Axiomatizing Minkowski Spacetime in First-Order Temporal Logic

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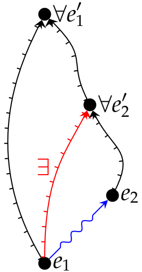
$\mathcal{L}, \mathfrak{M}$

Language

$$\tau ::= \; x \;\mid\; \tau_1 + \tau_2 \;\mid\; \tau_1 \cdot \tau_2$$

$$\varphi ::= \; a = b \;\mid\; \tau = \tau' \;\mid\; \tau \leq \tau' \;\mid\; e = e' \;\mid\; \mathrm{P}(e, a, \tau) \;\mid\; \neg\varphi \;\mid\; \varphi \wedge \psi \;\mid\; \exists x \varphi \;\mid\; \exists a \varphi \;\mid\; \exists e \varphi$$



We can define inertiality and all the causal relations:

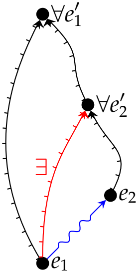
$$e \ll e' \overset{\text{def}}{\Leftrightarrow} \exists a \exists x, y (\mathrm{P}(e, a, x) \wedge \mathrm{P}(e', a, y) \wedge x < y)$$

$$e \rightsquigarrow e' \overset{\text{def}}{\Leftrightarrow} \neg e_1 \ll e_2 \wedge \forall e'_1, e'_2 (e_1 \ll e'_1 \wedge e_1 \ll e'_2 \ll e'_1 \rightarrow e_1 \ll e'_2)$$

$$e \prec e' \overset{\text{def}}{\Leftrightarrow} e \ll e' \vee e \rightsquigarrow e'$$

$$\mathrm{In}(a) \overset{\text{def}}{\Leftrightarrow} (\forall e, e' \in \mathrm{wline}_a)(\forall b \in \mathrm{D}_e \cap \mathrm{D}_{e'}) \quad |a(e) - a(e')| \geq |b(e) - b(e')|$$

LANGUAGE



$$\tau ::= \; x \;\mid\; \tau_1 + \tau_2 \;\mid\; \tau_1 \cdot \tau_2$$

$$\varphi ::= \; a = b \;\mid\; \tau = \tau' \;\mid\; \tau \leq \tau' \;\mid\; e = e' \;\mid\; \mathrm{P}(e, a, \tau) \;\mid\; \neg\varphi \;\mid\; \varphi \wedge \psi \;\mid\; \exists x\varphi \;\mid\; \exists a\varphi \;\mid\; \exists e\varphi$$

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$$e \rightsquigarrow e' \overset{\text{def}}{\Leftrightarrow} \neg e_1 \ll e_2 \wedge \forall e_1', e_2'(e_1 \ll e_1' \wedge e_1 \ll e_2' \ll e_1' \rightarrow e_1 \ll e_2')$$

$$e \prec e' \overset{\text{def}}{\Leftrightarrow} e \ll e' \vee e \rightsquigarrow e'$$

$$\mathrm{In}(a) \overset{\text{def}}{\Leftrightarrow} (\forall e, e' \in \mathrm{wline}_a)(\forall b \in \mathrm{D}_e \cap \mathrm{D}_{e'}) \quad |a(e) - a(e')| \geq |b(e) - b(e')|$$

$$a(e) = \tau \overset{\text{def}}{\Leftrightarrow} \mathrm{P}(a, e, \tau)$$

$$e\mathcal{E}a \overset{\text{def}}{\Leftrightarrow} \exists x\, \mathrm{P}(a, e, x)$$

$$\mathrm{wline}_a \overset{\text{def}}{=} \{e : \exists x\, \mathrm{P}(a, e, x)\}$$

$$\mathrm{D}_e \overset{\text{def}}{=} \{a : \exists x\, \mathrm{P}(a, e, x)\}$$

$$a \approx b \overset{\text{def}}{\Leftrightarrow} \forall e(e\mathcal{E}a \leftrightarrow e\mathcal{E}b)$$

$$e \underline{\ll} e' \overset{\text{def}}{\Leftrightarrow} e \ll e' \vee e = e'$$

$$e \underline{\rightsquigarrow} e' \overset{\text{def}}{\Leftrightarrow} e \rightsquigarrow e' \vee e = e'$$

$$e \preceq e' \overset{\text{def}}{\Leftrightarrow} e \prec e' \vee e = e'$$

$$\overrightarrow{e_1 e_2 e_3} \overset{\text{def}}{\Leftrightarrow} e_1 \rightsquigarrow e_2 \wedge e_2 \rightsquigarrow e_3 \wedge e_1 \rightsquigarrow e_3$$

D_e: domain of event e $\quad$ $a \approx b$: cohabitation $\quad$ $\overrightarrow{e_1 e_2 e_3}$: dir. lightlike betweenness

INTENDED MODELS

$$\mathfrak{M}^c = \left(\mathbb{R}^4, \mathbb{C}, \mathbb{R}, \prec^{\mathfrak{M}^c}, \mathrm{In}^{\mathfrak{M}^c} +, \cdot, \leq, \mathrm{P}^{\mathfrak{M}^c}\right)$$

