Axiomatizing Minkowski Spacetime in First-Order Temporal Logic

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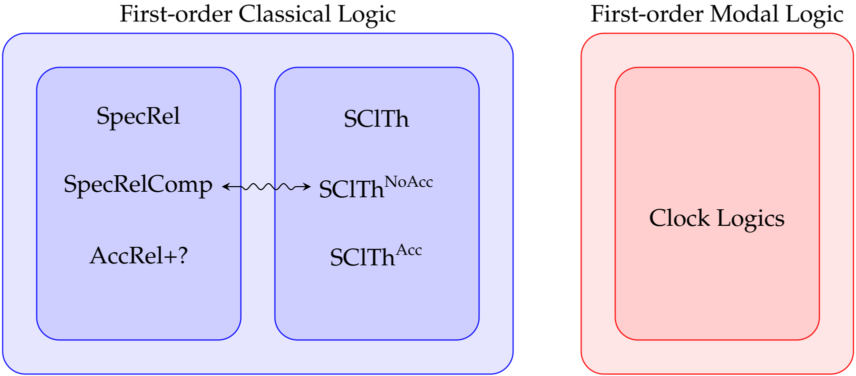
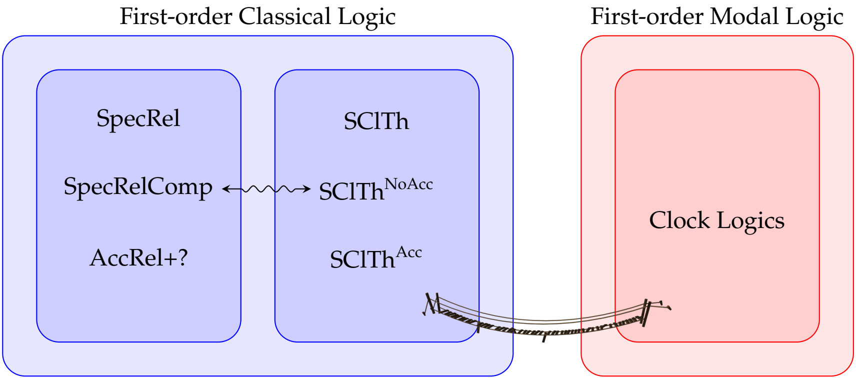
LOGREL GROUP MEETING
March 20, 2015

Big Picture

Big picture


First-order Classical Logic
SpecRel
SpecRelComp
AccRel+?
SClTh
SClTh^NoAcc
SClTh^Acc
First-order Modal Logic
Clock Logics

BIG PICTURE



that is a **two-way bridge**
if we aim Minkowski spacetimes!

1) max-*n*-zigzag connected
2) there is set of timelike curves (clocks) s.t.
   a) they are everywhere
   b) none of them are closed
   c) chronological confluence prop.

Abstract

I will present a first-order temporal logic which has the following properties:

1) **Strong Expressive Power:** It can express the basic paradigmatic relativistic effects of kinematics such as time dilation, length contraction, twin paradox, etc.
2) **Operationality:** The coordinatization itself is definable using metric tense operators with signalling procedures.
3) **Completeness and Decidability:** The set of formulas that are valid on the 4D Minkowski spacetime is recursively axiomatizable and decidable.
4) A (first-order modal variant of a) definitional equivalence can be proved w.r.t. the axiom system SpecRel ∪ Comp (Now just SRC) of HB of Spatial Logics.

So far it seems that the presented framework is flexible enough to allow for similar (expressive, operational, axiomatizable) results in general relativity and branching spacetimes.

Clock logic

Two-sorted modal predicate logic:

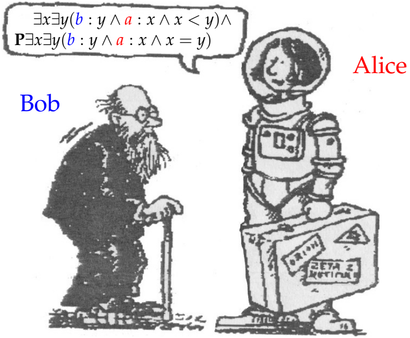
- Terms: $\tau ::= x \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$
- Formulas:

$$\varphi ::= \tau_1 \leq \tau_2 \mid \tau_1 = \tau_2 \mid a{:}\tau \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{F}\varphi \mid \mathbf{P}\varphi \mid \exists x \varphi \mid \exists a \varphi$$

$a{:}\tau$ ↓ pointing statements: clock a shows time τ

$\mathbf{F}\varphi$ ↓ φ will be true in the causal future

$\exists a \varphi$ ↓ there is a clock in the actual event for which φ



worlds :	events
alternative relation :	irreflexive causal future
domains :	one universal domain for math, varying domains for clocks
meaning of math :	rigid predicates and rigid terms
meaning of clock terms :	intensional objects / non-rigid designators / individual concepts / functions eating worlds

Clock Models

$$\mathfrak{M} \stackrel{\text{def}}{=} \Big(\underset{\substack{\downarrow \\ \text{events}}}{W}, \overset{\substack{\text{causality relation} \\ \uparrow}}{\prec}, \underset{\substack{\downarrow \\ \text{numbers}}}{U}, \overset{\substack{\text{clocks} \\ \uparrow}}{\mathbb{C}}, \underbrace{[\![+]\!]^{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}}}_{\text{binary functions on } U}, \underset{\substack{\cap\!| \\ U^2}}{[\![\leq]\!]^{\mathfrak{M}}} \Big)$$

clocks $\subseteq \{\alpha : W \to U$ partial functions$\}$

Truth:

$$\begin{aligned}
\mathfrak{M}, \mu, \gamma, w \models a{:}\tau &\stackrel{\text{def}}{\Leftrightarrow} [\![a]\!]^{\mathfrak{M}}_{\gamma}(w) = [\![\tau]\!]^{\mathfrak{M}}_{\mu},\\
\mathfrak{M}, \mu, \gamma, w \models \mathbf{F}\varphi &\stackrel{\text{def}}{\Leftrightarrow} w' \succ w\ \mathfrak{M}, \mu, \gamma, w' \models \varphi,\\
\mathfrak{M}, \mu, \gamma, w \models \mathbf{P}\varphi &\stackrel{\text{def}}{\Leftrightarrow} w' \prec w\ \mathfrak{M}, \mu, \gamma, w' \models \varphi,\\
\mathfrak{M}, \mu, \gamma, w \models \exists x\varphi &\stackrel{\text{def}}{\Leftrightarrow} (\exists u \in U)\ \mathfrak{M}, \mu[x \mapsto u], \gamma, w \models \varphi,\\
\mathfrak{M}, \mu, \gamma, w \models \exists a\varphi &\stackrel{\text{def}}{\Leftrightarrow} (\exists \alpha \in \mathbb{C}_w)\ \mathfrak{M}, \mu, \gamma[a \mapsto \alpha], w \models \varphi.
\end{aligned}$$

where $\mathbb{C}_w \stackrel{\text{def}}{=} \{\alpha \in \mathbb{C} : \alpha \text{ is defined in } w\}$

Validity:

$$\mathfrak{M} \models \varphi \stackrel{\text{def}}{\Leftrightarrow} (\forall \mu, \gamma, w)\ \mathfrak{M}, \mu, \gamma, w \models \varphi$$

Minkowski model with inertial clocks

$$\mathfrak{Mink} = \left(W, \prec, U, \mathbb{C}, [\![+]\!]^{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}}, [\![\leq]\!]^{\mathfrak{M}}\right)$$

- $\left(U, [\![+]\!]^{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}}, [\![\leq]\!]^{\mathfrak{M}}\right) \stackrel{\text{def}}{=} \mathbb{R}$ is the field of reals.
- $W = \mathbb{R}^4$
- $w \prec w'$ iff $\mu(w - w') \geq 0$ and $w_n < w'_n$ where $\mu(\vec{w}) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n-1} w_i^2\right) - w_n^2$.
- $\mathbb{C} = \{\alpha : \alpha^{-1}$ is a timelike line$\}$ s.t. all of them use the measure system of $\mathbb{R}$, i.e.,

$$(\forall \alpha \in \mathbb{C})(\forall w, v \in \mathrm{dom}(\alpha)) \qquad \mu(w, v) = |\alpha(w) - \alpha(v)|$$

2-sorted temporal

$$
\begin{aligned}
\tau \; &::= \; x \\
&\quad\;\; \tau + \tau' \\
&\quad\;\; \tau \cdot \tau' \\
a \; &::= \; a \\
\varphi \; &::= \; \tau = \tau' \\
&\quad\;\; \tau \leq \tau' \\
&\quad\;\; a : \tau \\
&\quad\;\; \neg\varphi \\
&\quad\;\; \varphi \wedge \psi \\
&\quad\;\; \mathbf{F}\varphi \\
&\quad\;\; \mathbf{P}\varphi \\
&\quad\;\; \exists x \varphi \\
&\quad\;\; \exists a \varphi
\end{aligned}
$$

2-sorted classical

$$
\begin{array}{lcl}
\tau & ::= & x \\
 & & \tau + \tau' \\
 & & \tau \cdot \tau' \\
b & ::= & b \\
\varphi & ::= & \tau = \tau' \\
 & & \tau \leq \tau' \\
 & & \mathrm{Ph}(b) \\
 & & \mathrm{IOb}(b) \\
 & & \mathrm{W}(b, c, x, y, z, t) \\
 & & \neg\varphi \\
 & & \varphi \wedge \psi \\
 & & \exists x \varphi
\end{array}
$$

2-sorted temporal

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\begin{array}{lcl}
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\end{array}
$$

SRC
Complete and finite scheme axiomatization of 4D Minkowski Spacetime with inertial observers

2-sorted classical

$$
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SRC
Complete and finite scheme axiomatization of 4D Minkowski Spacetime with inertial observers

2-sorted classical

$$\begin{aligned} \tau &::= x \mid \tau + \tau' \mid \tau \cdot \tau' \\ b &::= b \\ \varphi &::= \tau = \tau' \mid \tau \leq \tau' \mid \mathrm{Ph}(b) \mid \mathrm{IOb}(b) \mid \mathrm{W}(b, c, x, y, z, t) \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \end{aligned}$$

3-sorted classical

$$\begin{aligned} \tau &::= x \mid \tau + \tau' \mid \tau \cdot \tau' \\ a &::= a \\ w &::= w \\ \varphi &::= \tau = \tau' \mid \tau \leq \tau' \mid w \prec w' \mid \mathrm{P}(w, a, \tau) \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \end{aligned}$$

Standard Translation (from 2-sorted temporal to 3-sorted classical)

2-sorted temporal

$$\begin{aligned} \tau &::= x \mid \tau + \tau' \mid \tau \cdot \tau' \\ a &::= a \\ \varphi &::= \tau = \tau' \mid \tau \leq \tau' \mid a : \tau \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{F}\varphi \mid \mathbf{P}\varphi \mid \exists x\varphi \mid \exists a\varphi \end{aligned}$$

SRC
Complete and finite scheme axiomatization of 4D Minkowski Spacetime with inertial observers

True formulas of Minkowski spacetimes with inertial clocks

2-sorted classical

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\end{aligned}
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Standard Translation (from 2-sorted temporal to 3-sorted classical)

2-sorted temporal

$$
\begin{aligned}
\tau &::= x \\
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\end{aligned}
$$

Definitional equivalence

SRC
Complete and finite scheme axiomatization of 4D Minkowski Spacetime with inertial observers

Tr(SRC)
True formulas of Minkowski spacetimes with inertial clocks

2-sorted classical

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Standard Translation (2-sorted temporal → 3-sorted classical)

2-sorted temporal

$$
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Definitional equivalence

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2-sorted classical

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2-sorted hybrid

$$\begin{array}{rrcl} & \tau & ::= & x \\ & & & \tau + \tau' \\ & & & \tau \cdot \tau' \\ & a & ::= & a \\ & \varphi & ::= & \tau = \tau' \\ & & & \tau \leq \tau' \\ & & & a : \tau \\ & & & \neg\varphi \\ & & & \varphi \wedge \psi \\ w & & & \mathbf{F}\varphi \\ @_w\varphi & & & \mathbf{P}\varphi \\ \mathbf{E}\varphi & & & \exists x \varphi \\ \downarrow w\, \varphi & & & \exists a \varphi \end{array}$$

