a_x,a_y,a_z}(e) = (t, 0, 0, 0)$$

(Note that t here is exactly $a(e)$.)

Proof. No matter how we choose a_x, a_y or a_z, the clock a_e can be chosen to be a itself, since $e \in \mathrm{wline}_a$. Since a is on all the axes (a, a_x), (a, a_y), (a, a_z), the distance of a from these lines are all 0, and $a(e)$ will be t. ■

Proposition 31 (Simple-AxPh)**.**

$$(\forall a \in \mathrm{In})\forall e, e'\ (\mathrm{v}_a(e, e') = 1 \leftrightarrow e \nearrow e')$$

Proof. The $\leftarrow$ direction is trivial by the definition of $\mathrm{sd}(,)$ and $\mathrm{et}(,)$; they produce the same number for lightlike related events. For the other direction, if $\mathrm{v}_a(e, e') = 1$, then take an iscm a' into e'. This a' can ping e', so there will be an event e'' on $\mathrm{wline}_{a'}$ such that $e \nearrow e''$. By 4, $e' = e''$. ■

Proposition 32 (Simple-AxEv)**.**

$$\forall e(\forall\langle a, a_x, a_y, a_z\rangle, \langle a', a'_x, a'_y, a'_z\rangle \in \mathrm{CoordSys})$$
$$\exists\vec{x}\ \mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \rightarrow \exists\vec{y}\ \mathrm{Coord}_{a',a'_x,a'_y,a'_z}(e) = \vec{y}$$

Proof. That is true by the totality of coordinatization, i.e., by Proposition 25. ■

Proposition 33 (Simple-AxSym)**.**

$$(\forall a, a' \in \mathrm{In})\forall e, e'\ (\mathrm{et}_a(e, e') = \mathrm{et}_{a'}(e, e') = 0 \rightarrow \mathrm{sd}_a(e, e') = \mathrm{sd}_{a'}(e, e'))$$

Proof. Here the local experimenters of a and a' coincide by Proposition 13. ■

Proposition 34 (Simple-AxThExp)**.**

$$\forall a\forall e, e' \quad (\mathrm{v}_a(e, e') < 1 \rightarrow (\exists a' \in \mathrm{In})e, e' \in \mathrm{wline}_{a'})$$

Proof. From AxPing, Propositions 5 and 4 and from the premise we have that there is an event $e'' \nwarrow e$ and a clock $a_{e'} \in \mathrm{D}_{e'} \cap \mathrm{D}_{e''}$. Now this event is in the chronological future of the causal future of e, so by AxChronology, it is in the chronological future of e as well. AxSecant then provides the existence of the desired inertial clock. ■

4 Geodetic-Inertial equivalence

Definition 17 (Geodetic)**.** *Geodetic* clocks are the fastest clocks between any two events on their worldline.

$$\mathrm{Geo}(a) \overset{\text{def}}{\Leftrightarrow} (\forall e, e' \in \mathrm{wline}_a)(\forall b \in \mathrm{D}_e \cap \mathrm{D}_{e'})|a(e) - a(e')| \geq |b(e) - b(e')|$$

UNDER CONSTRUCTION

5 Appendix: Definitional Equivalence of $\mathrm{SClTh}_{\mathrm{NoAcc}}$ and SpecRelComp

5.1 Language of SpecRelComp

Definition 18 (Language of SpecRelComp)**.**

- Body sort:
  - Body variables: $b_1, b_2, \dots \in BVar$
  - Body predicates: $\mathrm{Ob}, \mathrm{IOb}, \mathrm{Ph}$
- Mathematical sort:
  - Mathematical variables: $x, y, z, \dots \in MVar$
  - Mathematical functions: $+, \cdot$
  - Mathematical predicate: $\leq$
- Connection between sorts:
  - Intersort predicate: W
- Mathematical terms:

$$\tau ::= x \mid \mathsf{r} \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$$

- Formulas:

$$\begin{aligned}\varphi ::= b = b' \mid \tau_1 = \tau_2 \mid \tau_1 \leq \tau_2 \\ \mathrm{Ob}(b) \mid \mathrm{IOb}(b) \mid \mathrm{Ph}(b) \mid \mathrm{W}(b, b', \tau_t, \tau_x, \tau_y, \tau_z) \\ \neg\varphi \mid \varphi \wedge \psi \mid \exists b\varphi \mid \exists x\varphi\end{aligned}$$

5.2 Axioms of SpecRelComp

UNDER CONSTRUCTIONFor axioms and models of SpecRelComp of the axiom system $\mathsf{SpecRel}\cup\mathsf{Comp}$ see [Andréka et al. 2007]. (Note that the language of that paper contains one more sort for events. This, however, is definable, for more details on that see [Andréka et al. 2001], or, since we have to define it anyway to prove the definitional equivalence with SClTh, see the proof of Thm. 35 on p. 25.