- $\mathbb{C}$ is the set of those $\alpha : \mathbb{R}^4 \to \mathbb{R}$ partial functions, for which α^{-1}-s are timelike curves that follows the measure system of $\mathbb{R}^4$, i.e.,
  - α^{-1}-s are continuously differentiable functions on $\mathbb{R}$ w.r.t. Euclidean metric:
  - $(\alpha^{-1})'$ is timelike: $\mu \circ \left(\alpha^{-1}\right)'(x) > 0$ for all $x \in \mathbb{R}$.
  - Measure system of $\mathbb{R}^4$: $\mu(\alpha^{-1}(x), \alpha^{-1}(x+y)) = y$ for all $x, y \in \mathbb{R}$.
- $\vec{x} \prec^{\mathfrak{M}^c} \vec{y} \overset{\text{def}}{\Leftrightarrow} \mu(\vec{x}, \vec{y}) \geq 0$ and $x_1 < y_1$,
- $\mathrm{In}^{\mathfrak{M}^c} \overset{\text{def}}{=} \{\alpha \in \mathbb{C} : \alpha^{-1} \text{ is a line}\}$
- $\mathrm{P}^{\mathfrak{M}^c} = \{\langle \vec{x}, \alpha, y\rangle \in \mathbb{R}^4 \times \mathbb{C}_I \times \mathbb{R} : \alpha(\vec{x}) = y\}$,

The non-accelerating intended model $\mathfrak{M}^c_{\mathrm{I}}$ is the largest submodel of $\mathfrak{M}^c$ whose domain of clocks is $\mathrm{In}^{\mathfrak{M}^c}$.

Math, Cont, Ext

Mathematics and Continuity

AxReals The mathematical sort forms a Euclidean field; an ordered field with square roots.

$$\begin{array}{llll} & (x+y)+z = x+(y+z) & & (x\cdot y)\cdot z = x\cdot(y\cdot z) \\ \exists 0 & x+0 = x & \exists 1 & x\cdot 1 = x \\ \exists(-x) & x+(-x) = 0 & x\neq 0 \to \exists x^{-1} & x\cdot x^{-1} = 1 \\ & x+y = y+x & & x\cdot y = y\cdot x \end{array}$$

$$x\cdot(x+y) = (x\cdot y)+(x\cdot z)$$

$$\begin{array}{rr} a\leq b\wedge b\leq a\to a=b & \\ a\leq b\wedge b\leq c\to a\leq c & a\leq b\to a+c\leq b+c \\ \neg a\leq b\to b\leq a & a\leq b\wedge 0\leq c\to a\cdot c\leq b\cdot c \end{array}$$

$$0\leq x\to\exists r\; r\cdot r = x$$

AxContinuity Infimum axiom scheme that allows substitutions of arbitrary formulas.

$$\exists x(\forall y\in\varphi)x\leq y\to\exists i(\forall y\in\varphi)(i\leq y\wedge\forall i'(\forall y\in\varphi)(i'\leq y\to i'\leq i))$$

AxExt

We do not distinguish between (1) indistinguishable clocks, (2) states of a particular clock in an event and (3) two events where a clock shows the same time.

$$(1) \quad \forall a, a' \quad \Big(\forall e \forall x (\mathrm{P}(e,a,x) \leftrightarrow \mathrm{P}(e,a',x))\Big) \rightarrow a = a'$$

$$(2) \quad \forall e, e' \forall a \forall x \quad \big(\mathrm{P}(e,a,x) \wedge \mathrm{P}(e',a,x)\big) \rightarrow e = e'$$

$$(3) \quad \forall e \forall a \forall x, y \quad \big(\mathrm{P}(e,a,x) \wedge \mathrm{P}(e,a,y)\big) \rightarrow x = y$$

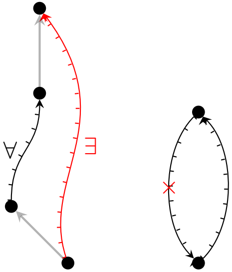
Chronology, Ping

AXCHRONOLOGY

Interiors of lightcones are filled with clocks crossing through the vertex, and all the clocks are ticking in the same direction.

$$(e_1 \preceq e_2 \wedge e_2 \ll e_3 \wedge e_3 \preceq e_4) \rightarrow e_1 \ll e_4$$

$$e_1 \ll e_2 \rightarrow \neg e_2 \ll e_1$$



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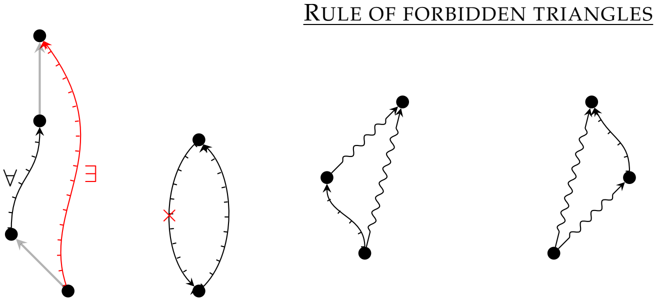

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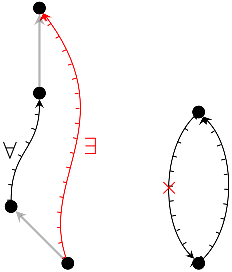
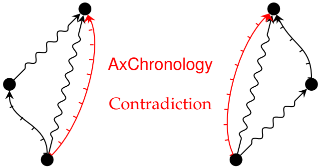


Rule of forbidden triangles



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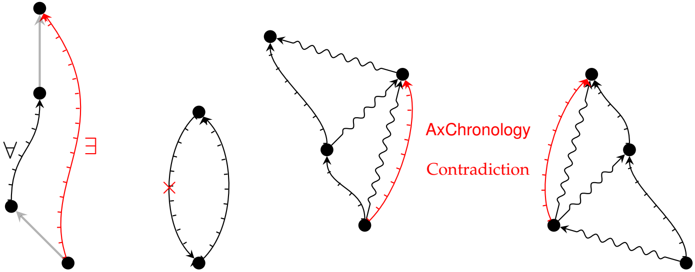
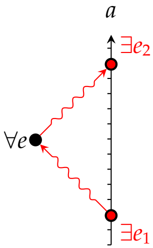


Corollary: If a (not necessarily inertial) clock can radar an event, then the elapsed time (distance) is unique.