2-sorted temporal

$$\begin{array}{rcl} \tau & ::= & x \\ & & \tau + \tau' \\ & & \tau \cdot \tau' \\ a & ::= & a \\ \varphi & ::= & \tau = \tau' \\ & & \tau \leq \tau' \\ & & a : \tau \\ & & \neg\varphi \\ & & \varphi \wedge \psi \\ & & \mathbf{F}\varphi \\ & & \mathbf{P}\varphi \\ & & \exists x \varphi \\ & & \exists a \varphi \end{array}$$

Definitional equivalence (SRC ⟷ Tr(SRC))

SRC
Complete and finite scheme axiomatization of 4D Minkowski Spacetime with inertial observers

Tr(SRC)
True formulas of Minkowski spacetimes with inertial clocks

Hybrid Clock logic of Minkowski Spacetime with inertial clocks

2-sorted classical

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 & & \varphi \wedge \psi \\
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2-sorted hybrid

$$
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\tau & ::= & x \\
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 & & a : \tau \\
 & & \neg\varphi \\
 & & \varphi \wedge \psi \\
w & & \mathbf{F}\varphi \\
@_w\varphi & & \mathbf{P}\varphi \\
\mathbf{E}\varphi & & \exists x \varphi \\
\downarrow w\, \varphi & & \exists a \varphi
\end{array}
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2-sorted temporal

$$
\begin{array}{lcl}
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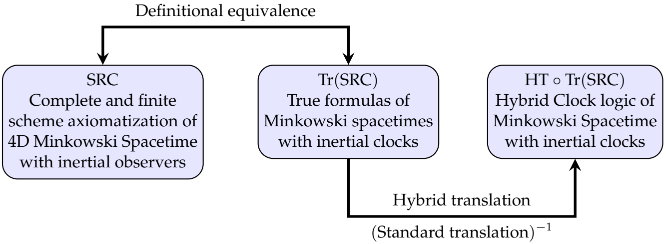
\end{array}
$$



2-sorted classical

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2-sorted hybrid

$$\begin{array}{rcl} \tau & ::= & x \\ & & \tau + \tau' \\ & & \tau \cdot \tau' \\ a & ::= & a \\ \varphi & ::= & \tau = \tau' \\ & & \tau \leq \tau' \\ & & a : \tau \\ & & \neg\varphi \\ & & \varphi \wedge \psi \\ w & & \mathbf{F}\varphi \\ @_w\varphi & & \mathbf{P}\varphi \\ \mathbf{E}\varphi & & \exists x \varphi \\ \downarrow w\, \varphi & & \exists a \varphi \end{array}$$

2-sorted temporal

$$\begin{array}{lcl} \tau & ::= & x \\ & & \tau + \tau' \\ & & \tau \cdot \tau' \\ a & ::= & a \\ \varphi & ::= & \tau = \tau' \\ & & \tau \leq \tau' \\ & & a : \tau \\ & & \neg\varphi \\ & & \varphi \wedge \psi \\ & & \mathbf{F}\varphi \\ & & \mathbf{P}\varphi \\ & & \exists x \varphi \\ & & \exists a \varphi \end{array}$$

Definitional equivalence

SRC
Complete and finite scheme axiomatization of 4D Minkowski Spacetime with inertial observers

Tr(SRC)
True formulas of Minkowski spacetimes with inertial clocks

HT ∘ Tr(SRC)
Hybrid Clock logic of Minkowski Spacetime with inertial clocks

Clock logic of Minkowski Spacetime with inertial clocks

Hybrid translation

(Standard translation)$^{-1}$

2-sorted classical

$$\begin{array}{lll} \tau & ::= & x \\ & & \tau + \tau' \\ & & \tau \cdot \tau' \\ b & ::= & b \\ \varphi & ::= & \tau = \tau' \\ & & \tau \leq \tau' \\ & & \mathrm{Ph}(b) \\ & & \mathrm{IOb}(b) \\ & & \mathrm{W}(b, c, x, y, z, t) \\ & & \neg\varphi \\ & & \varphi \wedge \psi \\ & & \exists x \varphi \end{array}$$

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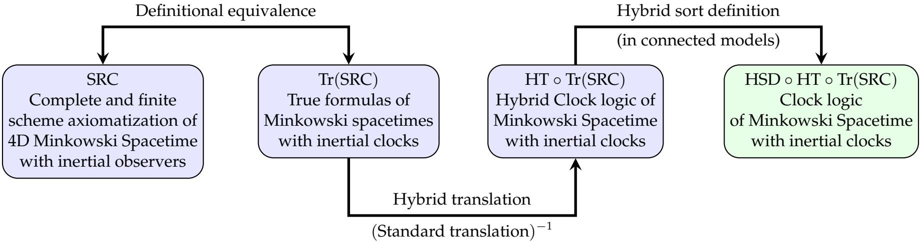
2-sorted hybrid

$$\begin{array}{lll} \tau & ::= & x \\ & & \tau + \tau' \\ & & \tau \cdot \tau' \\ a & ::= & a \\ \varphi & ::= & \tau = \tau' \\ & & \tau \leq \tau' \\ & & a : \tau \\ & & \neg\varphi \\ & & \varphi \wedge \psi \\ w & & \mathbf{F}\varphi \\ @_w\varphi & & \mathbf{P}\varphi \\ \mathbf{E}\varphi & & \exists x \varphi \\ \downarrow w\, \varphi & & \exists a \varphi \end{array}$$

2-sorted temporal

$$\begin{array}{lll} \tau & ::= & x \\ & & \tau + \tau' \\ & & \tau \cdot \tau' \\ a & ::= & a \\ \varphi & ::= & \tau = \tau' \\ & & \tau \leq \tau' \\ & & a : \tau \\ & & \neg\varphi \\ & & \varphi \wedge \psi \\ & & \mathbf{F}\varphi \\ & & \mathbf{P}\varphi \\ & & \exists x \varphi \\ & & \exists a \varphi \end{array}$$



2-sorted classical

$$
\begin{aligned}
\tau &::= x \mid \tau + \tau' \mid \tau \cdot \tau' \\
b &::= b \\
\varphi &::= \tau = \tau' \mid \tau \leq \tau' \mid \mathrm{Ph}(b) \mid \mathrm{IOb}(b) \mid \mathrm{W}(b, c, x, y, z, t) \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x \varphi
\end{aligned}
$$

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\begin{aligned}
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\end{aligned}
$$

2-sorted hybrid

$$
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\tau &::= x \mid \tau + \tau' \mid \tau \cdot \tau' \\
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\varphi &::= \tau = \tau' \mid \tau \leq \tau' \mid a : \tau \mid \neg\varphi \mid \varphi \wedge \psi \mid w \mid \mathbf{F}\varphi \mid @_w\varphi \mid \mathbf{P}\varphi \mid \mathbf{E}\varphi \mid \exists x \varphi \mid {\downarrow} w\, \varphi \mid \exists a \varphi
\end{aligned}
$$

2-sorted temporal

$$
\begin{aligned}
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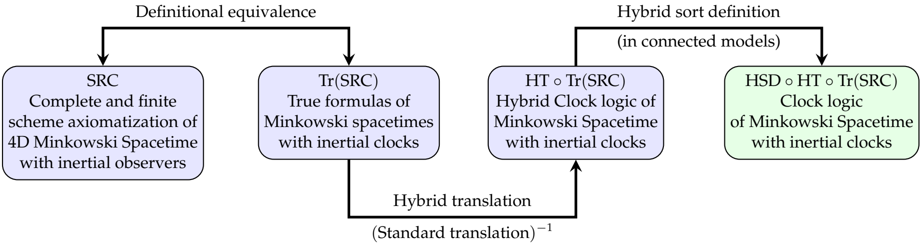
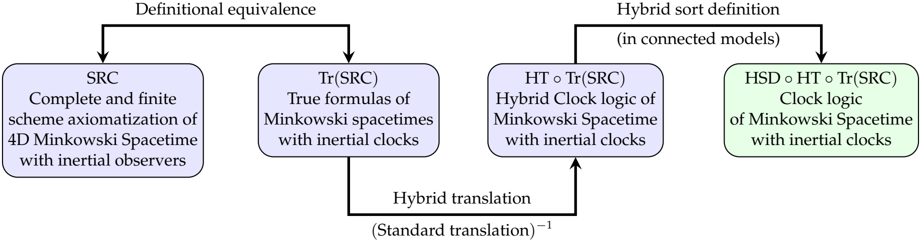
a &::= a \\
\varphi &::= \tau = \tau' \mid \tau \leq \tau' \mid a : \tau \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{F}\varphi \mid \mathbf{P}\varphi \mid \exists x \varphi \mid \exists a \varphi
\end{aligned}
$$



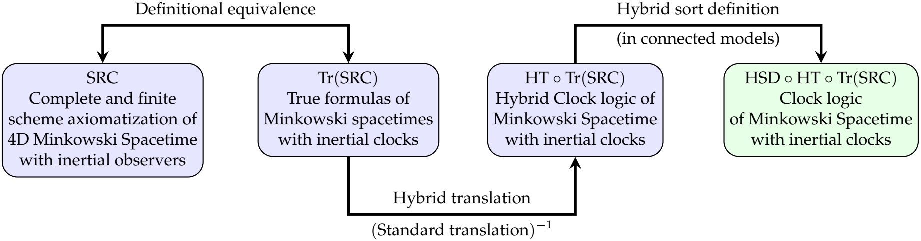
Hybrid sort definition

$$
\begin{aligned}
e_i &\mapsto a_{2i+1}\!:\!x_{2i+1} \\
a_i\!:\!x_i &\mapsto a_{2i}\!:\!x_{2i} \\
\exists v_i\varphi &\mapsto \exists v_{2i}\varphi \\
\mathbf{E}\varphi &\mapsto \mathbf{PF}\varphi \\
@_{e_i}\varphi &\mapsto \mathbf{PF}\big(a_{2i+1}\!:\!x_{2i+1} \wedge \varphi\big) \\
\downarrow e_i\varphi &\mapsto \exists a_{2i+1}\exists x_{2i+1}\big(a_{2i+1}\!:\!x_{2i+1} \wedge \varphi\big)
\end{aligned}
$$



Hybrid translation

$$
\begin{array}{rcl}
w_1 = w_2 & \mapsto & @_{w_1} w_2 \\
w_1 \prec w_2 & \mapsto & @_{w_1} \mathbf{F} w_2 \\
P(w, a, x) & \mapsto & @_w \, a{:}x \\
a_1 = a_2 & \mapsto & \mathbf{A} \forall x (a_1{:}x \leftrightarrow a_2{:}x) \\
\exists w \varphi & \mapsto & \mathbf{E} {\downarrow} w \varphi \\
\exists a \varphi & \mapsto & \mathbf{E} \exists a \varphi
\end{array}
$$



Tr from the definitional equivalence

$$
\begin{aligned}
b &\mapsto (c, c_x, c_y, c_z, w, v).\\
\mathrm{Ph}(b) &\mapsto w \rightsquigarrow v\\
\mathrm{IOb}(b) &\mapsto \neg w \rightsquigarrow v \wedge \mathrm{CoordSys}(c, c_x, c_y, c_z)\\
b = b' &\mapsto (w \rightsquigarrow w' \wedge \mathrm{lline}(w, v) = \mathrm{lline}(w', v')) \vee\\
&\quad \vee (\neg w \rightsquigarrow v \wedge \mathrm{wline}_c = \mathrm{wline}_{c'}\\
&\quad\quad \wedge(\mathrm{Between}(c, c_x, c'_x) \vee \mathrm{Between}(c, c'_x, c_x))\wedge\\
&\quad\quad\quad (\mathrm{Between}(c, c_y, c'_y) \vee \mathrm{Between}(c, c'_y, c_y))\wedge\\
&\quad\quad\quad (\mathrm{Between}(c, c_z, c'_z) \vee \mathrm{Between}(c, c'_z, c_z)))\\
\mathrm{W}(b, b', \vec{x}) &\mapsto (\exists w \in \mathrm{wline}_{c'}\ \mathrm{Coord}_{c,c_x,c_y,c_z}(w) = \vec{x}\\
\exists b\varphi &\mapsto \exists c, c_x, c_y, c_z \exists w, v((w \rightsquigarrow v \vee \mathrm{CoordSys}(c, c_x, c_y, c_z)) \wedge \varphi)
\end{aligned}
$$