5.3 Definitional equivalence with SpecRelComp

5.3.1 Plan

Theorem 35. SpecRelComp *and* SClTh *are definitionally equivalent, i.e., there are translations*

$$\begin{aligned}\mathrm{STC}_\xi &: \mathcal{L}_{\mathrm{SpecRelComp}} \to \mathcal{L}_{\mathrm{SClTh}} \\ \mathrm{CTS}_\zeta &: \mathcal{L}_{\mathrm{SClTh}} \to \mathcal{L}_{\mathrm{SpecRelComp}}\end{aligned}$$

and model-transformations

$$\begin{aligned}\mathrm{stc} &: \mathrm{Mod}(\mathrm{SpecRelComp}) \to \mathrm{Mod}(\mathrm{SClTh}) \\ \mathrm{cts} &: \mathrm{Mod}(\mathrm{SClTh}) \to \mathrm{Mod}(\mathrm{SpecRelComp})\end{aligned}$$

and assignment transformations f_ρ *and* g_ϱ *such that the followings hold for all* $\mathfrak{M}_s \in \mathrm{Mod}(\mathrm{SpecRelComp})$ *and* $\mathfrak{M}_c \in \mathrm{Mod}(\mathrm{SClTh})$ *and for any* $\varphi_s \in \mathcal{L}_{\mathrm{SpecRelComp}}$*,* $\varphi_c \in \mathcal{L}_{\mathrm{SClTh}}$*:*

$$\mathrm{stc}(\mathfrak{M}_s) \models \varphi_c \quad [\eta] \iff \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi_c)[f_\rho(\eta)] \tag{16}$$

$$\mathfrak{M}_c \models \mathrm{STC}_\zeta(\varphi_s)[g_\varrho(\mu)] \iff \mathrm{cts}(\mathfrak{M}_c) \models \varphi_s \quad [\mu] \tag{17}$$

$$\mathrm{SClTh} \vdash \varphi_c \implies \mathrm{SpecRelComp} \vdash \mathrm{CTS}_\xi(\varphi_c) \tag{18}$$

$$\mathrm{SpecRelComp} \vdash \varphi_s \implies \mathrm{SClTh} \vdash \mathrm{STC}_\zeta(\varphi_s) \tag{19}$$

Proof. 1. **Definition of** stc. Let

$$\mathfrak{M}_s = \left(B, \mathrm{IOb}^{\mathfrak{M}_s}, \mathrm{Ph}^{\mathfrak{M}_s}, \mathfrak{Q}, \mathrm{W}^{\mathfrak{M}_s}\right)$$

be an arbitrary but fixed model of SpecRel. We will introduce the transformation stc : Mod(SpecRel) $\to$ Mod(CTh), i.e., we will construct the corresponding CTh model $\mathrm{stc}(\mathfrak{M})$ from the information that $\mathfrak{M}$ contains. Such a CTh model will be given as

$$\mathrm{stc}(\mathfrak{M}_s) \stackrel{\mathrm{def}}{=} \left(\mathrm{stc}_W(\mathfrak{M}_s), \mathrm{stc}_\prec(\mathfrak{M}_s), \mathrm{IOb}^{\mathfrak{M}_s}, \mathfrak{Q}, \mathrm{stc}_\mathrm{P}(\mathfrak{M}_s)\right)$$

where the three undefined entity are the domain of events, the causality relation and the meaning of the pointing relation, respectively.

(a) **The event domain** $\mathrm{stc}_W(\mathfrak{M}_s)$**.** The idea is that an event will be identified as the set of bodies occurring *there*. To express the word 'there' in SpecRelComp, we have to use the worldview predicate W with parameters. To *name* the elements of the universe of the defined sort, we will use sets defined with 5 parameters:

$$ev_{o,t,x,y,z} \stackrel{\mathrm{def}}{=} \{b \in B \, : \, (o, b, t, x, y, z) \in \mathrm{W}^{\mathfrak{M}_s}\}$$

But we know that the *same* event can occur in different observers' different coordinate points. So we factorize over that set with the following equivalence relation.

$$\langle o_1, t_1, x_1, y_1, z_1\rangle \stackrel{e}{\simeq} \langle o_2, t_2, x_2, y_2, z_2\rangle \stackrel{\mathrm{def}}{\Leftrightarrow} \mathrm{w}^{\mathfrak{M}_s}_{o_1 o_2}(t_1, x_1, y_1, z_1) = (t_2, x_2, y_2, z_2)$$

Where $\mathrm{w}^{\mathfrak{M}_s}$ is the meaning of the worldview transformation defined in SpecRelComp. Now we are ready to define the universe of $\mathrm{stc}(\mathfrak{M}_s)$:

$$\mathrm{stc}_W(\mathfrak{M}_s) \stackrel{\mathrm{def}}{=} \{\langle o, t, x, y, z\rangle / \stackrel{e}{\simeq} \, : \, o \in \mathrm{IOb}^{\mathfrak{M}_s} \wedge t, x, y, z \in Q\}$$

(b) **The causality relation** $\mathrm{stc}_\prec(\mathfrak{M}_s)$ We use the usual definition of Minkowski distance, which is easily definable in SpecRelComp

$$\mu(\vec{x}, \vec{y}) \stackrel{\mathrm{def}}{\Leftrightarrow} (\vec{x}_1 - \vec{y}_1)^2 - (\vec{x}_2 - \vec{y}_2)^2 - (\vec{x}_3 - \vec{y}_3)^2 - (\vec{x}_4 - \vec{y}_4)^2$$

$$\mathrm{stc}_\prec(\mathfrak{M}_s) \stackrel{\mathrm{def}}{=} \Big\{ \langle (o, \vec{x})/\stackrel{e}{\simeq}, (o', \vec{x}')/\stackrel{e}{\simeq}\rangle \in \mathrm{stc}_W(\mathfrak{M}_s)^2 \, : \, \mu^{\mathfrak{M}_s}(\mathrm{w}^{\mathfrak{M}_s}_{oo'}(\vec{x}), \vec{x}') \geq 0 \text{ and } \left(\mathrm{w}^{\mathfrak{M}_s}_{oo'}(\vec{x})\right)_1 < \vec{x}'_1 \Big\}$$