AxPing

Every inertial clock can send and receive a signal to any event.

$$(\forall a \in \mathrm{In})\forall e(\exists e_1, e_2 \in \mathrm{wline}_a) \qquad e_1 \underline{\rightsquigarrow} e \underline{\rightsquigarrow} e_2$$



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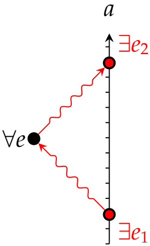

<u>Corollary</u>: By AxChronology, $\delta^i(a, e) = \tau$ is a total function

Kronheimer-Penrose axioms

$(e_1 \preceq e_2 \wedge e_2 \ll e_3 \wedge e_3 \preceq e_4) \rightarrow e_1 \ll e_4$ AxChronology

$e_1 \ll e_2 \rightarrow \neg e_2 \ll e_1$ AxChronology

KRONHEIMER-PENROSE AXIOMS

$(e_1 \preceq e_2 \wedge e_2 \ll e_3 \wedge e_3 \preceq e_4) \to e_1 \ll e_4$	AXCHRONOLOGY
$e_1 \ll e_2 \to \neg e_2 \ll e_1$	AXCHRONOLOGY
$(e_1 \prec e_2 \wedge e_2 \ll e_3) \to e_1 \ll e_3$	$e_1 \neq e_2$ and $e_3 = e_4$

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Transitivity of causality $\prec$ means the following:

$$\frac{e_1 \rightsquigarrow e_2 \vee e_1 \ll e_2 \qquad e_2 \rightsquigarrow e_3 \vee e_2 \ll e_3}{e_1 \rightsquigarrow e_3 \vee e_1 \ll e_3}$$

Here we have:

KRONHEIMER-PENROSE AXIOMS

$$
\begin{array}{ll}
(e_1 \preceq e_2 \wedge e_2 \ll e_3 \wedge e_3 \preceq e_4) \rightarrow e_1 \ll e_4 & \text{AxChronology} \\
e_1 \ll e_2 \rightarrow \neg e_2 \ll e_1 & \text{AxChronology} \\
(e_1 \prec e_2 \wedge e_2 \ll e_3) \rightarrow e_1 \ll e_3 & e_1 \neq e_2 \text{ and } e_3 = e_4 \\
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\end{array}
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$$

Here we have:

$$
e_1 \ll e_2 \wedge e_2 \ll e_3 \Rightarrow e_1 \ll e_3 \qquad \text{AxChronology}
$$

Kronheimer-Penrose axioms

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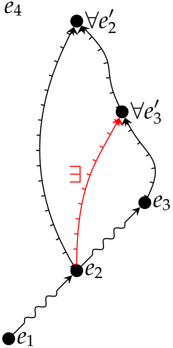

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$(e_1 \rightsquigarrow e_2 \wedge e_2 \rightsquigarrow e_3 \wedge \neg e_1 \ll e_3) \Rightarrow e_1 \rightsquigarrow e_3$	see the figure

(By transitivity of $\ll$ we know that every e_1' in the timelike future of e_3 and e_1 is in the timelike future of e_3 and e_2.)

Kronheimer-Penrose axioms

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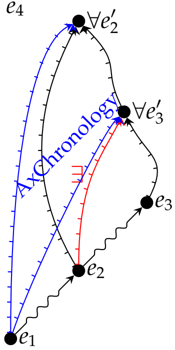

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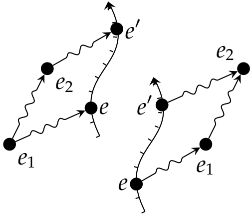
STRAIGHT SIGNALS ARRIVE SOONER



$$\forall a \forall e_1, e_2, e, e' (e \mathcal{E} a \wedge e' \mathcal{E} a \wedge e_1 \underline{\rightsquigarrow} e \wedge e_1 \underline{\rightsquigarrow} e_2 \underline{\rightsquigarrow} e') \rightarrow a(e) \leq a(e')$$

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Indirectly by the Kronheimer-Penrose axioms.