Coordinatization

Language

$$\tau ::= x \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$$

$$\varphi ::= a = b \mid \tau = \tau' \mid \tau \leq \tau' \mid e = e' \mid e \prec e' \mid \mathrm{In}(a) \mid \mathrm{P}(e, a, \tau) \mid \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \mid \exists a\varphi \mid \exists e\varphi$$

Now we have a primitive predicate for inertiality but it is eliminable by identifying them with *geodetics*:

$$\mathrm{Geo}(a) \overset{\text{def}}{\Leftrightarrow} (\forall e, e' \in \mathrm{wline}_a)(\forall b \in \mathrm{D}_e \cap \mathrm{D}_{e'}) \quad |a(e) - a(e')| \geq |b(e) - b(e')|$$

LANGUAGE

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$$a(e) = \tau \overset{\text{def}}{\Leftrightarrow} \mathrm{P}(a, e, \tau)$$

$$e\mathcal{E}a \overset{\text{def}}{\Leftrightarrow} \exists x\, \mathrm{P}(a, e, x)$$

$$\mathrm{wline}_a \overset{\text{def}}{=} \{e : \exists x\, \mathrm{P}(a, e, x)\}$$

$$\mathrm{D}_e \overset{\text{def}}{=} \{a : \exists x\, \mathrm{P}(a, e, x)\}$$

$$a \approx b \overset{\text{def}}{\Leftrightarrow} \forall e(e\mathcal{E}a \leftrightarrow e\mathcal{E}b)$$

$$e \ll e' \overset{\text{def}}{\Leftrightarrow} e \prec e' \wedge \exists a(e\mathcal{E}a \wedge e'\mathcal{E}a)$$

$$e \underline{\ll} e' \overset{\text{def}}{\Leftrightarrow} e \ll e' \vee e = e'$$

$$e \rightsquigarrow e' \overset{\text{def}}{\Leftrightarrow} e \prec e' \wedge \neg\exists a(e\mathcal{E}a \wedge e'\mathcal{E}a)$$

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$$\overrightarrow{e_1 e_2 e_3} \overset{\text{def}}{\Leftrightarrow} e_1 \rightsquigarrow e_2 \wedge e_2 \rightsquigarrow e_3 \wedge e_1 \rightsquigarrow e_3$$

D_e: domain of event e
$a \approx b$: cohabitation

$\overrightarrow{e_1 e_2 e_3}$: directed lightlike betweenness

Intended models

$$\mathfrak{M}^c = \left(\mathbb{R}^4, \mathbb{C}, \mathbb{R}, \prec^{\mathfrak{M}^c}, \mathrm{In}^{\mathfrak{M}^c} +, \cdot, \leq, \mathrm{P}^{\mathfrak{M}^c}\right)$$

- $\mathbb{C}$ is the set of those $\alpha : \mathbb{R}^4 \to \mathbb{R}$ partial functions, for which α^{-1}-s are timelike curves that follows the measure system of $\mathbb{R}^4$, i.e.,
  - α^{-1}-s are continuously differentiable functions on $\mathbb{R}$ w.r.t. Euclidean metric:
  - $(\alpha^{-1})'$ is timelike: $\mu \circ \left(\alpha^{-1}\right)'(x) > 0$ for all $x \in \mathbb{R}$.
  - Measure system of $\mathbb{R}^4$: $\mu(\alpha^{-1}(x), \alpha^{-1}(x+y)) = y$ for all $x, y \in \mathbb{R}$.
- $\vec{x} \prec^{\mathfrak{M}^c} \vec{y} \stackrel{\text{def}}{\Leftrightarrow} \mu(\vec{x}, \vec{y}) \geq 0$ and $x_1 < y_1$,
- $\mathrm{In}^{\mathfrak{M}^c} \stackrel{\text{def}}{=} \{\alpha \in \mathbb{C} : \alpha^{-1} \text{ is a line}\}$
- $\mathrm{P}^{\mathfrak{M}^c} = \{\langle \vec{x}, \alpha, y\rangle \in \mathbb{R}^4 \times \mathbb{C}_I \times \mathbb{R} : \alpha(\vec{x}) = y\}$,

The non-accelerating intended model $\mathfrak{M}^c_{\mathrm{I}}$ is the largest submodel of $\mathfrak{M}^c$ whose domain of clocks is $\mathrm{In}^{\mathfrak{M}^c}$.

Goals

- Construct coordinate systems for inertial clocks.

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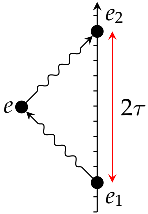
SPACE

$$\color{red}{\text{Distance of events}}\text{: } \delta^i(a,e)=\tau \overset{\text{def}}{\Leftrightarrow} \text{In}(a) \wedge (\exists e_1, e_2 \in \text{wline}_a) \\ \left(e_1 \underset{=}{\rightsquigarrow} e \underset{=}{\rightsquigarrow} e_2 \wedge a(e_1) - a(e_2) = 2\cdot\tau\right)$$



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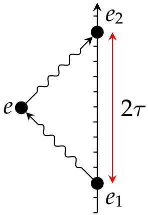
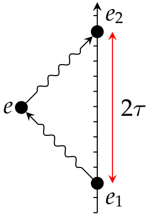


SPACE



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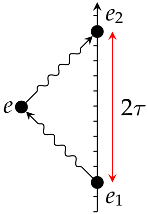

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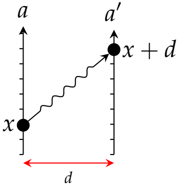
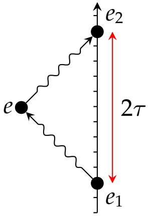
Clocks a and a' are inertial synchronised co-movers iff a' shows $x + \delta^i(a,a')$ whenever a' sees that a shows x.

$$a \overset{\text{syn}}{\uparrow\uparrow} a' \overset{\text{def}}{\Leftrightarrow} (\forall w \in \mathrm{D}_a)(\forall w' \in \mathrm{D}'_a)\left(w \underset{=}{\rightsquigarrow} w' \to a'(w') = a(w) + \delta^i(a,a')\right)$$



Space



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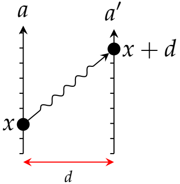

Space of a: $\mathrm{Space}_a \overset{\text{def}}{=} \{a' : a \overset{\text{syn}}{\uparrow\uparrow} a'\}$

Geometry

$$\mathrm{B}(a_1, a_2, a_3) \overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) + \delta^i(a_2, a_3) = \delta^i(a_1, a_3)$$

Geometry

$$\begin{aligned} \mathrm{B}(a_1, a_2, a_3) &\overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) + \delta^i(a_2, a_3) = \delta^i(a_1, a_3) \\ a_1 a_2 \equiv a_3 a_4 &\overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) = \delta^i(a_3, a_4) \end{aligned}$$

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\begin{aligned}
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a_1 a_2 \equiv a_3 a_4 &\overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) = \delta^i(a_3, a_4) \\
C(a_1, a_2, a_3) &\overset{\text{def}}{\Leftrightarrow} B(a_1, a_2, a_3) \vee B(a_3, a_1, a_2) \vee B(a_2, a_3, a_1)
\end{aligned}
$$

Geometry

$$
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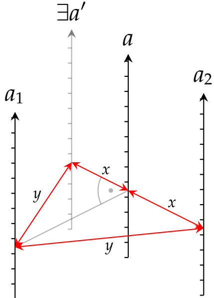
\mathrm{B}(a_1, a_2, a_3) &\overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) + \delta^i(a_2, a_3) = \delta^i(a_1, a_3) \\
a_1 a_2 \equiv a_3 a_4 &\overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) = \delta^i(a_3, a_4) \\
\mathrm{C}(a_1, a_2, a_3) &\overset{\text{def}}{\Leftrightarrow} \mathrm{B}(a_1, a_2, a_3) \vee \mathrm{B}(a_3, a_1, a_2) \vee \mathrm{B}(a_2, a_3, a_1) \\
\mathrm{Ort}(a, a_1, a_2) &\overset{\text{def}}{\Leftrightarrow} \delta^i(a, a_1) > 0 \wedge \delta^i(a_1, a_2) > 0 \wedge \delta^i(a, a_2) > 0 \\
&\qquad \wedge \exists a' \, (\mathrm{B}(a_2, a, a') \wedge a a_2 \equiv a a' \wedge a_1 a_2 \equiv a_1 a)
\end{aligned}
$$



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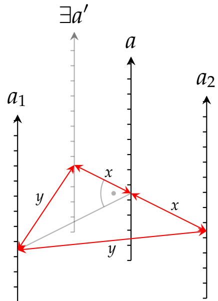
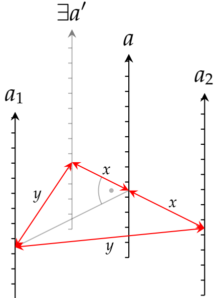
\delta^i(a, (a_1, a_2)) = \tau &\overset{\text{def}}{\Leftrightarrow} \exists a' (\mathrm{Ort}(a', a, a_1) \wedge \mathrm{Ort}(a', a, a_2) \wedge \delta^i(a, a') = \tau)
\end{aligned}
$$

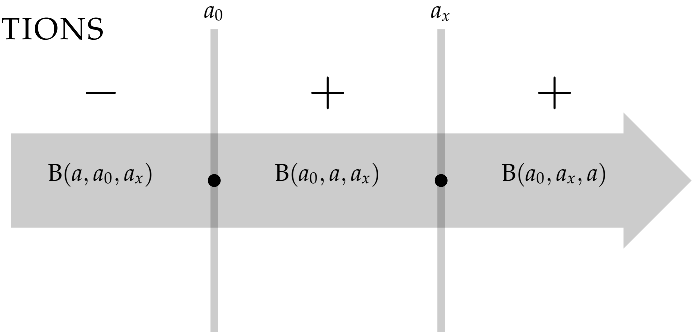


GEOMETRY

$$
\begin{aligned}
\mathrm{B}(a_1,a_2,a_3) &\overset{\text{def}}{\Leftrightarrow} \delta^i(a_1,a_2)+\delta^i(a_2,a_3)=\delta^i(a_1,a_3)\\
a_1a_2 \equiv a_3a_4 &\overset{\text{def}}{\Leftrightarrow} \delta^i(a_1,a_2)=\delta^i(a_3,a_4)\\
\mathrm{C}(a_1,a_2,a_3) &\overset{\text{def}}{\Leftrightarrow} \mathrm{B}(a_1,a_2,a_3)\vee \mathrm{B}(a_3,a_1,a_2)\vee \mathrm{B}(a_2,a_3,a_1)\\
\mathrm{Ort}(a,a_1,a_2) &\overset{\text{def}}{\Leftrightarrow} \delta^i(a,a_1)>0\wedge \delta^i(a_1,a_2)>0\wedge \delta^i(a,a_2)>0\\
&\qquad \wedge\exists a'\,(\mathrm{B}(a_2,a,a')\wedge aa_2\equiv aa'\wedge a_1a_2\equiv a_1a)\\
\delta^i(a,(a_1,a_2))=\tau &\overset{\text{def}}{\Leftrightarrow} \exists a'(\mathrm{Ort}(a',a,a_1)\wedge \mathrm{Ort}(a',a,a_2)\wedge \delta^i(a,a')=\tau)\\
\mathrm{CoordSys}(a,a_x,a_y,a_z) &\overset{\text{def}}{\Leftrightarrow} \mathrm{Ort}(a,a_x,a_y)\wedge \mathrm{Ort}(a,a_y,a_z)\wedge \mathrm{Ort}(a,a_x,a_z)
\end{aligned}
$$



DIRECTIONS



Sign or direction of a point *a on the line given by the ray* (a_0, a_x) is:

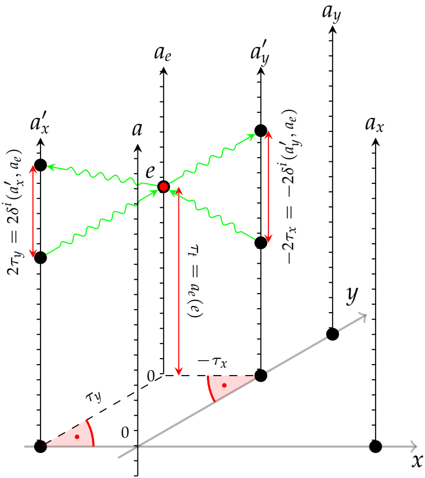
$$\operatorname{sign}^-_{a_0,a_x}(a) = \tau \overset{\text{def}}{\Leftrightarrow} (a \neq a_0 \wedge B(a, a_0, a_x) \wedge \tau = -1) \vee (a = a_0 \wedge \tau = 0) \vee \\ (a \neq a_0 \wedge (B(a_0, a, a_x) \vee B(a_0, a_x, a)) \wedge \tau = 1)$$