(c) **Meaning of pointing** $\mathrm{stc_P}(\mathfrak{M}_s)$ Pointing statements comes straight from the worldview-transformation:

$$\mathrm{stc_P}(\mathfrak{M}_s) \stackrel{\mathrm{def}}{=} \Big\{\langle (o,t,x,y,z)/\simeq, o', t'\rangle \;:\; \mathrm{w}^{\mathfrak{M}_s}_{oo'}(t,x,y,z) = (t',0,0,0)\Big\}$$

2. **Definition of** ξ. Note that to quantify over $\mathrm{stc}_W(\mathfrak{M}_s)$, it is enough if we can quantify over the representants, i.e., over $\mathrm{IOb} \times Q^4$. But to do so, we'll need variables. For mathematical variables we map mathematical variables, for event variable e, we map a 5-tuples of ***different*** variables $(b, x_t, x_x, x_y, x_z) \in Var_b \times Var_m^4$, and for clock variable c we body variables $b \in Var_b$ in a way that no variable will be the representative of two different variables[4]:

$$\xi : \begin{array}{rcl} x_i & \mapsto & x_{5i} \\ a_i & \mapsto & b_{2i} \\ e_i & \mapsto & \langle b_{2i+1}, x_{5i+1}, x_{5i+2}, x_{5i+3}, x_{5i+4}\rangle \end{array}$$

For mathematical terms we define $\hat{\xi}$ to be the induced substitution based on ξ:

$$\begin{array}{rcl} \hat{\xi}(x) & \stackrel{\mathrm{def}}{=} & \xi(x) \\ \hat{\xi}(\tau+\tau') & \stackrel{\mathrm{def}}{=} & \hat{\xi}(\tau)+\hat{\xi}(\tau') \\ \hat{\xi}(\tau\cdot\tau') & \stackrel{\mathrm{def}}{=} & \hat{\xi}(\tau)\cdot\hat{\xi}(\tau') \end{array}$$

3. **Definition of** CTS_ξ

$$\begin{array}{lcl} \mathrm{CTS}_\xi(e=e') & \stackrel{\mathrm{def}}{=} & \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) = \xi_{2-5}(e') \\ \mathrm{CTS}_\xi(e\prec e') & \stackrel{\mathrm{def}}{=} & \mu\Big(\mathrm{w}_{\xi_1(e)\xi_1(e')}\big(\xi_{2-5}(e)\big), \xi_{2-5}(e')\Big) \geq 0 \,\wedge \\ & & \quad\wedge\, \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) \neq \xi_{2-5}(e') \\ \mathrm{CTS}_\xi(\tau=\tau') & \stackrel{\mathrm{def}}{=} & \hat{\xi}(\tau) = \hat{\xi}(\tau') \\ \mathrm{CTS}_\xi(\tau\leq\tau') & \stackrel{\mathrm{def}}{=} & \hat{\xi}(\tau) \leq \hat{\xi}(\tau') \\ \mathrm{CTS}_\xi(\mathrm{P}(e,a,\tau)) & \stackrel{\mathrm{def}}{=} & \mathrm{w}_{\xi_1(e)\xi(a)}(\xi_{2-5}(e)) = (\hat{\xi}(\tau),0,0,0) \\ \mathrm{CTS}_\xi(\neg\varphi) & \stackrel{\mathrm{def}}{=} & \neg\mathrm{CTS}_\xi(\varphi) \\ \mathrm{CTS}_\xi(\varphi\wedge\psi) & \stackrel{\mathrm{def}}{=} & \mathrm{CTS}_\xi(\varphi)\wedge\mathrm{CTS}_\xi(\psi) \\ \mathrm{CTS}_\xi(\exists e\varphi) & \stackrel{\mathrm{def}}{=} & \exists\xi_1(e)\exists\xi_2(e)\exists\xi_3(e)\exists\xi_4(e)\exists\xi_5(e)\mathrm{CTS}_\xi(\varphi) \\ \mathrm{CTS}_\xi(\exists a\varphi) & \stackrel{\mathrm{def}}{=} & \exists\xi(a)\big(\mathrm{IOb}(v(a))\wedge\mathrm{CTS}_\xi(\varphi)\big) \\ \mathrm{CTS}_\xi(\exists x\varphi) & \stackrel{\mathrm{def}}{=} & \exists\xi(x)\mathrm{CTS}_\xi(\varphi) \end{array}$$

[4]The latter is an important constraint: suppose that $\xi_1(e_1) = \xi(a_1)$, i.e., the variable b_1 represents an inertial observer and a maybe different observer that coordinatizes the event e_1 in $\xi_{2,5}(e_1)$. It is easy to find a model of SpecRelComp with an assignment such that these two observers are different. Then their difference will not be expressible since the transformated assignment η will not be able differentiate between them, since we used the same variable b_1 to represent 'them'. This failure could be conjectured also syntactically, if we imagine the situation when we try to translate a formula $\exists e_1\exists a_1\varphi$, because that would result in a formula that starts with

$$\exists\xi_1(e_1)\exists\xi_2(e_1)\exists\xi_3(e_1)\exists\xi_4(e_1)\exists\xi_5(e_1)\exists\xi(a_1)(\ldots\mathrm{CTS}_\xi(\varphi)\ldots)$$

but here, by $\xi_1(e_1) = \xi(a_1)$, we have a *vacuous quantification*, which was not present in $\exists e_1\exists a_1\varphi$. The position in the proof when we will refer to that 'well-separated' property of ξ is when we are going to discuss the formulas $\exists e\varphi$.