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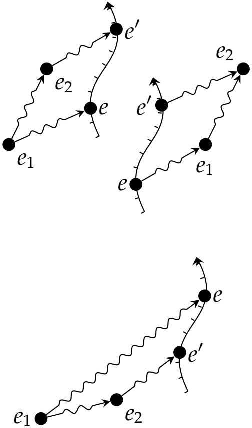

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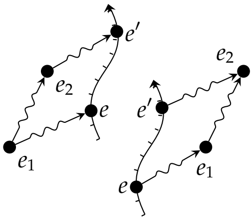
Straight signals arrive sooner



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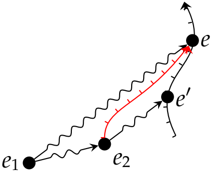
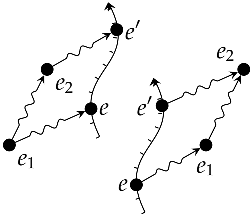


Straight signals arrive sooner



$$\forall a \forall e_1, e_2, e, e' \big(e \mathcal{E} a \wedge e' \mathcal{E} a \wedge e_1 \underline{\rightsquigarrow} e \wedge e_1 \underline{\rightsquigarrow} e_2 \underline{\rightsquigarrow} e'\big) \rightarrow a(e) \leq a(e')$$

$$\forall a \forall e_1, e_2, e, e' \big(e \mathcal{E} a \wedge e' \mathcal{E} a \wedge e' \underline{\rightsquigarrow} e_2 \wedge e \underline{\rightsquigarrow} e_1 \underline{\rightsquigarrow} e_2\big) \rightarrow a(e) \leq a(e')$$

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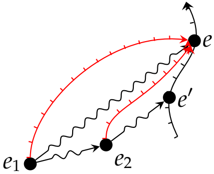
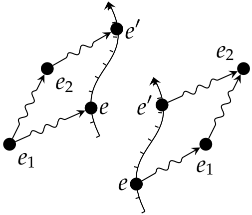


STRAIGHT SIGNALS ARRIVE SOONER



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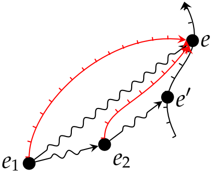

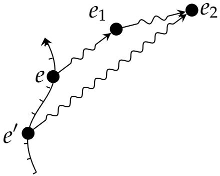

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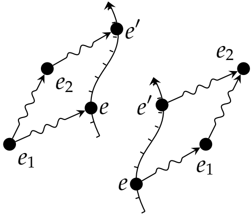

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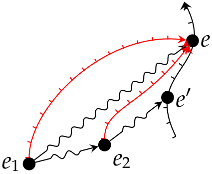

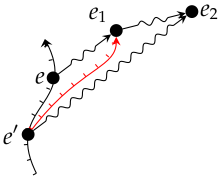

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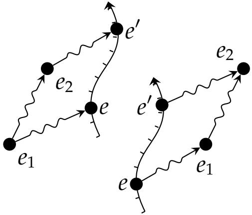

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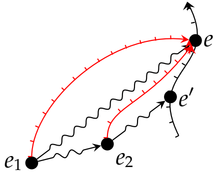
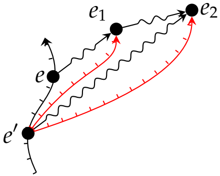


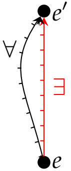


AxFull, AxSecant, AxLocExp

AxFull: Every number occurs as a state of any clock in an event.

$$\forall a \forall x \exists e \quad \mathrm{P}(e, a, x)$$

AxFull, AxSecant, AxLocExp



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AxSecant: Any two events that share a clock share an inertial clock as well.

$$e \ll e' \to (\exists a \in \mathrm{In})(e\mathcal{E}a \wedge e'\mathcal{E}a)$$

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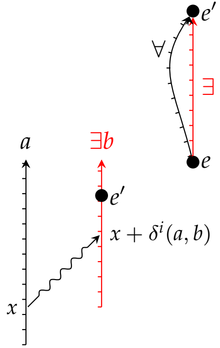

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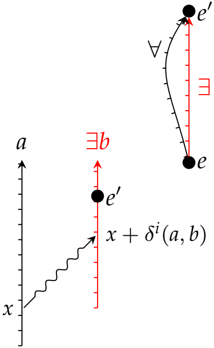

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PROPOSITION: There is a clock in every event,

$$\forall e \exists c \quad e\mathcal{E}c$$

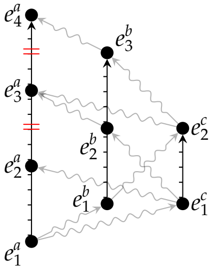
Let e be an arbitrary but fixed event.

- Logical Tautology: $\exists a\, a = a$.
- AxFull: that a shows the number 0 in some event e_0.
- AxSecant: There is an inertial clock a_i in e_0 as well.
- AxLocExp: There is an inertial comover c of a_i in e as well.

AxRound

Given comoving observers a, b and c, the travelling time of simultaneously sent signals on the route $\langle a, b, c, a\rangle$ and $\langle a, c, b, a\rangle$ are (the same, namely,) the average of the travelling time of the $\langle a, c, a\rangle$ and $\langle a, b, c, b, a\rangle$.

$$b \overset{i}{\uparrow\uparrow} a \overset{i}{\uparrow\uparrow} c \wedge \begin{pmatrix} e_1^a, e_2^a, e_3^a, e_4^a \in \mathrm{wline}_a \\ e_1^b, e_2^b, e_3^b \in \mathrm{wline}_b \\ e_1^c, e_2^c \in \mathrm{wline}_c \end{pmatrix} \wedge \begin{pmatrix} e_1^a \rightsquigarrow e_1^b \rightsquigarrow e_2^c \rightsquigarrow e_3^b \rightsquigarrow e_4^a \\ e_1^a \rightsquigarrow e_1^c \rightsquigarrow e_2^b \rightsquigarrow e_3^a \\ e_1^a \rightsquigarrow e_1^c \rightsquigarrow e_2^a \end{pmatrix} \rightarrow a(e_3^a) = \frac{a(e_2^a) + a(e_4^a)}{2}$$