For other points the direction is the direction of the projection of that point:

$$\operatorname{sign}_{a_0,a_x}(a) = \tau \overset{\text{def}}{\Leftrightarrow} \operatorname{sign}^-_{a_0,a_x}(a) = \tau \vee \\ \vee \exists a' (\operatorname{Ort}(a', a, a_0) \wedge \operatorname{Ort}(a', a, a_x) \wedge \operatorname{sign}^-_{a_0,a_x}(a') = \tau)$$

Coordinatization

The event e will be coordinatized on the spatiotemporal position $\langle \tau_t, \tau_x, \tau_y, \tau_z \rangle$ by the coordinate system $\langle a, a_x, a_y, a_z \rangle$ iff there is a synchronized co-mover a_e of a that shows the time τ_t in e and $\tau_d = \mathrm{sign}_{a,a_d}(a_e) \cdot \delta^i(a_e, a, a_d)$ for $d \in \{x, y, z\}$.



$$\mathrm{Coord}_{a,a_x,a_y,a_z}(e) = (\tau_t, \tau_x, \tau_y, \tau_z) \overset{\text{def}}{\Leftrightarrow}$$

$$(\exists a_e \in \mathrm{Space}_a)\Big(\mathrm{CoordSys}(a, a_x, a_y, a_z) \wedge \mathrm{P}(e, a_e, \tau_t) \wedge$$

$$\mathrm{sign}_{a,a_x}(a_e) \cdot \delta^i(a_e, (a, a_x)) = \tau_x \wedge$$

$$\mathrm{sign}_{a,a_y}(a_e) \cdot \delta^i(a_e, (a, a_y)) = \tau_y \wedge$$

$$\mathrm{sign}_{a,a_z}(a_e) \cdot \delta^i(a_e, (a, a_z)) = \tau_z\Big)$$

Axioms

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- Axioms are still useful in the practice of high-level proofs.
- Axioms are about a group of agents performing experiments.
  (the result will be part of the distributed knowledge of the group)
  (weak operationalism)
- Axioms are about arbitrary particular agent performing its own experiments.
  (the result will be known by the agent) (strong operationalism)

AxReals

The mathematical sort forms a real closed field.

$$
\begin{array}{llll}
 & (x+y)+z = x+(y+z) & & (x\cdot y)\cdot z = x\cdot(y\cdot z) \\
\exists 0 & x+0 = x & \exists 1 & x\cdot 1 = x \\
\exists(-x) & x+(-x) = 0 & x \neq 0 \rightarrow \exists x^{-1} & x\cdot x^{-1} = 1 \\
 & x+y = y+x & & x\cdot y = y\cdot x
\end{array}
$$

$$x\cdot(x+y) = (x\cdot y)+(x\cdot z)$$

$$
\begin{array}{ll}
a \leq b \wedge b \leq a \rightarrow a = b & a \leq b \rightarrow a+c \leq b+c \\
a \leq b \wedge b \leq c \rightarrow a \leq c & a \leq b \wedge 0 \leq c \rightarrow a\cdot c \leq b\cdot c \\
\neg a \leq b \rightarrow b \leq a &
\end{array}
$$

$$0 \leq x \rightarrow \exists r\; r\cdot r = x$$

$$\exists x(\forall y \in \varphi) x \leq y \rightarrow \exists i(\forall y \in \varphi)(i \leq y \wedge \forall i'((\forall y \in \varphi)(i' \leq y \rightarrow i' \leq i))$$

$$\exists x(\forall y \in \varphi) x \geq y \rightarrow \exists s(\forall y \in \varphi)(s \geq y \wedge \forall s'((\forall y \in \varphi)(s' \geq y \rightarrow s' \geq s))$$

AxFull

Every number occurs as a state of any clock in an event.

$$\forall a \forall x \exists e \quad \mathrm{P}(e, a, x)$$

AxExt

We do not distinguish between (1) indistinguishable clocks, (2) states of a particular clock in an event and (3) two events where a clock shows the same time.

$$
\begin{aligned}
&(1) \quad \forall a, a' \qquad \Big(\forall e \forall x (\mathrm{P}(e,a,x) \leftrightarrow \mathrm{P}(e,a',x))\Big) \;\rightarrow\; a = a' \\
&(2) \quad \forall e \forall a \forall x, y \qquad \big(\mathrm{P}(e,a,x) \wedge \mathrm{P}(e,a,y)\big) \;\rightarrow\; x = y \\
&(3) \quad \forall e, e' \forall a \forall x \qquad \big(\mathrm{P}(e,a,x) \wedge \mathrm{P}(e',a,x)\big) \;\rightarrow\; e = e'
\end{aligned}
$$

AxForward

Clocks are ticking forward.

$$\forall a (\forall e, e' \in \mathrm{wline}_a) \quad \big(e \prec e' \leftrightarrow a(e) < a(e')\big)$$

AxSynchron

All clocks occupying the same worldline (i.e., cohabitants) use the same measure system, and for every clock, and delay, there is a cohabitant clock with that delay.

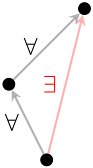
$$\forall a(\forall b \approx a)\exists x(\forall e \in \mathrm{wline}_a) \quad a(e) = b(e) + x$$

$$\forall a \forall x(\exists b \approx a)(\forall e \in \mathrm{wline}_a) \quad a(e) = b(e) + x$$

AxCausality

Causality is transitive.

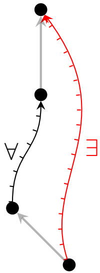
$$(e_1 \prec e_2 \wedge e_2 \prec e_3) \rightarrow e_1 \prec e_3$$



AxChronology

Interiors of lightcones are filled with clocks crossing through the vertex.

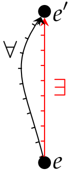
$$(e_1 \preceq e_2 \wedge e_2 \ll e_3 \wedge e_3 \preceq e_4) \rightarrow e_1 \ll e_4$$



AxSecant

Any two events that share a clock share an inertial clock as well.

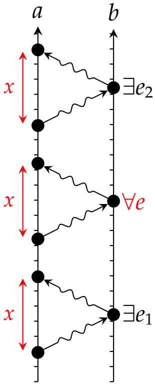
$$e \ll e' \rightarrow (\exists a \in \mathrm{In})(e \mathcal{E} a \land e' \mathcal{E} a))$$



AxInComoving

If an inertial clock measures an other inertial clock with the same distance twice, then they are comoving.

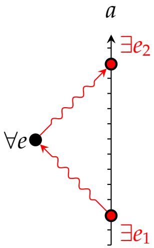
$$\big(e_1 \mathcal{E} b \wedge e_2 \mathcal{E} b \wedge e_1 \neq e_2 \wedge \delta^i(a, e_1) = \delta^i(a, e_2) \wedge a, b \in \mathrm{In}\big) \rightarrow a \overset{\mathrm{i}}{\uparrow\uparrow} b$$



AxPing

Every inertial clock can send and receive a signal to any event.

$$(\forall a \in \mathrm{In})\forall e(\exists e_1, e_2 \in \mathrm{wline}_a) \qquad e_1 \leadsto e \leadsto e_2$$

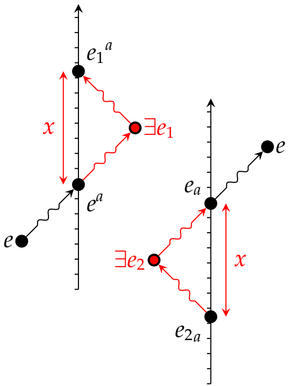


AxRays

For every observer, for any positive x and every direction (given by a light signal) there are lightlike separated events in the past and the future whose distances are exactly x.

$$(\forall x > 0)\forall a \forall e \exists e_1 \exists e_2 (\exists e^a, e_a \in \mathrm{wline}_a)$$

$$\overrightarrow{e_2 e_a e} \wedge \delta^i(a, e_2) = x \wedge \overrightarrow{e e^a e_1} \wedge \delta^i(a, e_1) = x$$

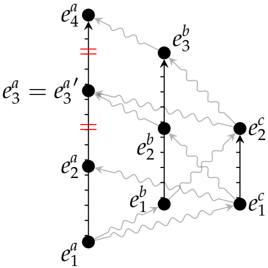


AxRound

Given comoving observers a, b and c, the travelling time of simultaneously sent signals on the route $\langle a,b,c,a\rangle$ and $\langle a,c,b,a\rangle$ are (the same, namely,) the average of the travelling time of the $\langle a,c,a\rangle$ and $\langle a,b,c,b,a\rangle$.

$$b \overset{i}{\uparrow\uparrow} a \overset{i}{\uparrow\uparrow} c \wedge \begin{pmatrix} e_1^a, e_2^a, e_3^a, e_3^{a\prime}, e_4^a \in \mathrm{wline}_a \\ e_1^b, e_2^b, e_3^b \in \mathrm{wline}_b \\ e_1^c, e_2^c \in \mathrm{wline}_c \end{pmatrix} \wedge \begin{pmatrix} e_1^a \rightsquigarrow e_1^b \rightsquigarrow e_2^c \rightsquigarrow e_3^a \\ e_1^a \rightsquigarrow e_1^c \rightsquigarrow e_2^b \rightsquigarrow e_3^{a\prime} \\ e_2^c \rightsquigarrow e_3^b \rightsquigarrow e_4^a \\ e_1^c \rightsquigarrow e_2^a \end{pmatrix} \rightarrow$$

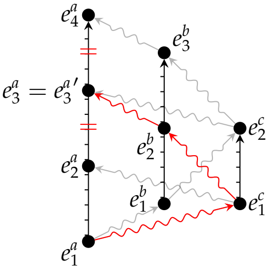
$$\rightarrow \left(a(e_3^a) = a(e_3^{a\prime}) = \frac{a(e_2^a) + a(e_4^a)}{2} \right)$$



AxRound

Given comoving observers a, b and c, the travelling time of simultaneously sent signals on the route $\langle a, b, c, a\rangle$ and $\langle a, c, b, a\rangle$ are (the same, namely,) the average of the travelling time of the $\langle a, c, a\rangle$ and $\langle a, b, c, b, a\rangle$.

$$b \overset{i}{\uparrow\uparrow} a \overset{i}{\uparrow\uparrow} c \wedge \begin{pmatrix} e_1^a, e_2^a, e_3^a, e_3^{a\prime}, e_4^a \in \text{wline}_a \\ e_1^b, e_2^b, e_3^b \in \text{wline}_b \\ e_1^c, e_2^c \in \text{wline}_c \end{pmatrix} \wedge \begin{pmatrix} e_1^a \rightsquigarrow e_1^b \rightsquigarrow e_2^c \rightsquigarrow e_3^a \\ e_1^a \rightsquigarrow e_1^c \rightsquigarrow e_2^b \rightsquigarrow e_3^{a\prime} \\ e_2^c \rightsquigarrow e_3^b \rightsquigarrow e_4^a \\ e_1^c \rightsquigarrow e_2^a \end{pmatrix} \rightarrow$$

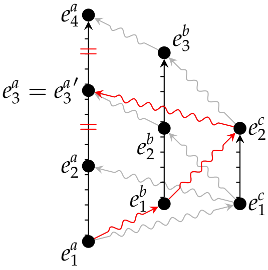


$$\rightarrow \left(a(e_3^a) = a(e_3^{a\prime}) = \frac{a(e_2^a) + a(e_4^a)}{2} \right)$$

AxRound

Given comoving observers a, b and c, the travelling time of simultaneously sent signals on the route $\langle a, b, c, a\rangle$ and $\langle a, c, b, a\rangle$ are (the same, namely,) the average of the travelling time of the $\langle a, c, a\rangle$ and $\langle a, b, c, b, a\rangle$.