4. **Definition of the assignment transformation** f_ρ Let ρ be an arbitrary choice function that chooses one representant from every equivalence class of $\mathrm{stc}_W(\mathfrak{M}_c)$, i.e., ρ satisfies the equation

$$\rho(\langle o,t,x,y,z\rangle/\overset{e}{\simeq}) \overset{e}{\simeq} \langle o,t,x,y,z\rangle \tag{20}$$

Now we define f_ρ to fit to ξ:

$$f_\rho(\eta): \begin{array}{rcl} b_{2i} & \mapsto & \eta(a_i) \\ b_{2i+1} & \mapsto & \rho_1 \circ \eta(e_i) \\ x_{5i} & \mapsto & \eta(x_i) \\ x_{5i+n} & \mapsto & \rho_{n+1} \circ \eta(e_i) \text{ for any } n \in \{1,2,3,4\} \end{array}$$

Now by the construction we have that

$$f_\rho(\eta) \circ \xi(e)/\overset{e}{\simeq} = \eta(e) \tag{21}$$

where we used the abbreviation

$$f_\rho(\eta)(\vec{v}) \overset{\text{def}}{=} \langle f_\rho(\eta)(\vec{v}_1), f_\rho(\eta)(\vec{v}_2), f_\rho(\eta)(\vec{v}_3), f_\rho(\eta)(\vec{v}_4), f_\rho(\eta)(\vec{v}_5)\rangle$$

By the construction of $f_\rho(\eta)$ we also have the equations

$$f_\rho(\eta) \circ \xi(a) = \eta(a) \tag{22}$$

and

$$f_\rho(\eta) \circ \xi(x) = \eta(x),$$

and if we take the natural extension $\hat{\eta}$ of the assignment function η for terms, i.e.,

$$\begin{array}{rcl} \hat{\eta}(x) & \overset{\text{def}}{=} & \eta(x) \\ \hat{\eta}(\tau+\tau') & \overset{\text{def}}{=} & \hat{\eta}(\tau) +^{\mathfrak{M}_s} \hat{\eta}(\tau') \\ \hat{\eta}(\tau\cdot\tau') & \overset{\text{def}}{=} & \hat{\eta}(\tau) \cdot^{\mathfrak{M}_s} \hat{\eta}(\tau') \end{array}$$

and we can generalize the above equation to

$$\widehat{f_\rho(\eta)} \circ \hat{\xi}(\tau) = \hat{\eta}(\tau) \tag{23}$$

5. **Proof of the equivalence** (16)

$$\mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi_c)[f_\rho(\eta)] \iff \mathrm{stc}(\mathfrak{M}_s) \models \varphi_c[\eta]$$

We prove this by induction on φ_c. Notice that the proof itself is of logical in nature; the proof goes through because the model-construction and the translations are defined to fit to each other. (So)

- $\varphi_c = e = e'$

$$\begin{array}{rll} & \mathfrak{M}_s \models \mathrm{CTS}_\xi(e=e')[f_\rho(\eta)] & \\ \iff & \mathfrak{M}_s \models \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) = (\xi_{2-5}(e'))[f_\rho(\eta)] & \text{def.of } \mathrm{CTS}_\xi \\ \iff & \mathrm{w}^{\mathfrak{M}_s}_{f_\rho(\eta)\circ\xi_1(e), f_\rho(\eta)\circ\xi_1(e')}(f_\rho(\eta)\circ\xi_{2-5}(e)) = f_\rho(\eta)\circ\xi_{2-5}(e') & \\ \iff & ev_{f_\rho(\eta)\circ\xi(e)} \overset{e}{\simeq} ev_{f_\rho(\eta)\circ\xi(e')} & \text{def.of } \overset{e}{\simeq} \\ \iff & \mathrm{stc}(\mathfrak{M}_s) \models e=e'[\eta] & (21) \end{array}$$

- $\varphi_c = e \prec e'$ is similar to $e = e'$.
- $\varphi_c = \tau = \tau'$

$$\begin{array}{lll} & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\tau = \tau')[f_\rho(\eta)] & \\ \iff & \mathfrak{M}_s \models \hat{\xi}(\tau) = \hat{\xi}(\tau')[f_\rho(\eta)] & \text{def.of CTS}_\xi \\ \iff & \mathfrak{M}_s \models \hat{\xi}(\tau) = \hat{\xi}(\tau')[f_\rho(\eta)] & \text{def.of } \hat{\xi} \\ \iff & \widehat{f_\rho(\eta)} \circ \hat{\xi}(\tau) = \widehat{f_\rho(\eta)} \circ \hat{\xi}(\tau') & \text{def.of} \models \\ \iff & \hat{\eta}(\tau) = \hat{\eta}(\tau') & (23) \\ \iff & \mathrm{stc}(\mathfrak{M}_s) \models \tau = \tau'[\eta] & \text{def.of} \models \end{array}$$

- $\varphi_c = \tau \leq \tau'$ is similar to $\tau = \tau'$
- $\varphi_c = \mathrm{P}(e, a, \tau)$

$$\begin{array}{lll} & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\mathrm{P}(e,a,\tau))[f_\rho(\eta)] & \\ \iff & \mathfrak{M}_s \models \mathrm{w}_{\xi_1(e)\xi(a)}(\xi_{2-5}(e)) = (\hat{\xi}(\tau), 0, 0, 0)[f_\rho(\eta)] & \text{def.of CTS}_\xi \\ \iff & \mathrm{w}^{\mathfrak{M}_s}_{f_\rho(\eta)\circ\xi_1(e), f_\rho(\eta)\circ\xi(a)}(f_\rho(\eta) \circ \xi_{2-5}(e)) = (\widehat{f_\rho(\eta)} \circ \hat{\xi}(\tau), 0, 0, 0) & \text{def.of} \models \\ \iff & \mathrm{w}^{\mathfrak{M}_s}_{f_\rho(\eta)\circ\xi_1(e), \eta(a)}(f_\rho(\eta) \circ \xi_{2-5}(e)) = (\hat{\eta}(\tau), 0, 0, 0) & (22), (23) \\ \iff & f_\rho(\eta) \circ \xi(e) \stackrel{e}{\simeq} \langle \eta(a), \hat{\eta}(\tau), 0, 0, 0 \rangle & \text{def.of } \stackrel{e}{\simeq} \\ \iff & \eta(e) \stackrel{e}{\simeq} \langle \eta(a), \hat{\eta}(\tau), 0, 0, 0 \rangle & (21) \\ \iff & \langle \eta(e), \eta(a), \hat{\eta}(\tau) \rangle \in \mathrm{stc}_{\mathrm{P}}(\mathfrak{M}_s) & \text{def.of stc}_{\mathrm{P}}(\mathfrak{M}_s) \\ \iff & \mathrm{stc}(\mathfrak{M}_s) \models \mathrm{P}(e, a, \tau)[\eta] & \text{def.of} \models \end{array}$$