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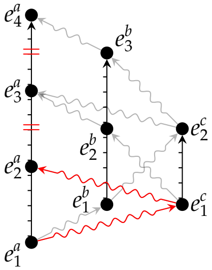

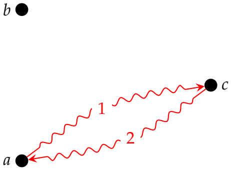

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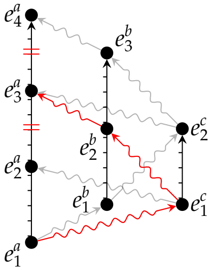
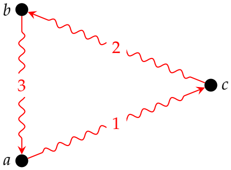
$$b \overset{\mathrm{i}}{\uparrow\uparrow} a \overset{\mathrm{i}}{\uparrow\uparrow} c \wedge \begin{pmatrix} e^a_1, e^a_2, e^a_3, e^a_4 \in \mathrm{wline}_a \\ e^b_1, e^b_2, e^b_3 \in \mathrm{wline}_b \\ e^c_1, e^c_2 \in \mathrm{wline}_c \end{pmatrix} \wedge \begin{pmatrix} e^a_1 \rightsquigarrow e^b_1 \rightsquigarrow e^c_2 \rightsquigarrow e^b_3 \rightsquigarrow e^a_4 \\ e^a_1 \rightsquigarrow e^c_1 \rightsquigarrow e^b_2 \rightsquigarrow e^a_3 \\ e^a_1 \rightsquigarrow e^c_1 \rightsquigarrow e^a_2 \end{pmatrix} \rightarrow a(e^a_3) = \frac{a(e^a_2) + a(e^a_4)}{2}$$





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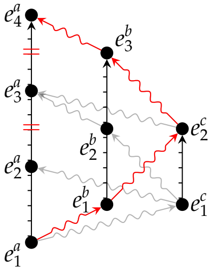
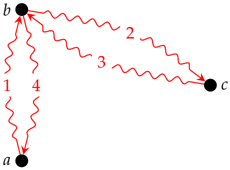




Metric theorem

$\overset{syn}{\uparrow\uparrow}$ is an equivalence relation and δ^i is a(n U-relative) metric on $\overset{syn}{\uparrow\uparrow}$ related clocks, i.e.,

$$a \overset{syn}{\uparrow\uparrow} a$$
$$a_1 \overset{syn}{\uparrow\uparrow} a_2 \Rightarrow a_2 \overset{syn}{\uparrow\uparrow} a_1$$
$$a_1 \overset{syn}{\uparrow\uparrow} a_2 \wedge a_2 \overset{syn}{\uparrow\uparrow} a_3 \Rightarrow a_1 \overset{syn}{\uparrow\uparrow} a_3$$

$$\delta^i(a, a) = 0$$
$$\delta^i(a_1, a_2) = 0 \Rightarrow a_1 = a_2$$
$$\delta^i(a_1, a_2) = \delta^i(a_2, a_1)$$
$$\delta^i(a_1, a_2) + \delta^i(a_2, a_3) \geq \delta^i(a_1, a_3)$$

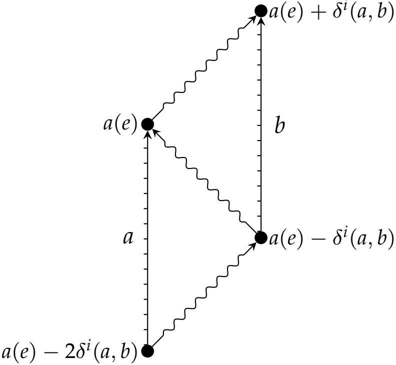
We prove these simultaneously.

REFLEXIVITY OF $\overset{\text{syn}}{\uparrow\uparrow}$, SELFDISTANCE=0

- **Self-distance**: By $e \mathrel{\underline{\rightsquigarrow}} e \mathrel{\underline{\rightsquigarrow}} e$ we have $\delta^i(a, e) = a(e) - a(e) = 0$. The truth of $\delta^i(a, a) = 0$ is trivially implied by that fact.
- **Reflexivity of $\overset{\text{syn}}{\uparrow\uparrow}$**: By the self-distance we have $a(e) = a(e) + 0$ whenever $e \mathrel{\underline{\rightsquigarrow}} e$, so $\overset{\text{syn}}{\uparrow\uparrow}$ is reflexive.

Symmetry of $\overset{\text{syn}}{\uparrow\uparrow}$ and δ^i

Suppose that $a_1 \overset{\text{syn}}{\uparrow\uparrow} a_2$.



IDENTITY OF INDISCERNIBLES

Take arbitrary iscm's a_1 and a_2 for which $\delta^i(a_1, a_2) = 0$, i.e.,

$$(\forall e \in \text{wline}_{a_2})(\exists w_1, w_2 \in \text{wline}_{a_1}) \qquad w_1 \underline{\rightsquigarrow} e \underline{\rightsquigarrow} w_2 \land$$

$$a_1(w_2) - a_1(w_2) = 0$$

but $a_1(w_1) = a_1(w_2) \overset{\text{AxExt}}{\Longrightarrow} w_1 = w_2$.