$$b \overset{i}{\uparrow\uparrow} a \overset{i}{\uparrow\uparrow} c \wedge \begin{pmatrix} e_1^a, e_2^a, e_3^a, e_3^{a\prime}, e_4^a \in \mathrm{wline}_a \\ e_1^b, e_2^b, e_3^b \in \mathrm{wline}_b \\ e_1^c, e_2^c \in \mathrm{wline}_c \end{pmatrix} \wedge \begin{pmatrix} e_1^a \rightsquigarrow e_1^b \rightsquigarrow e_2^c \rightsquigarrow e_3^a \\ e_1^a \rightsquigarrow e_1^c \rightsquigarrow e_2^b \rightsquigarrow e_3^{a\prime} \\ e_2^c \rightsquigarrow e_3^b \rightsquigarrow e_4^a \\ e_1^c \rightsquigarrow e_2^a \end{pmatrix} \rightarrow$$

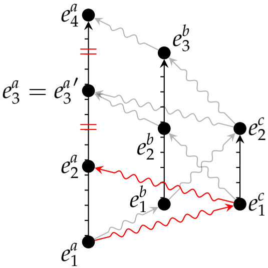


$$\rightarrow \left(a(e_3^a) = a(e_3^{a\prime}) = \frac{a(e_2^a) + a(e_4^a)}{2} \right)$$

AxRound

Given comoving observers a, b and c, the travelling time of simultaneously sent signals on the route $\langle a, b, c, a\rangle$ and $\langle a, c, b, a\rangle$ are (the same, namely,) the average of the travelling time of the $\langle a, c, a\rangle$ and $\langle a, b, c, b, a\rangle$.

$$b \overset{i}{\uparrow\uparrow} a \overset{i}{\uparrow\uparrow} c \wedge \begin{pmatrix} e_1^a, e_2^a, e_3^a, e_3^{a\prime}, e_4^a \in \mathrm{wline}_a \\ e_1^b, e_2^b, e_3^b \in \mathrm{wline}_b \\ e_1^c, e_2^c \in \mathrm{wline}_c \end{pmatrix} \wedge \begin{pmatrix} e_1^a \rightsquigarrow e_1^b \rightsquigarrow e_2^c \rightsquigarrow e_3^a \\ e_1^a \rightsquigarrow e_1^c \rightsquigarrow e_2^b \rightsquigarrow e_3^{a\prime} \\ e_2^c \rightsquigarrow e_3^b \rightsquigarrow e_4^a \\ e_1^c \rightsquigarrow e_2^a \end{pmatrix} \rightarrow$$



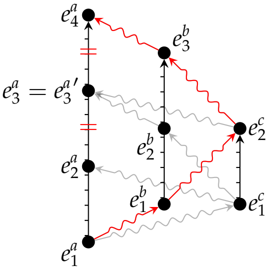
$$\rightarrow \left(a(e_3^a) = a(e_3^{a\prime}) = \frac{a(e_2^a) + a(e_4^a)}{2} \right)$$

AxRound

Given comoving observers a, b and c, the travelling time of simultaneously sent signals on the route $\langle a, b, c, a\rangle$ and $\langle a, c, b, a\rangle$ are (the same, namely,) the average of the travelling time of the $\langle a, c, a\rangle$ and $\langle a, b, c, b, a\rangle$.

$$b \overset{i}{\uparrow\uparrow} a \overset{i}{\uparrow\uparrow} c \wedge \left(\begin{array}{r} e_1^a, e_2^a, e_3^a, e_3^{a\prime}, e_4^a \in \mathrm{wline}_a \\ e_1^b, e_2^b, e_3^b \in \mathrm{wline}_b \\ e_1^c, e_2^c \in \mathrm{wline}_c \end{array} \right) \wedge \left(\begin{array}{l} e_1^a \rightsquigarrow e_1^b \rightsquigarrow e_2^c \rightsquigarrow e_3^a \\ e_1^a \rightsquigarrow e_1^c \rightsquigarrow e_2^b \rightsquigarrow e_3^{a\prime} \\ e_2^c \rightsquigarrow e_3^b \rightsquigarrow e_4^a \\ e_1^c \rightsquigarrow e_2^a \end{array} \right) \to$$

$$\to \left(a(e_3^a) = a(e_3^{a\prime}) = \frac{a(e_2^a) + a(e_4^a)}{2} \right)$$

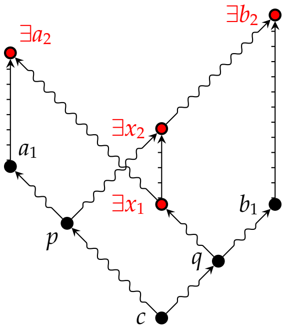


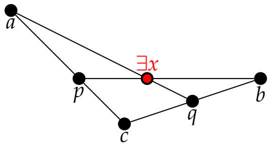
AxPasch

Pasch-axiom for light signals.

$$\Big(a \overset{\mathrm{i}}{\uparrow\uparrow} b \wedge (\exists a_1 \in \mathrm{wline}_a)(\exists b_1 \in \mathrm{wline}_b)(\overrightarrow{cpa_1} \wedge \overrightarrow{cqb_1})\Big) \rightarrow$$

$$\rightarrow (\exists x \overset{\mathrm{i}}{\uparrow\uparrow} a)(\exists x_1, x_2 \in \mathrm{wline}_x)(\exists a_2 \in \mathrm{wline}_a)(\exists b_2 \in \mathrm{wline}_b)(\overrightarrow{px_2b_2} \wedge \overrightarrow{qx_1a_2})$$





Ax5Segment

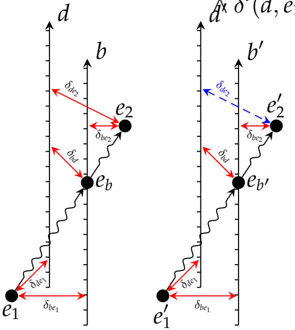
If two pairs of observers b,d and b',d' measures two pair of lightlike separated events e_1, e_2 and e_1', e_2' to the same distances, respectively, and the lightline crosses the worldlines of b and b', respectively, then the distances b-d and b'-d' are the same (for all of them).

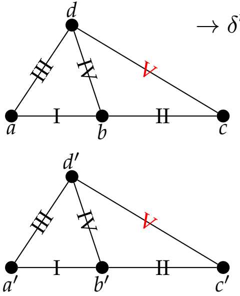
$$d \overset{i}{\uparrow\uparrow} d' \wedge b \overset{i}{\uparrow\uparrow} b' \wedge e_b \mathcal{E} b \wedge e_b' \mathcal{E} b' \wedge \overrightarrow{e_1 e_b e_2} \wedge \overrightarrow{e_1' e_b' e_2'} \wedge$$

$$\wedge\, \delta^i(b, e_1) = \delta^i(b', e_1') \wedge \delta^i(b, e_2) = \delta^i(b', e_2') \wedge$$

$$\wedge\, \delta^i(d, e_1) = \delta^i(d', e_1') \wedge \delta^i(b, d) = \delta^i(b', d')) \rightarrow$$

$$\rightarrow \delta^i(d, e_2) = \delta^i(d', e_2')$$

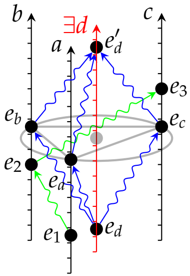




AxCircle

For every three non-collinear inertial observer there is a fourth one that measures them with the same distance.

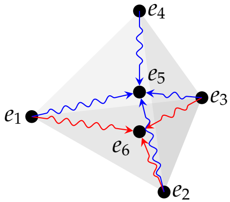
$$(\forall a,b,c \in \mathrm{In})\big((a \overset{\mathrm{i}}{\uparrow\uparrow} b \overset{\mathrm{i}}{\uparrow\uparrow} c \wedge$$
$$\wedge\ \exists e_1,e_2,e_3(e_1\mathcal{E}a \wedge e_2\mathcal{E}b \wedge e_3\mathcal{E}c \wedge e_1 \rightsquigarrow e_2 \rightsquigarrow e_3 \wedge \neg e_1 \rightsquigarrow e_3)) \rightarrow$$
$$\rightarrow \exists d \exists e_a,e_b,e_c,e_d,e_d'(e_a\mathcal{E}a \wedge e_b\mathcal{E}b \wedge e_c\mathcal{E}c \wedge e_d\mathcal{E}d \wedge e_d'\mathcal{E}d \wedge$$
$$\wedge\ e_d \rightsquigarrow e_a \rightsquigarrow e_d' \wedge e_d \rightsquigarrow e_b \rightsquigarrow e_d' \wedge e_d \rightsquigarrow e_c \rightsquigarrow e_d')\big)$$



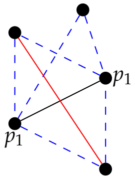
AxMinDim : n

The dimension of the spacetime is at least n. The formula says that $n-1$ lightcones never intersect in only one event.

$$\forall e_1,\ldots,e_n \left(\bigwedge_{i\leq n-1} e_i \rightsquigarrow e_n \rightarrow \exists e_{n+1} \left(\bigwedge_{i\leq n-1} e_i \rightsquigarrow e_n \wedge e_n \neq e_{n+1} \right)\right)$$



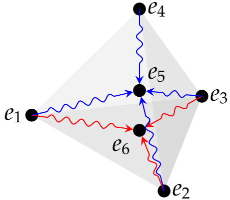
Tarski's lower n-dimensional axiom: Centers of circumscribed spheres around $n-1$ points cannot be covered with a line.



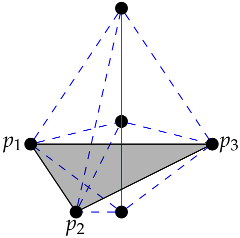
AxMaxDim : n

The dimension of the spacetime is at most n. The formula says that there are n lightcones that intersect at most in one event.

$$\exists e_1, \ldots, e_{n+1} \left(\bigwedge_{i \le n} e_i \rightsquigarrow e_{n+1} \wedge \forall e_{n+2} \left(\bigwedge_{i \le n} e_i \rightsquigarrow e_{n+2} \rightarrow e_{n+1} = e_{n+2} \right) \right)$$



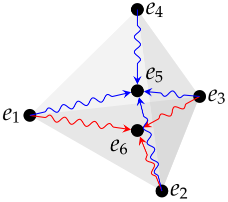
Tarski's upper n-dimensional axiom: Centers of circumscribed spheres around n points are on a line.



Ax4Dim

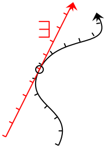
The dimension of the spacetime is exactly 4; 3 lightcones never intersect in only one event and there are 4 lightcones intersect in at most one event.

$$\mathsf{AxMinDim} : 4 \wedge \mathsf{AxMaxDim} : 4$$



AxTangent

For every event of every clock there is an inertial clock that shares that event and the local instantaneous velocity of that observer.



AxNoAcceleration

Every clock is inertial.

$$\forall a \quad \mathrm{In}(a)$$

AxAcceleration

For every coordinate system and every definable timelike curve there is a clock having that wordline in the coordinate system.