- $\varphi_c = \neg\varphi$ and $\varphi_c = \varphi \wedge \psi$ are straightforward.
- $\varphi_c = \exists e \varphi$ Here we will need a lemma:

  Lemma 36.

$$\mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)[\xi(e) \mapsto \langle b, t, x, y, z \rangle]] \iff$$
$$\iff \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e \mapsto \langle b, t, x, y, z \rangle / \stackrel{e}{\simeq}])]$$

Proof. We use the following abbreviations:

$$g \stackrel{\text{def}}{=} f_\rho(\eta)[\xi(e) \mapsto \langle b, t, x, y, z \rangle] \quad \text{and} \quad g' \stackrel{\text{def}}{=} f_\rho(\eta[e \mapsto \langle b, t, x, y, z \rangle / \stackrel{e}{\simeq}])$$

It is clear that

$$g(v) \ = \ g'(v) \text{ for every variable } v \text{ not occurring in } \xi(e) \tag{24}$$

Now we cannot be sure whether $g' \circ \xi(e) = \langle b, t, x, y, z \rangle$, but we know from the equations (20) and (21) that

$$g' \circ \xi(e) \stackrel{e}{\simeq} \langle b, t, x, y, z \rangle. \tag{25}$$

By the construction of ξ, in the formula $\mathrm{CTS}_\xi(\varphi)$ the sole purpose of any variable that occur in $\xi(e)$ is to represent the event e and are not used to represent bodies or numbers; the variables that refers to numbers as numbers and bodies as bodies, are shifted to positions x_{5i} and b_{2i} but no variable of $\xi(e)$ has even indexes by the construction of ξ. This observation will be enough to prove Lemma 36.

We prove by induction on the construction of φ. The observation (24) make every case of this induction trivial in which e does not occur, so it is enough to check the cases $e = e'$, $e' = e$, $e \prec e'$, $e' \prec e$, $\mathrm{P}(e, a, \tau)$, $\exists e \varphi$ where e and e' are different variables.

– $e = e'$:

$$
\begin{array}{lll}
 & \mathfrak{M}_s \models \mathrm{CTS}_\xi(e = e')[g] & \text{assumption} \\
\iff & \mathfrak{M}_s \models \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) = \xi_{2-5}(e')[g] & \text{def.of } \mathrm{CTS}_\xi \\
\iff & \mathrm{w}^{\mathfrak{M}_s}_{g\circ\xi_1(e)g\circ\xi_1(e')}(g \circ \xi_{2-5}(e)) = g \circ \xi_{2-5}(e') & \text{def.of } \models \\
\iff & g \circ \xi(e) \stackrel{e}{\simeq} g \circ \xi(e') & \text{def.of } \stackrel{e}{\simeq} \\
\iff & \langle b, t, x, y, z\rangle \stackrel{e}{\simeq} g' \circ \xi(e') & \text{def.of } g\text{, (24)} \\
\iff & g' \circ \xi(e) \stackrel{e}{\simeq} g' \circ \xi(e') & \text{(25)} \\
\iff & \mathfrak{M}_s \models \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) = \xi_{2-5}(e')[g'] & \text{def.of } \models \\
\iff & \mathfrak{M}_s \models \mathrm{CTS}_\xi(e = e')[g'] & \text{def.of } \mathrm{CTS}_\xi
\end{array}
$$

– $e' = e$, $e \prec e'$, $e' \prec e$ and $\mathrm{P}(e, a, \tau)$ are similar to $e = e'$.

– $\exists e\varphi$:

$$
\begin{array}{lll}
 & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\exists e\varphi)[g] & \text{assumption} \\
\iff & \mathfrak{M}_s \models \exists\xi(e)\mathrm{CTS}_\xi(\varphi)[g] & \text{def.of } \mathrm{CTS}_\xi \\
\iff & (\exists\langle b', t', x', y', z'\rangle \in B \times Q^4) & \\
 & \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[g[\xi(e) \mapsto \langle b', t', x', y', z'\rangle]] & \text{def.of } \models \\
\iff & (\exists\langle b', t', x', y', z'\rangle \in B \times Q^4) & \text{modification } [\xi(e) \mapsto \\
 & \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)[\xi(e) \mapsto \langle b', t', x', y', z'\rangle]] & \text{overwrote the one in} \\
\iff & (\exists\langle b', t', x', y', z'\rangle \in B \times Q^4) & \\
 & \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e \mapsto \langle b', t', x', y', z'\rangle/\stackrel{e}{\simeq}])] & \text{(nested) ind.hip.} \\
\iff & (\exists\langle b', t', x', y', z'\rangle \in B \times Q^4) & \\
 & \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e \mapsto \langle b, t, x, y, z\rangle/\stackrel{e}{\simeq}][e \mapsto \langle b', t', x', y', z'\rangle/\stackrel{e}{\simeq}])] & \text{substitution will be c} \\
\iff & (\exists\langle b', t', x', y', z'\rangle \in B \times Q^4) & \\
 & \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e \mapsto \langle b, t, x, y, z\rangle/\stackrel{e}{\simeq}])[\xi(e) \mapsto \langle b', t', x', y', z'\rangle])] & \text{(nested)ind.hip.} \\
\iff & (\exists\langle b', t', x', y', z'\rangle \in B \times Q^4) & \text{def.of } g \\
 & \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[g'[\xi(e) \mapsto \langle b', t', x', y', z'\rangle]] & \\
\iff & \mathfrak{M}_s \models \exists\xi(e)\mathrm{CTS}_\xi(\varphi)[g'] & \text{def.of } \models \\
\iff & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\exists e\varphi)[g'] & \text{def.of } \mathrm{CTS}_\xi
\end{array}
$$