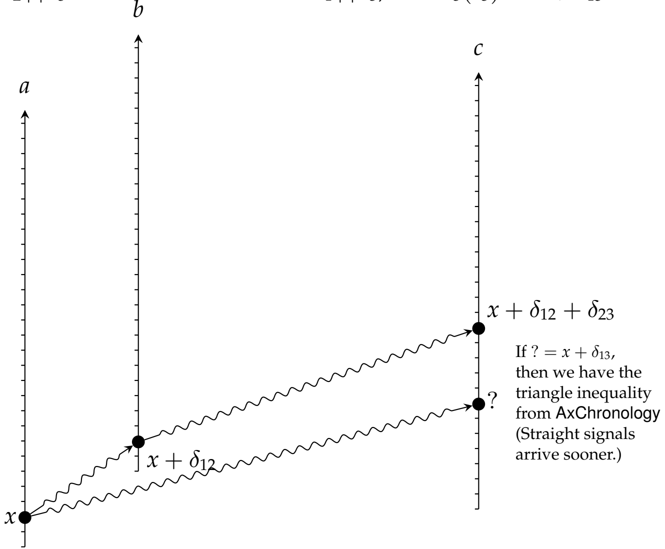
- $w_1 \rightsquigarrow e$ and $e \rightsquigarrow w_2 = w_1$ is false by AxChronology (transitivity and irreflexivity of causality).
- $w_1 = e \rightsquigarrow w_2$ or $w_2 = e \rightsquigarrow w_1$ are impossible too since w_1 and w_2 share a. (definition of $\rightsquigarrow$)

So the only possiblity is that $w_1 = e = w_2$.
Since this is true for all $e \in \text{wline}_{a_2}$,

$$\text{wline}_{a_2} \subseteq \text{wline}_{a_1} \overset{\delta^i \text{ sym.}}{\Longrightarrow} \text{wline}_{a_2} = \text{wline}_{a_1} \overset{\text{AxExt}, a_1 \overset{syn}{\uparrow\uparrow} a_2}{\Longrightarrow} a_1 = a_2$$

Transitivity of $\overset{syn}{\uparrow\uparrow}$, triangle inequality

Asssume $a_1 \overset{syn}{\uparrow\uparrow} a_2 \overset{syn}{\uparrow\uparrow} a_3$. We have to show that $a_1 \overset{syn}{\uparrow\uparrow} a_3$, here $a_3(e_3) = x + d_{13}$.



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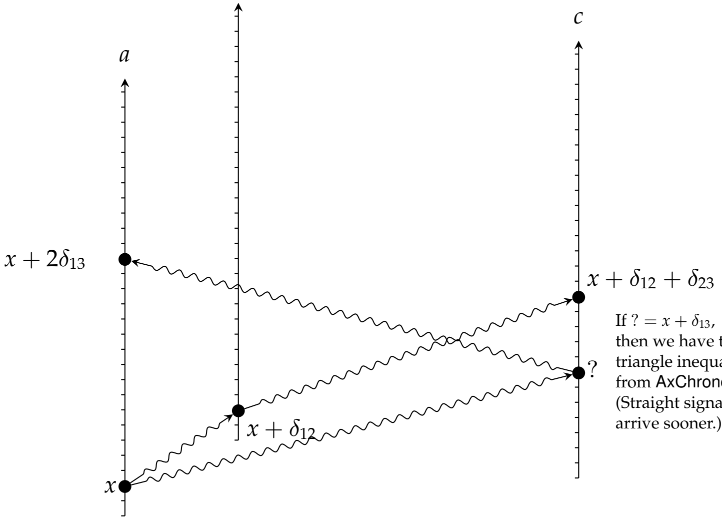

If $? = x + \delta_{13}$, then we have the triangle inequality from AxChronology (Straight signals arrive sooner.)

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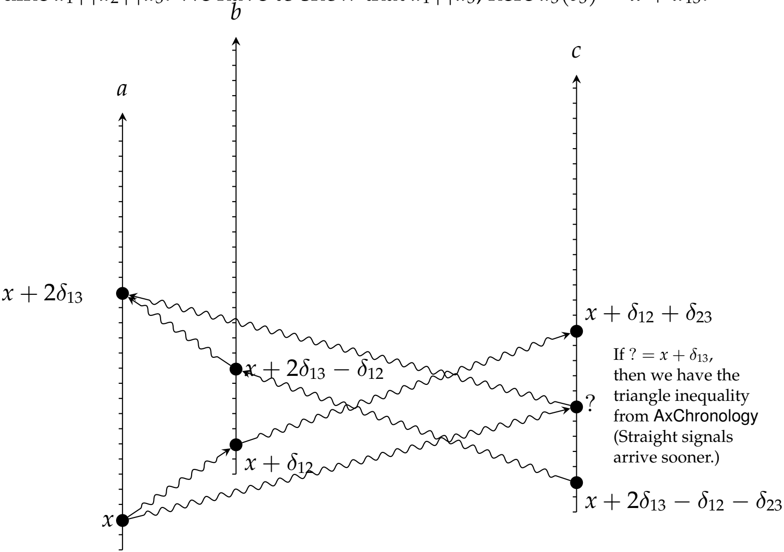