Axiom systems

We define SClTh to be the following sets of axioms.

$$\mathrm{SClTh} \stackrel{\mathrm{def}}{=} \left\{ \begin{array}{llll} \mathsf{AxFull} & \mathsf{AxCausality} & \mathsf{AxRay} & \mathsf{Ax5Seg} \\ \mathsf{AxExt} & \mathsf{AxChronology} & \mathsf{AxPing} & \mathsf{AxCircle} \\ \mathsf{AxForward} & \mathsf{AxSecant} & \mathsf{AxRound} & \mathsf{Ax4Dim} \\ \mathsf{AxSynchron} & \mathsf{AxInComovement} & \mathsf{AxPasch} & \mathsf{AxTangent} \end{array} \right\}$$

$$\mathrm{SClTh}^{\mathrm{NoAcc}} \stackrel{\mathrm{def}}{=} (\mathrm{SClTh} - \{\mathsf{AxSecant}, \mathsf{AxTangent}\}) \cup \{\mathsf{AxNoAcceleration}\}$$

$$\mathrm{SClTh}^{\mathrm{Acc}} \stackrel{\mathrm{def}}{=} \mathrm{SClTh} \cup \{\mathsf{AxAcceleration}\}$$

Theorems

Plan

1 Kronheimer-Penrose axioms (Immediate)

PLAN

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2. Signalling (radar-distance) is unique (Immediate)

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5. Straight signals arrive sooner (Nice)

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7. There are no two ISCM/points in an event. (simple)

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7. There are no two ISCM/points in an event. (simple)
8. ‘Equivalence’ of $\overrightarrow{e_a e_b e_c}$ and $\mathrm{B}(a, b, c)$. (simple)

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7. There are no two ISCM/points in an event. (simple)
8. 'Equivalence' of $\overrightarrow{e_a e_b e_c}$ and $\mathrm{B}(a, b, c)$. (simple)
9. Tarski's axioms. (Mostly trivial)

PLAN

1. Kronheimer-Penrose axioms (Immediate)
2. Signalling (radar-distance) is unique (Immediate)
3. There is an ISCM/point in every event (Nice)
4. There is a point in every event (Immediate)
5. Straight signals arrive sooner (Nice)
6. $\overset{\text{syn}}{\uparrow\uparrow}$ is an eq.rel. and δ^i is a metric on $\overset{\text{syn}}{\uparrow\uparrow}$-related clocks (Important, long)
7. There are no two ISCM/points in an event. (simple)
8. 'Equivalence' of $\overrightarrow{e_a e_b e_c}$ and $\mathrm{B}(a, b, c)$. (simple)
9. Tarski's axioms. (Mostly trivial)
10. Coordinatization is a bijection between W and Q^4. (simple)

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11. Radar-based spatial distance and elapsed time defines the same quantities as coordinate based definition. (Tarski)
12. Proving 'Simple-SpecRel'. (simple)

Kronheimer-Penrose axioms

$$
\begin{aligned}
& e \preceq e \\
& (e_1 \preceq e_2 \wedge e_2 \preceq e_3) \rightarrow e_1 \preceq e_3 \\
& (e_1 \preceq e_2 \wedge e_2 \preceq e_1) \rightarrow e_1 = e_2 \\
& \neg e \ll e \\
& e_1 \ll e_2 \rightarrow e_1 \preceq e_2 \\
& (e_1 \preceq e_2 \wedge e_2 \ll e_3) \rightarrow e_1 \ll e_3 \\
& (e_1 \ll e_2 \wedge e_2 \preceq e_3) \rightarrow e_1 \ll e_3 \\
& e_1 \underset{=}{\rightsquigarrow} e_2 \leftrightarrow (e_1 \preceq e_2 \wedge \neg e_1 \ll e_2)
\end{aligned}
$$

Are consequences of

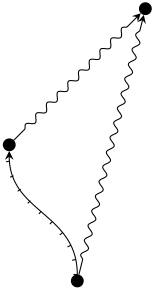
$$\neg e \prec e \tag{1}$$

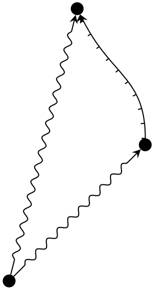
$$(e_1 \prec e_2 \wedge e_2 \ll e_3) \rightarrow e_1 \ll e_3 \tag{2}$$

$$(e_1 \ll e_2 \wedge e_2 \prec e_3) \rightarrow e_1 \ll e_3 \tag{3}$$

- (1) comes from **AxForward**; $e \prec e$ would lead to $a(e) < a(e)$.
- (2): is **AxChronology** where $e_1 \neq e_2$ and $e_3 = e_4$.
- (3): is **AxChronology** where $e_1 = e_2$ and $e_3 \neq e_4$.

Forbidden triangles





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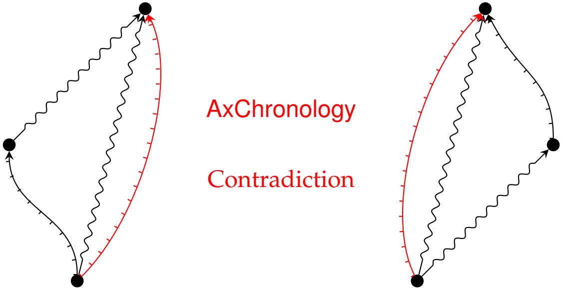

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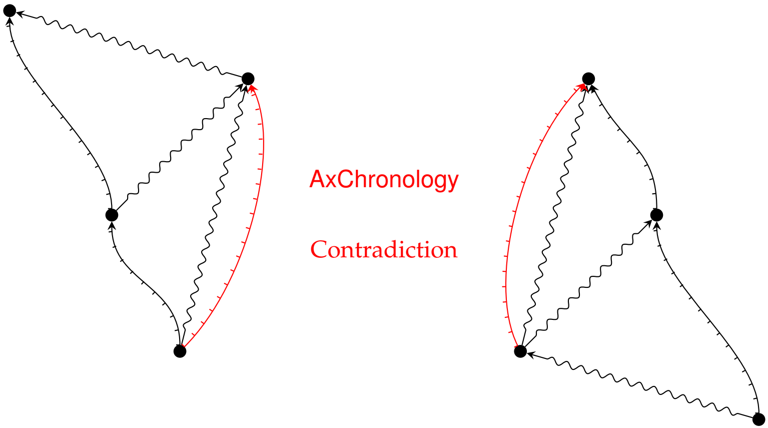
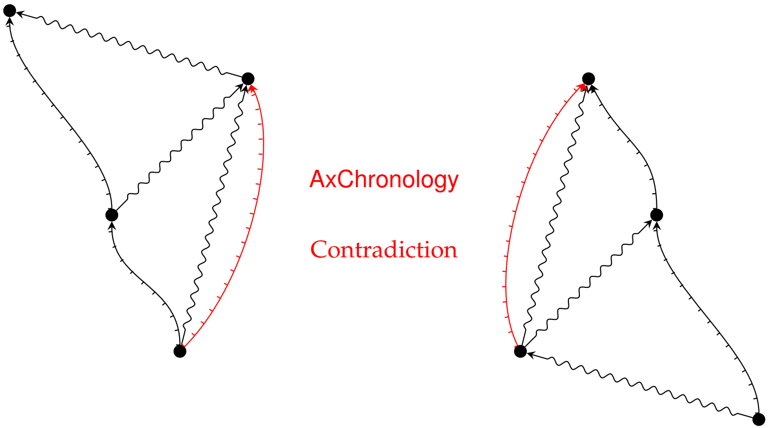
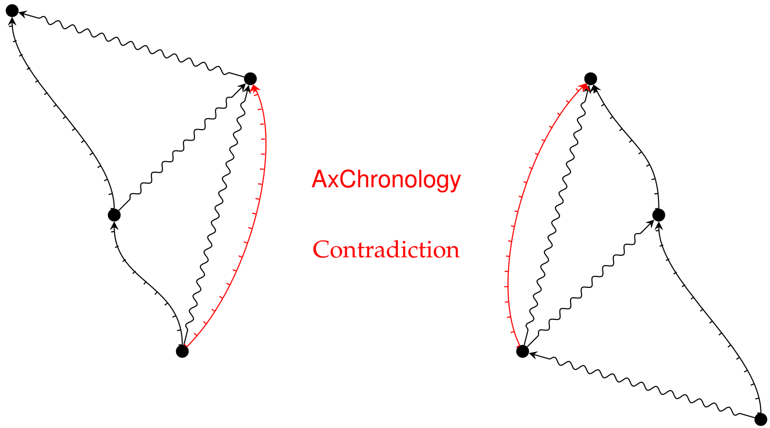


Forbidden triangles



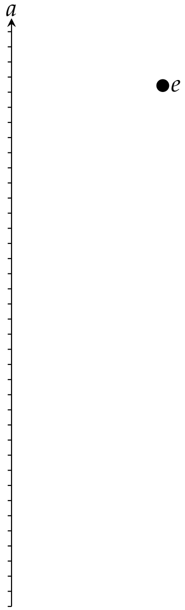
<u>Corollary</u>: If a (not necessarily inertial) clock can radar an event, then the elapsed time (distance) is unique.

Forbidden triangles



COROLLARY: If a (not necessarily inertial) clock can radar an event, then the elapsed time (distance) is unique. COROLLARY: $\delta^i(a,e) = \tau$ is a total function by AxPing

EXISTENCE OF ISCMS IN EVENTS



For every inertial observer, there is a synchronized inertial observer in any event.

$$(\forall a \in \mathrm{In})\forall e \exists b \qquad e\mathcal{E}b \wedge a \overset{\text{syn}}{\uparrow\uparrow} b$$

Let $a \in \mathrm{In}$ and e be arbitrary. If $e \in \mathrm{wline}_a$ then we are ready. Suppose now that $e \notin \mathrm{wline}_a$. By AxPing, there are $e_a, e^a \in \mathrm{wline}_a$ s.t. $e_a \nearrow e \nearrow e^a$. Let $x \overset{\text{def}}{=} a(e^a) - a(e_a)$. Note that $\delta^i(a, e) = x$ is true. By AxCausality and AxForward and by the assumption that $e \notin \mathrm{wline}_a$, this x is strictly positive. By AxRay, there is an e_0 s.t. e_0 is 1 distance away from a and $\overrightarrow{e_0 e_a e}$. By AxPing, there is an $e_{0_a} \in \mathrm{wline}_a$ s.t. $e_{0_a} \nearrow e_0$. By AxRay again, there is an event e_b s.t. $\overrightarrow{e_b e_{0_a} e_0}$ and $\delta^i(a, e_b) = x$. Since $e_b \nearrow e_{0_a} \ll e_a \nearrow e$, by AxChronology we have $e_b \ll e$. By AxSecant, there is an inertial clock b through e_b and e. Now since both a and b are inertial and $\delta^i(a, e_b) = x$ and $\delta^i(a, e) = x$, by AxInCoMovement, $a \overset{\text{i}}{\uparrow\uparrow} b$, and by AxSynchron again, there is an a-synchronized b' cohabitant of b here as well; that is the clock having delay x.

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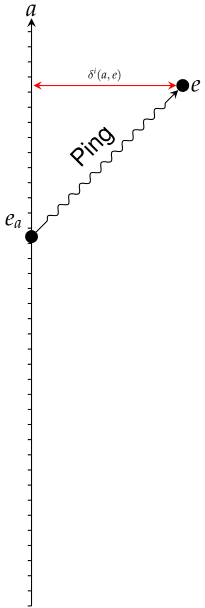

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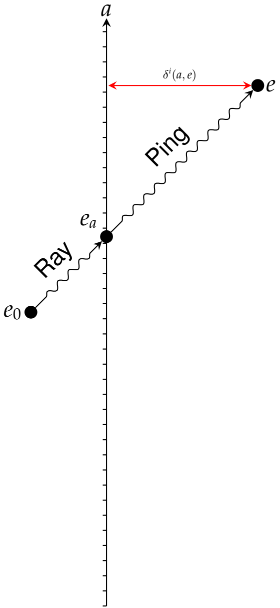

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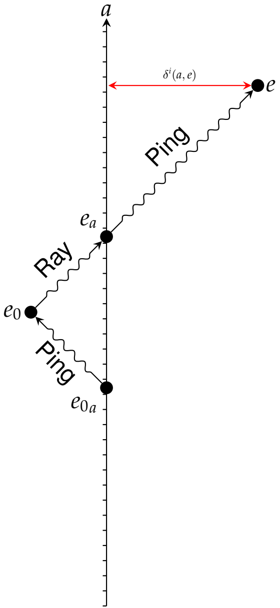

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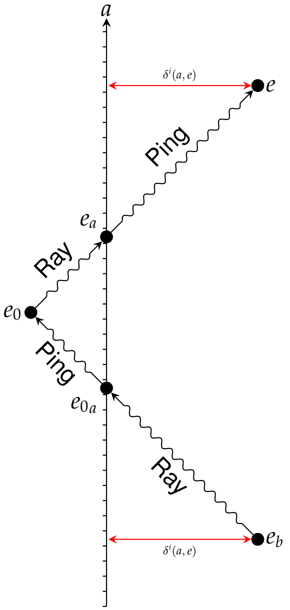

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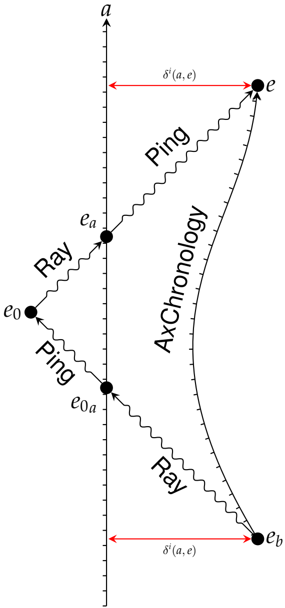

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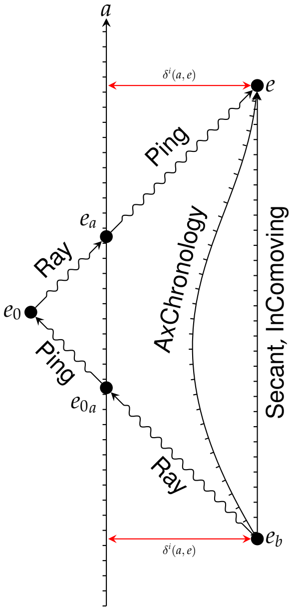
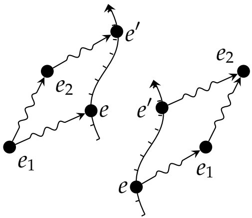


There is a clock in every event.

$$\forall e \exists c \quad e\mathcal{E}c$$

Let e be an arbitrary event. There is a clock a in some event e_0 by AxFull (and by the tautology $\exists a\, a = a$). By AxSecant, there is an inertial clock at e_0 as well. By the previous proposition, there is an inertial comover of a at e.