This ends the proof of Lemma 36. ■

Now the main induction:

$$
\begin{array}{lll}
 & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\exists e\varphi)[f_\rho(\eta)] & \\
\iff & \mathfrak{M}_s \models \exists\xi(e)\mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)] & \text{def.of } \mathrm{CTS}_\xi \\
\iff & (\exists\langle b, t, x, y, z\rangle \in B \times Q^4) & \\
 & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)[\xi(e) \mapsto \langle b, t, x, y, z\rangle]] & \text{def.of } \models \\
\iff & (\exists\langle b, t, x, y, z\rangle \in B \times Q^4) & \\
 & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e \mapsto \langle b, t, x, y, z\rangle/\stackrel{e}{\simeq}])] & \text{Lemma 36} \\
\iff & (\exists\langle b, t, x, y, z\rangle \in B \times Q^4) & \\
 & \qquad \mathrm{stc}(\mathfrak{M}_s) \models \varphi[\eta[e \mapsto \langle b, t, x, y, z\rangle/\stackrel{e}{\simeq}]] & \text{ind.hip.} \\
\iff & \mathrm{stc}(\mathfrak{M}_s) \models \exists e\varphi[\eta] &
\end{array}
$$

- $\varphi_c = \exists a\varphi$ Now the main induction:

$$\begin{array}{lll}
& \mathfrak{M}_s \models \mathrm{CTS}_\xi(\exists a\varphi)[f_\rho(\eta)] & \\
\iff & \mathfrak{M}_s \models \exists\xi(a)(\mathrm{IOb}(\xi(a)) \wedge \mathrm{CTS}_\xi(\varphi))[f_\rho(\eta)] & \text{def.of CTS}_\xi \\
\iff & (\exists b \in \mathrm{IOb}^{\mathfrak{M}_s}) & \\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)[\xi(a) \mapsto b]] & \text{def.of} \models \\
\iff & (\exists b \in \mathrm{IOb}^{\mathfrak{M}_s}) & \\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[a \mapsto b])] & (22) \\
\iff & (\exists b \in \mathrm{IOb}^{\mathfrak{M}_s})\,\mathrm{stc}(\mathfrak{M}_s) \models \varphi[\eta[a \mapsto b]] & \text{ind.hip.} \\
\iff & \mathrm{stc}(\mathfrak{M}_s) \models \exists a\varphi[\eta] & \text{def.of} \models
\end{array}$$

- $\varphi_c = \exists x\varphi$ is similar to $\exists a\varphi$.

6. **Definition of** cts. Let

$$\mathfrak{M}_c = \left(W, \prec^{\mathfrak{M}_c}, C, \mathfrak{Q}, \mathrm{P}^{\mathfrak{M}_c}\right)$$

be an arbitrary but fixed model of SClTh. We will introduce the transformation cts : Mod(SClTh) $\to$ Mod(SpecRelComp), i.e., we will construct the corresponding SpecRelComp model cts$(\mathfrak{M}_c)$ from the information that $\mathfrak{M}_c$ contains. Such a SpecRelComp model will be given as

$$\mathrm{cts}(\mathfrak{M}_c) \stackrel{\text{def}}{=} (\mathrm{cts}_B(\mathfrak{M}_c), \mathrm{stc}_{\mathrm{IOb}}(\mathfrak{M}_c), \mathrm{stc}_{\mathrm{Ph}}(\mathfrak{M}_c), \mathfrak{Q}, \mathrm{stc}_{\mathrm{W}}(\mathfrak{M}_c))$$

where the four undefined entity will be body domain and the meanings of predicates IOb, Ph and W, respectively.

(a) **Body domain** $\mathrm{cts}_B(\mathfrak{M}_c)$. The first idea would be that a body will be identified with a set of events (the worldline). Even if we have a predicate variable sort for that purpose, we do not have the quantifiers for that sort, and thus we cannot translate the formulas of the form $\exists b\varphi$. We will sort out a lot of worldlines; we keep only those that are worldlines of observers or photons. In models of SpecRelComp, there are no other worldlines anyway. The worldlines of observers seems to be easy, the set

$$\{w \in W \,:\, (\exists q \in Q)(w, c, q) \in \mathrm{P}^{\mathfrak{M}_c}\}$$

seems to be a fine candidate. But this won't be enough, since a SpecRelComp observer is very different from a clock. If we take a closer look on the axioms about the interaction of IOb and W, a SpecRelComp observer knows where is forward, where is right, where is up, while a clock does not know this alone; it needs (mutually orthogonal) partners c_x, c_y, c_z to represent these directions. So an observer is a coordinate system rather than a body drifting alone in the Minkowski spacetime. Thus we are going to identify an inertial observer with a 4-tuple of clocks c, c_x, c_y, c_z:

$$wl_{c,c_x,c_y,c_z} \stackrel{\text{def}}{=} \{e \in W \,:\, (\exists q \in Q)(e, c, q) \in \mathrm{P}^{\mathfrak{M}_c} \text{ and } (c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c}\}$$

where $\mathrm{CoordSys}^{\mathfrak{M}_c}$ is the meaning of the formula defined on p. 7.