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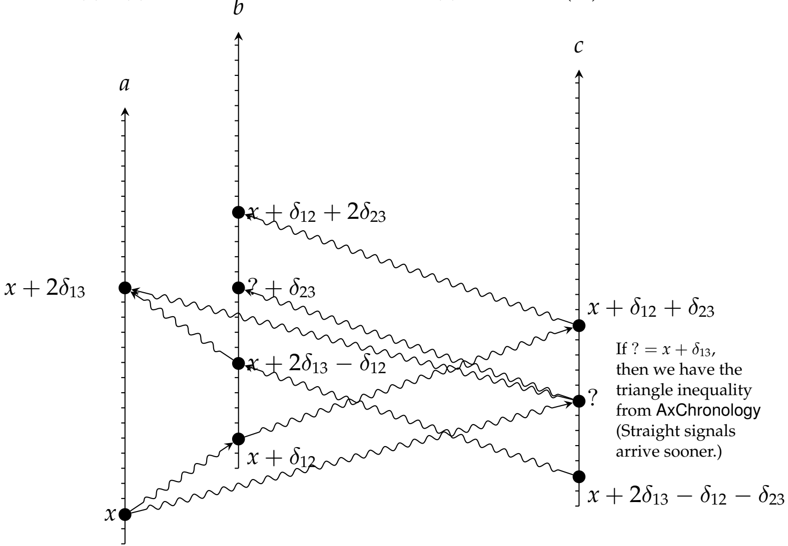

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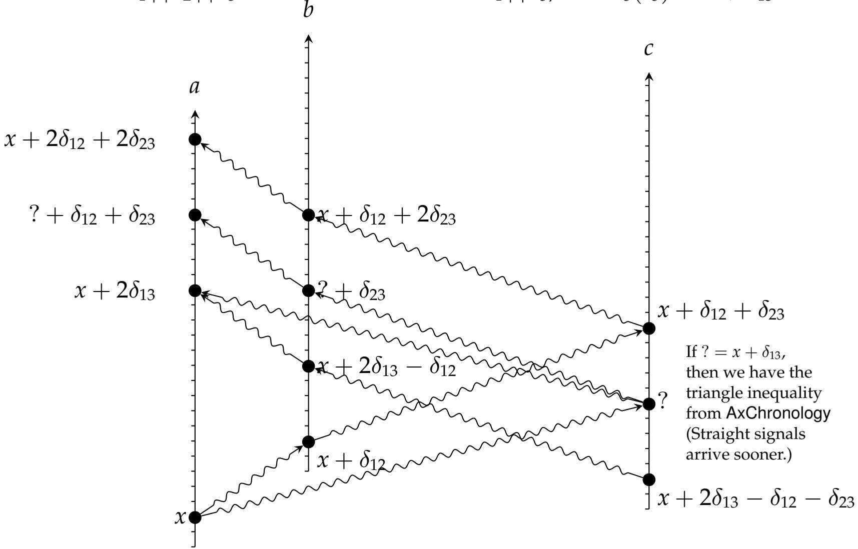

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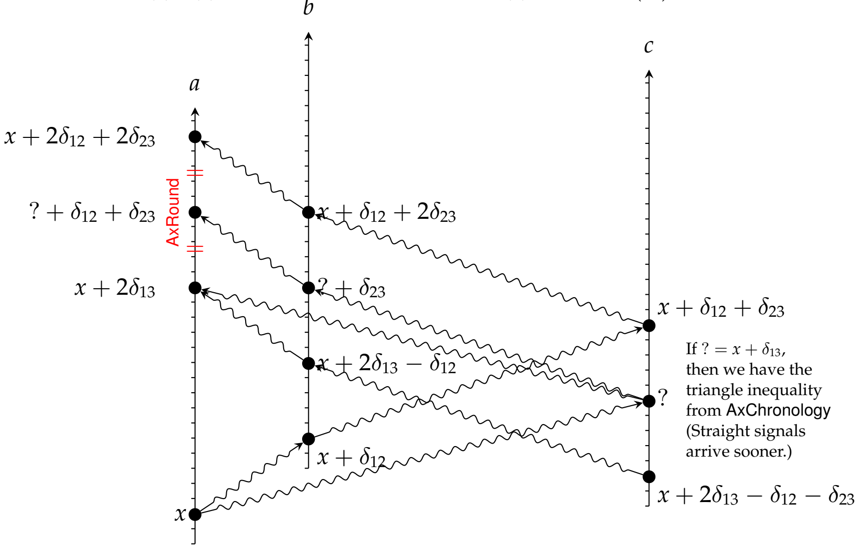

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No clock has two ISCMs at the same event

$$(\forall a \in \mathrm{In})\forall e(\forall a_1, a_2 \in \mathrm{D}_e) \quad a_1 \overset{\text{syn}}{\uparrow\uparrow} a \overset{\text{syn}}{\uparrow\uparrow} a_2 \Rightarrow a_1 = a_2$$

Let $e \in \mathrm{wline}_{a_1} \cap \mathrm{wline}_{a_2}$ be arbitrary but fixed. Let a_1 and a_2 be inertial comovers of a occurring at e.

- Transitivity $\overset{\text{syn}}{\uparrow\uparrow}$: $a_1 \overset{\text{syn}}{\uparrow\uparrow} a_2$.