STRAIGHT SIGNALS ARRIVE SOONER



$$\forall a \forall e_1, e_2, e, e' (e \mathcal{E} a \wedge e' \mathcal{E} a \wedge e_1 \underset{=}{\rightsquigarrow} e \wedge e_1 \underset{=}{\rightsquigarrow} e_2 \underset{=}{\rightsquigarrow} e') \rightarrow a(e) \leq a(e')$$

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Indirectly by the Kronheimer-Penrose axioms.

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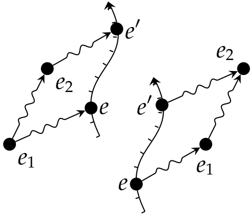

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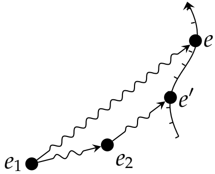

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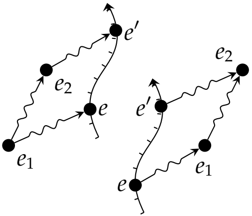

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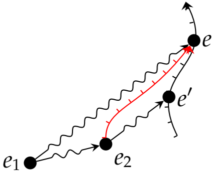
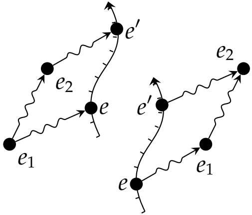


Straight signals arrive sooner



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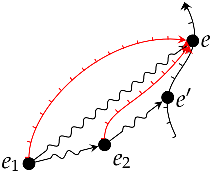

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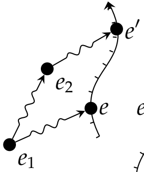

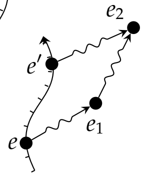

$$\forall a \forall e_1, e_2, e, e' (e \mathcal{E} a \wedge e' \mathcal{E} a \wedge e_1 \underset{=}{\rightsquigarrow} e \wedge e_1 \underset{=}{\rightsquigarrow} e_2 \underset{=}{\rightsquigarrow} e') \rightarrow a(e) \leq a(e')$$

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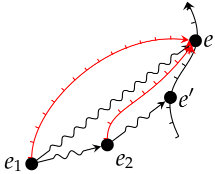
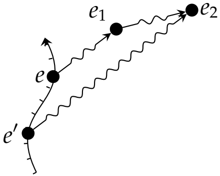


Indirectly by the Kronheimer-Penrose axioms.



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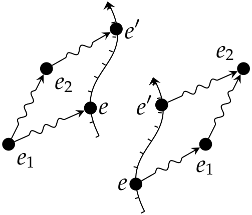

$$\forall a \forall e_1, e_2, e, e' (e \mathcal{E} a \wedge e' \mathcal{E} a \wedge e_1 \underset{=}{\rightsquigarrow} e \wedge e_1 \underset{=}{\rightsquigarrow} e_2 \underset{=}{\rightsquigarrow} e') \rightarrow a(e) \leq a(e')$$

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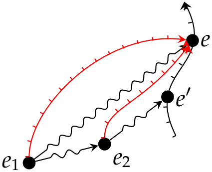
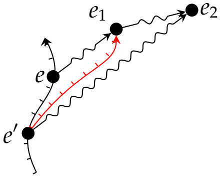
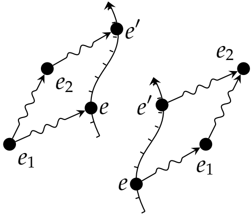




STRAIGHT SIGNALS ARRIVE SOONER



$$\forall a \forall e_1, e_2, e, e' \big(e\mathcal{E}a \wedge e'\mathcal{E}a \wedge e_1 \underset{=}{\rightsquigarrow} e \wedge e_1 \underset{=}{\rightsquigarrow} e_2 \underset{=}{\rightsquigarrow} e'\big) \rightarrow a(e) \leq a(e')$$

$$\forall a \forall e_1, e_2, e, e' \big(e\mathcal{E}a \wedge e'\mathcal{E}a \wedge e' \underset{=}{\rightsquigarrow} e_2 \wedge e \underset{=}{\rightsquigarrow} e_1 \underset{=}{\rightsquigarrow} e_2\big) \rightarrow a(e) \leq a(e')$$

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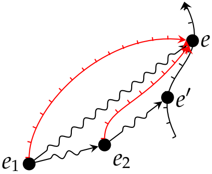
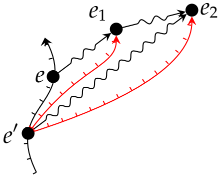




METRIC THEOREM

$\overset{\text{syn}}{\uparrow\uparrow}$ is an equivalence relation and δ^i is a(n U-relative) metric on $\overset{\text{syn}}{\uparrow\uparrow}$ related clocks, i.e.,

$$a \overset{\text{syn}}{\uparrow\uparrow} a$$

$$a_1 \overset{\text{syn}}{\uparrow\uparrow} a_2 \Rightarrow a_2 \overset{\text{syn}}{\uparrow\uparrow} a_1$$

$$a_1 \overset{\text{syn}}{\uparrow\uparrow} a_2 \wedge a_2 \overset{\text{syn}}{\uparrow\uparrow} a_3 \Rightarrow a_1 \overset{\text{syn}}{\uparrow\uparrow} a_3$$

$$\delta^i(a, a) = 0$$

$$\delta^i(a_1, a_2) = 0 \Rightarrow a_1 = a_2$$

$$\delta^i(a_1, a_2) = \delta^i(a_2, a_1)$$

$$\delta^i(a_1, a_2) + \delta^i(a_2, a_3) \geq \delta^i(a_1, a_3)$$

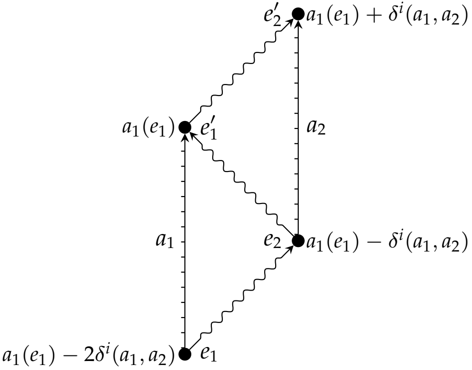
We prove these simultaneously.

Reflexivity of $\overset{\text{syn}}{\uparrow\uparrow}$, selfdistance=0

- **Self-distance**: By $e \underset{=}{\leadsto} e \underset{=}{\leadsto} e$ we have $\delta^i(a,e) = a(e) - a(e) = 0$. The truth of $\delta^i(a,a) = 0$ is trivially implied by that fact.
- **Reflexivity of** $\overset{\textbf{syn}}{\uparrow\uparrow}$: By the self-distance we have $a(e') = a(e) + 0$ whenever $e \underset{=}{\leadsto} e'$, so $\overset{\text{syn}}{\uparrow\uparrow}$ is reflexive.

Symmetry of $\overset{\text{syn}}{\uparrow\uparrow}$ and δ^i

Suppose that $a_1 \overset{\text{syn}}{\uparrow\uparrow} a_2$.



IDENTITY OF INDISCERNIBLES

Take arbitrary iscm's a_1 and a_2 for which $\delta^i(a_1, a_2) = 0$, i.e.,

$$(\forall e \in \mathrm{wline}_{a_2})(\exists w_1, w_2 \in \mathrm{wline}_{a_1}) w_1 \underline{\rightsquigarrow} e \underline{\rightsquigarrow} w_2 \wedge a_1(w_2) - a_1(w_2) = 0$$

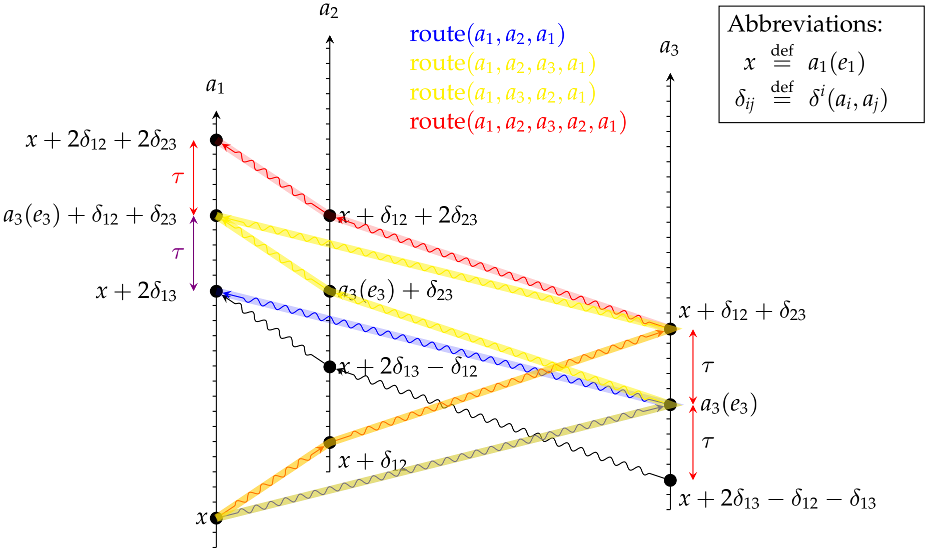
but that means that $a_1(w_1) = a_1(w_2)$, so $w_1 = w_2$. Observe that

- $w_1 \rightsquigarrow e$ and $e \rightsquigarrow w_2 = w_1$ are impossible (AxCausality, irreflexivity).
- $w_1 = e \rightsquigarrow w_2$ or $w_2 = e \rightsquigarrow w_1$ are impossible too since w_1 and w_2 share a.

So the only possiblity is that $w_1 = e = w_2$.
Since this is true for all $e \in \mathrm{wline}_{a_2}$, we have that $\mathrm{wline}_{a_2} \subseteq \mathrm{wline}_{a_1}$. By symmetry of δ^i we have that $\mathrm{wline}_{a_2} = \mathrm{wline}_{a_1}$. Now since a_1 and a_2 are iscms, they show the same numbers in the same events, therefore $a_1 = a_2$.

TRANSITIVITY OF $\overset{\text{syn}}{\uparrow\uparrow}$

We have to show that $a_3(e_3) = x + d_{13}$.



Triangle inequality

By AxPing, we can take $e_1 \in \mathrm{wline}_{a_1}$, $e_2 \in \mathrm{wline}_{a_2}$, $e_3, e_3^* \in \mathrm{wline}_{a_3}$ s.t. $e_1 \underset{=}{\rightsquigarrow} e_2 \underset{=}{\rightsquigarrow} e_3$ and $e_1 \rightsquigarrow e_3^*$. Since $\overset{\mathrm{syn}}{\uparrow\uparrow}$ is an eq.rel, we have that

$$a_3(e_3) = a_1(e_1) + \delta^i(a_1, a_2) + \delta^i(a_2, a_3)$$
$$a_3(e_3^*) = a_1(e_1) + \delta^i(a_1, a_3)$$

Since straight signals arrive sooner, $a(e_3^*) \leq a(e_3)$, so

$$a_1(e_1) + \delta^i(a_1, a_3) \leq a_1(e_1) + \delta^i(a_1, a_2) + \delta^i(a_2, a_3)$$

No clock has two ISCMs at the same event

$$(\forall a \in \mathrm{In})\forall e(\forall a_1, a_2 \in \mathrm{D}_e) \quad a_1 \overset{\text{syn}}{\upuparrows} a \overset{\text{syn}}{\upuparrows} a_2 \Rightarrow a_1 = a_2$$

Let $e \in \mathrm{wline}_{a_1} \cap \mathrm{wline}_{a_2}$ be arbitrary but fixed. Let a_1 and a_2 be inertial comovers of a occurring at e.