But the worldlines of photons must be different, since no observer can travel as fast as the light, and there are no terms for photons in the language of SClTh. We will use the relation of light-like separation instead. Using that relation we can identify every photon with a pair of lightlike separated events $e_1 \nearrow^{\mathfrak{M}_c} e_2$. Let us define (in the object language) the lightline determined by (e_1, e_2):

$$e \in \mathrm{lline}(e_1, e_2) \overset{\mathrm{def}}{\Leftrightarrow} e_1 \nearrow e_2 \nearrow e \vee e_1 \nearrow e \nearrow e_2 \vee e \nearrow e_1 \nearrow e_2$$

Now take the meaning of that formula, i.e., let

$$\mathrm{lline}^{\mathfrak{M}_c}(w_1, w_2) \overset{\mathrm{def}}{=} \left\{ w \in W \ : \ \begin{array}{l} w_1 \nearrow^{\mathfrak{M}_c} w_2 \nearrow^{\mathfrak{M}_c} w \text{ or} \\ w_1 \nearrow^{\mathfrak{M}_c} w \nearrow^{\mathfrak{M}_c} w_2 \text{ or} \\ w \nearrow^{\mathfrak{M}_c} w_1 \nearrow^{\mathfrak{M}_c} w_2 \end{array} \right\}$$

Now we can merge the two concepts of wordlines (worldline of a photon as the lightline defined by two lightlike events, and the worldline of a observer defined via 5 clocks) in the following way: a body is determined by a 5-tuple $(c, c_x, c_y, c_z, c_t, e_1, e_2)$ in the following way: if $e_1 \nearrow^{\mathfrak{M}_c} e_2$ then it is the light-line determined by e_1, e_2, otherwise it is the worldline of c:

$$\begin{aligned} & wl_{c,c_x,c_y,c_z,e_1,e_2} \overset{\mathrm{def}}{=} \\ & \quad \left\{ e \in W \ : \ \left[e_1 \nearrow e_2 \text{ and } \left(\begin{array}{l} e_1 \nearrow^{\mathfrak{M}_c} e_2 \nearrow^{\mathfrak{M}_c} e \text{ or} \\ e_1 \nearrow^{\mathfrak{M}_c} e \nearrow^{\mathfrak{M}_c} e_2 \text{ or} \\ e \nearrow^{\mathfrak{M}_c} e_1 \nearrow^{\mathfrak{M}_c} e_2 \end{array} \right) \right] \right. \\ & \qquad \qquad \left. \text{or } \left[\begin{array}{c} \text{not } e_1 \nearrow e_2 \text{ and} \\ (\exists q \in Q)(e, c, q) \text{ and} \\ (c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c} \end{array} \right] \right\} \end{aligned}$$

According to that definition it is not true that every 6-tuple $c, c_x, c_y, c_z, e_1, e_2$ determines a body: this can happen when $e_1 \nearrow e_2$ is not true and the 4 clock do not constitute a coordinate system. So the set of suitable 6-tuples to name bodies will be

$$\begin{aligned} WL \overset{\mathrm{def}}{=} \{ \langle c, c_x, c_y, c_z, e_1, e_2 \rangle \in C^4 \times W^2 \ : & \\ e_1 \nearrow^{\mathfrak{M}_c} e_2 \text{ or } (c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c} \} & \end{aligned}$$

Also note that, according to the definitions, if both $e_1 \nearrow^{\mathfrak{M}_c} e_2$ and $(c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c}$, i.e., it is both capable of referring to a photon and an inertial observer, then we always refer with that tuple to the photon.

But a lot of 6-tuple can name the same worldline, so we have to find a suitable equivalence relation to factorize over the set of bodies to create the final domain of bodies.

$$
\begin{array}{c}
(c, c_x, c_y, c_z, e_1, e_2) \overset{b}{\simeq} (c', c'_x, c'_y, c'_z, e'_1, e'_2) \overset{\text{def}}{\Leftrightarrow} \\
[e_1 \nearrow^{\mathfrak{M}_c} e_2 \text{ and } \mathrm{lline}^{\mathfrak{M}_c}(e_1, e_2) = \mathrm{lline}^{\mathfrak{M}_c}(e'_1, e'_2)] \\
\text{or} \left[\begin{array}{c} \text{not } e_1 \nearrow^{\mathfrak{M}_c} e_2 \text{ and} \\ wl_{c,c_x,c_y,c_z} = wl_{c',c'_x,c'_y,c'_z} \text{ and} \\ (c, c_x, c'_x), (c, c_y, c'_y), (c, c_z, c'_z) \in \mathrm{C}^{\mathfrak{M}_c} \end{array} \right]
\end{array}
$$

where C is the meaning of the collinearity relation, see Def. 7. This is an equivalence relation (definable in the language of clock logic). So the domain of the bodies will be

$$
\mathrm{cts}_B(\mathfrak{M}_c) \overset{\text{def}}{=} WL/\overset{b}{\simeq}.
$$