- transitivity $\overset{\text{syn}}{\upuparrows}$: $\quad a_1 \overset{\text{syn}}{\upuparrows} a_2$.

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- transitivity $\overset{\mathrm{syn}}{\uparrow\uparrow}$: $a_1 \overset{\mathrm{syn}}{\uparrow\uparrow} a_2$.
- Selfdistance: $\delta^i(a_1, e) = \delta^i(a_2, e) = 0$.

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- Since $a_1 \overset{\text{syn}}{\uparrow\uparrow} a_2$ implies comovement, i.e., constant distance: $\delta^i(a_1, a_2) = 0$.

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- Since $a_1 \overset{\text{syn}}{\uparrow\uparrow} a_2$ implies comovement, i.e., constant distance: $\quad \delta^i(a_1, a_2) = 0$.
- Identity of indiscernibles: $\quad a_1 = a_2$.

'Equivalence' of betweens

For any three distinct inertial comovers a, b and c, the clock b is between a and c iff a can send a light signal to c through b.

$$\forall a_0 (\forall a, b, c \in \mathrm{Space}_{a_0})$$
$$a \neq b \neq c \wedge \mathrm{B}(a, b, c) \leftrightarrow \exists e_a, e_b, e_c (e_a \mathcal{E} a \wedge e_b \mathcal{E} b \wedge e_c \mathcal{E} c \wedge \overrightarrow{e_a e_b e_c})$$

$\Leftarrow$: Since we have iscm observers

$$\begin{aligned} c(e_c) &= a(e_a) + \delta^i(a, b) + \delta^i(a, c) && \text{by } e_a \leadsto e_b \leadsto e_c \\ &= a(e_a) + \delta^i(a, c) && \text{by } e_a \leadsto e_c \end{aligned}$$

therefore $\delta^i(a, b) + \delta^i(b, c) = \delta^i(a, c)$.

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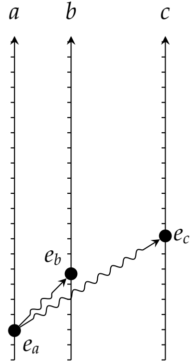

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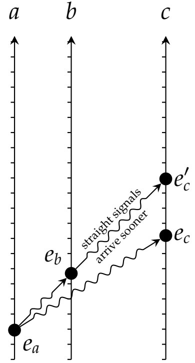

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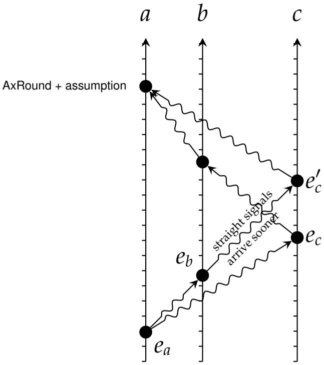

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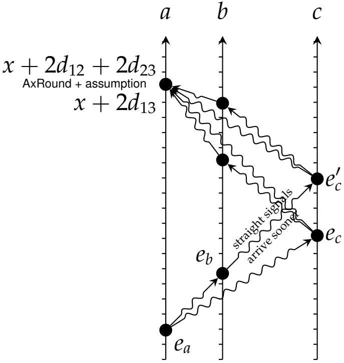

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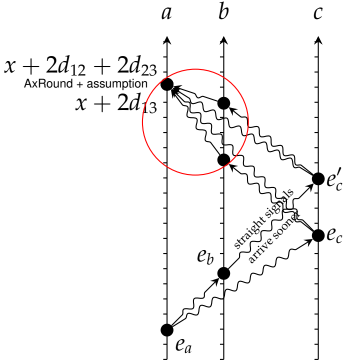

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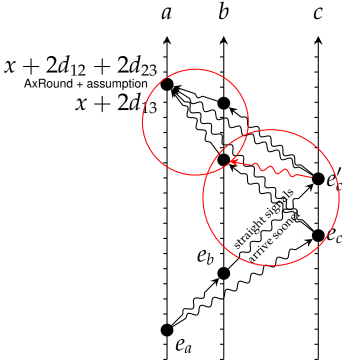

TARSKI'S AXIOMATIZATION OF GEOMETRY

1. $ab \equiv ba$ (Reflexivity for $\equiv$)
2. $(ab \equiv pq \wedge ab \equiv rs) \rightarrow pq \equiv rs$ (Transitivity for $\equiv$)
3. $ab \equiv cc \rightarrow a = b$ (Identity for $\equiv$)
4. $\exists x(B(qax) \wedge ax \equiv bc)$ (Segment Construction)
5. $(a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge$ $\wedge\, ad \equiv a'd' \wedge bd \equiv b'd') \rightarrow cd \equiv c'd'$ (Five-segment)
6. $B(aba) \rightarrow a = b$ (Identity for B)
7. $(B(apc) \wedge B(bqc)) \rightarrow \exists x(B(pxb) \wedge B(qxa))$ (Pasch)

8^n. $\exists a, b, c, p_1, \ldots p_{n-1}\left(\bigwedge_{i<j<n} p_i \neq p_j \wedge \bigwedge_{1<i<n} (ap_1 \equiv ap_i \wedge bp_1 \equiv bp_i \wedge cp_1 \equiv cp_i) \wedge \wedge\, \neg(B(abc) \vee B(bca) \vee B(cab))\right)$ (Lower n-dimension)

9^n. $\left(\bigwedge_{i<j<n} p_i \neq p_j \wedge \bigwedge_{1<i<n} (ap_1 \equiv ap_i \wedge bp_1 \equiv bp_i \wedge cp_1 \equiv cp_i)\right) \rightarrow \rightarrow (B(abc) \vee B(bca) \vee B(cab))$ (Upper n-dimension)

10_2. $B(abc) \vee B(bca) \vee B(cab) \vee \exists x(ax \equiv bx \wedge ax \equiv cx)$ (Circumscribed tr.)

11. $\exists a \forall x, y(\alpha \wedge \beta \rightarrow B(axy)) \rightarrow \exists b \forall x, y(\alpha \wedge \beta \rightarrow B(aby))$ (Continuity scheme)

where α and β are first-order formulas, the first of which does not contain any free occurrences of a, b and y and the second any free occurrences of a, b, x.

Every event is coordinatized with a 4-tuple. (Totality)

Let e be an arbitrary event. Since there exactly one iscm there, we have a synchronized comover a_e of a in e. Then by definition, $a_e(e)$ will be the time coordinate. We can use Tarski's axioms to conclude that there are (unique) a'_x, a'_y and a'_z that are projections of the point a_e to the lines (a, a_x), (a, a_y) and (a, a_z), respectively. By AxPing, these projections can ping a_e, i.e., they can measure the spatial distance between them and a_e (and e), and thus we will have the spatial coordinates of e as well.

Corollary: No event has two different coordinates. (Functionality)

EVERY 4-TUPLE IS A COORDINATE OF AN EVENT. (SURJECTIVITY)

Let (t, x, y, z) be an arbitrary 4-tuple. It follows from Tarski's axioms that there are planes there are inertial comovers a'_x, a'_y and a'_z of a on the axes (a, a_x), (a, a_y) and (a, a_z), respectively, such that $\delta^i(a, a_x) = x$, $\delta^i(a, a_y) = y$ and $\delta^i(a, a_t) = t$. For all $i \in \{x, y, z\}$ Let P_i denote the plane that contains a'_i and is orthogonal to the line (a, a_i). Then by Tarski's axioms, these planes has one unique intersection, a_e. By the definition of the Coord, any event of wline_{a_e} are coordinatized on the spatial coordinates (x, y, z). Now we know from Ax-Full that there is an event e of wline_{a_e} such that $a(e) = t$.

COROLLARY: No 4-tuple is a coordinatization of two different events.

(Injectivity)

Spatial distance

$$\mathrm{sd}_a(e, e') = \tau \overset{\text{def}}{\Leftrightarrow} (\exists a' \in \mathrm{Space}_a)(a \in \mathrm{D}_e \wedge \delta^i(a, e') = \tau)$$

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$$\mathrm{sd}_a(e,e') = \tau \iff (\exists \langle a_x, a_y, a_z \rangle \in \mathrm{CoordSys(a)})\exists \vec{x}\vec{y}$$
$$\mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \wedge \mathrm{Coord}_{a,a_x,a_y,a_z}(e') = \vec{y} \wedge \tau = |\vec{x}_{2-4} - \vec{y}_{2-4}|$$

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Pythagoras's theorem:

$$\delta^i(a_e, a_{e'})^2 = \delta^i(a_e, b)^2 + \delta^i(b, a_{e'})^2$$

where $b \in \mathrm{Space_a}$ is a clock with which

$$\mathrm{Ort}(a'_x, a, b) \wedge \mathrm{Ort}(a'_y, a, b) \wedge \mathrm{Ort}(a'_z, a, b)$$

where a'_x, a'_y, a'_z, are the projections of a_e to the axes of the coordinate system (see the figure of coordinatization).

Elapsed time

$$\mathrm{et}_a(e, e') = \tau \stackrel{\text{def}}{\Leftrightarrow} (\exists b, b' \in \mathrm{Space}_a)\ |b(e) - b'(e')| = \tau$$

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$$\mathrm{et}_a(e, e') = \tau \overset{\text{def}}{\Leftrightarrow} (\exists b, b' \in \mathrm{Space}_a)\ |b(e) - b'(e')| = \tau$$

$$\mathrm{et}_a(e, e') = \tau \iff (\exists \langle a_x, a_y, a_z \rangle \in \mathrm{CoordSys(a)}) \exists \vec{x}, \vec{y}$$
$$\mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \wedge \mathrm{Coord}_{a,a_x,a_y,a_z}(e') = \vec{y} \wedge \tau = |\vec{x}_1 - \vec{y}_1|$$

Elapsed time

$$\mathrm{et}_a(e, e') = \tau \overset{\mathrm{def}}{\Leftrightarrow} (\exists b, b' \in \mathrm{Space}_a)\ |b(e) - b'(e')| = \tau$$

$$\mathrm{et}_a(e, e') = \tau \iff (\exists \langle a_x, a_y, a_z \rangle \in \mathrm{CoordSys(a)}) \exists \vec{x}, \vec{y}$$
$$\mathrm{Coord}_{a, a_x, a_y, a_z}(e) = \vec{x} \wedge \mathrm{Coord}_{a, a_x, a_y, a_z}(e') = \vec{y} \wedge \tau = |\vec{x}_1 - \vec{y}_1|$$

The clocks that measures the time in the events are the same in both definitions by Proposition ‘there are no two iscms in one event’, so practically, both formula refer to the same measurement.

Speed

$$\mathrm{v}_a(e, e') \stackrel{\text{def}}{=} \frac{\mathrm{sd}_a(e, e')}{\mathrm{et}_a(e, e')}$$

SIMPLE SPECREL

Simple-AxSelf $\forall a(\forall e \in \text{wline}_a)(\forall\langle a_x, a_y, a_z\rangle \in \text{CoordSys(a)})$
$\exists t\ \text{Coord}_{a,a_x,a_y,a_z}(e) = (t, 0, 0, 0)$

Simple-AxPh $(\forall a \in \text{In})\forall e, e' \qquad \text{v}_a(e, e') = 1 \leftrightarrow e \rightsquigarrow e'$

Simple-AxEv $\forall e(\forall\langle a, a_x, a_y, a_z\rangle, \langle a', a'_x, a'_y, a'_z\rangle \in \text{CoordSys})$
$\exists\vec{x}\ \text{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \rightarrow \exists\vec{y}\ \text{Coord}_{a',a'_x,a'_y,a'_z}(e) = \vec{y}$

Simple-AxSym $(\forall a, a' \in \text{In})\forall e, e'$
$\text{et}_a(e, e') = \text{et}_{a'}(e, e') = 0 \rightarrow \text{sd}_a(e, e') = \text{sd}_{a'}(e, e')$

Simple-AxThExp $\forall a \forall e, e' \quad \text{v}_a(e, e') < 1 \rightarrow (\exists a' \in \text{In})e, e' \in \text{wline}_{a'}$