(b) **meaning of observer predicate** $\mathrm{cts}_{\mathrm{IOb}}(\mathfrak{M}_c)$

$$
\begin{array}{l}
\mathrm{cts}_{\mathrm{IOb}}(\mathfrak{M}_c) \overset{\text{def}}{=} \{(c, c_x, c_y, c_z, e_1, e_2)/\overset{b}{\simeq} \in \mathrm{cts}_{\mathrm{B}} : \\
\qquad \text{not } e_1 \nearrow^{\mathfrak{M}} e_2 \text{ and } (c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c}\}
\end{array}
$$

(c) **meaning of photon predicate** $\mathrm{cts}_{\mathrm{Ph}}$

$$
\mathrm{cts}_{\mathrm{Ph}}(\mathfrak{M}_c) \overset{\text{def}}{=} \left\{ (c, c_x, c_y, c_z, e_1, e_2)/\overset{b}{\simeq} \in \mathrm{cts}_{\mathrm{B}} : e_1 \nearrow^{\mathfrak{M}_c} e_2 \right\}
$$

(d) **meaning of worldview relation** $\mathrm{cts}_{\mathrm{W}}(\mathfrak{M}_c)$

$$
\begin{array}{l}
\mathrm{cts}_{\mathrm{W}}(\mathfrak{M}_c) \overset{\text{def}}{=} \\
\{(c, c_x, c_y, c_z, e_1, e_2)/\overset{b}{\simeq}, (c', c'_x, c'_y, c'_z, e'_1, e'_2)/\overset{b}{\simeq}, \vec{x} \in \mathrm{cts}_{\mathrm{B}}(\mathfrak{M}_c)^2 \times Q^4 : \\
\qquad (\exists e \in wl_{c',c'_x,c'_y,c'_z,e'_1,e'_2}) \mathrm{Coord}^{\mathfrak{M}_c}(c, c_x, c_y, c_z, e) = \vec{x}\}
\end{array}
$$

where $\mathrm{Coord}^{\mathfrak{M}_c}$ is the meaning of the coordinatization relation defined on p. 7.

7. **Definition of** ζ. By the above construction of the domains, we choose ζ to be

$$
\zeta : \begin{array}{ccc} x_i & \mapsto & x_i \\ b_i & \mapsto & \langle a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}, e_{2i}, e_{2i+1} \rangle \end{array}
$$

8. **Definition of** STC_ζ.

$$
\begin{array}{ll}
\mathrm{STC}_\zeta(b=b') & \stackrel{\text{def}}{=} (\zeta_5(b) \mathrel{\rightsquigarrow} \zeta_6(b') \wedge \mathrm{lline}(\zeta_{5-6}(b)) = \mathrm{lline}(\zeta_{5-6}(b'))) \vee \\
 & \vee \big(\neg\zeta_5(b) \mathrel{\rightsquigarrow} \zeta_6(b) \wedge \mathrm{wline}_{\zeta_1(b)} = \mathrm{wline}_{\zeta_1(b')} \wedge \\
 & \wedge \mathrm{C}(\zeta_{1,2}(b), \zeta_2(b')) \wedge \mathrm{C}(\zeta_{1,3}(b), \zeta_3(b')) \wedge \mathrm{C}(\zeta_{1,4}(b), \zeta_4(b'))\big) \\
\mathrm{STC}_\zeta(\tau=\tau') & \stackrel{\text{def}}{=} \tau = \tau' \\
\mathrm{STC}_\zeta(\tau\le\tau') & \stackrel{\text{def}}{=} \tau \le \tau' \\
\mathrm{STC}_\zeta(\mathrm{IOb}(b)) & \stackrel{\text{def}}{=} \mathrm{CoordSys}(\zeta_{1-4}(b)) \wedge \neg\zeta_5(b) \mathrel{\rightsquigarrow} \zeta_6(b) \\
\mathrm{STC}_\zeta(\mathrm{Ph}(b)) & \stackrel{\text{def}}{=} \zeta_5(b) \mathrel{\rightsquigarrow} \zeta_6(b) \\
\mathrm{STC}_\zeta(\mathrm{W}(b,b',\vec{\tau})) & \stackrel{\text{def}}{=} (\exists e \in \mathrm{wline}_{\zeta(b')})\mathrm{Coord}_{\zeta_{1-4}(b)}(e) = \vec{\tau} \\
\mathrm{STC}_\zeta(\neg\varphi) & \stackrel{\text{def}}{=} \neg\mathrm{STC}_\zeta(\varphi) \\
\mathrm{STC}_\zeta(\varphi\wedge\psi) & \stackrel{\text{def}}{=} \mathrm{STC}_\zeta(\varphi) \wedge \mathrm{STC}_\zeta(\psi) \\
\mathrm{STC}_\zeta(\exists b\varphi) & \stackrel{\text{def}}{=} \exists\zeta(b)\Big(\big(\zeta_5(b) \mathrel{\rightsquigarrow} \zeta_6(b) \vee \mathrm{CoordSys}(\zeta_{1-4}(b))\big) \wedge \mathrm{STC}_\zeta(\varphi)\Big) \\
\mathrm{STC}_\zeta(\exists x\varphi) & \stackrel{\text{def}}{=} \exists x \mathrm{STC}_\zeta(\varphi)
\end{array}
$$

9. Proof of the equivalence (17). Similar to step 5. (including a lemma like Lemma 36 in case of $\exists b$).

10. Proof of (19): Proving SpecRelComp in SClTh According to Propositions 29 and 27, every translation of every axiom of SpecRelComp is equivalent to its 'simple-'version described in Section **??**, so we are already done.

11. Proof of (18): Proving SClTh in SpecRelComp. This proof itself consists only of standard analytical geometrical calculations and basic facts about Minkowski geometry. Since in this report we focus on logical issues and signalling procedures in a logical environment, we omit this proof. UNDER CONSTRUCTION

■

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