NO CLOCK HAS TWO ISCMS AT THE SAME EVENT

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- Selfdistance: $\delta^i(a_1, e) = \delta^i(a_2, e) = 0$.

No clock has two ISCMs at the same event

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- Selfdistance: $\quad \delta^i(a_1, e) = \delta^i(a_2, e) = 0$.
- Since $a_1 \overset{syn}{\uparrow\uparrow} a_2$ implies comovement, i.e., constant distance: $\quad \delta^i(a_1, a_2) = 0$.

NO CLOCK HAS TWO ISCMS AT THE SAME EVENT

$$(\forall a \in \mathrm{In})\forall e(\forall a_1, a_2 \in \mathrm{D}_e) \quad a_1 \overset{\text{syn}}{\uparrow\uparrow} a \overset{\text{syn}}{\uparrow\uparrow} a_2 \Rightarrow a_1 = a_2$$

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- Identity of indiscernibles: $a_1 = a_2$.

'Equivalence' of betweens

For any iscms $a \neq b \neq c$ the followings are equivalent:

- $\mathrm{B}(a, b, c)$
- $\overrightarrow{e_a e_b e_c}$ for some e_a, e_b, e_c events of a, b, c, respectively.

⇑: Since we have iscm observers

$$\begin{aligned} c(e_c) &= a(e_a) + \delta^i(a,b) + \delta^i(a,c) && \text{by } e_a \rightsquigarrow e_b \rightsquigarrow e_c \\ &= a(e_a) + \delta^i(a,c) && \text{by } e_a \rightsquigarrow e_c \end{aligned}$$

therefore $\delta^i(a,b) + \delta^i(b,c) = \delta^i(a,c)$.

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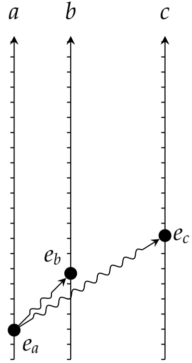

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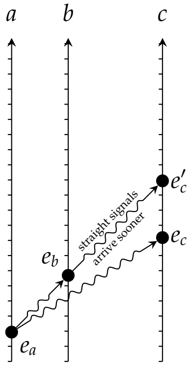

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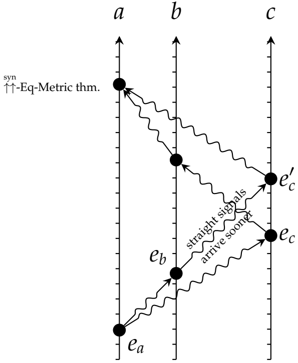

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$$\overbrace{\delta^i(a,b) + \delta^i(b,c)}^{\delta^i(a,c)} + \underbrace{\delta^i(a,c)}_{\delta^i(a,b)+\delta^i(b,c)}$$
$$\|$$
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$$\|$$
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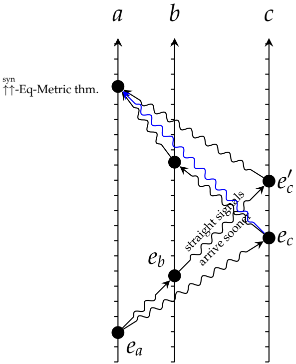

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$$||$$
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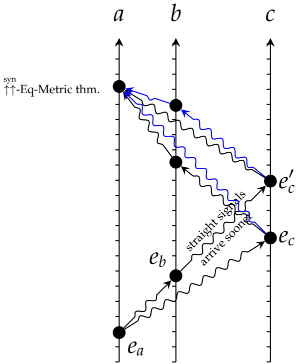

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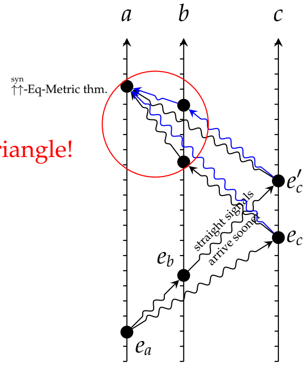
Forbidden triangle!



TARSKI'S AXIOMATIZATION OF GEOMETRY

1. $ab \equiv ba$ (Reflexivity for $\equiv$)
2. $(ab \equiv pq \wedge ab \equiv rs) \to pq \equiv rs$ (Transitivity for $\equiv$)
3. $ab \equiv cc \to a = b$ (Identity for $\equiv$)
4. $\exists x(B(qax) \wedge ax \equiv bc)$ (Segment Construction)
5. $(a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge$ $\wedge\, ad \equiv a'd' \wedge bd \equiv b'd') \to cd \equiv c'd'$ (Five-segment)
6. $B(aba) \to a = b$ (Identity for B)
7. $(B(apc) \wedge B(bqc)) \to \exists x(B(pxb) \wedge B(qxa))$ (Pasch)

8^n. $\exists a, b, c, p_1, \ldots p_{n-1}\left(\bigwedge_{i<j<n} p_i \neq p_j \wedge \bigwedge_{1<i<n} (ap_1 \equiv ap_i \wedge bp_1 \equiv bp_i \wedge cp_1 \equiv cp_i) \wedge \wedge \neg(B(abc) \vee B(bca) \vee B(cab))\right)$ (Lower n-dimension)

9^n. $\left(\bigwedge_{i<j<n} p_i \neq p_j \wedge \bigwedge_{1<i<n} (ap_1 \equiv ap_i \wedge bp_1 \equiv bp_i \wedge cp_1 \equiv cp_i)\right) \to \to (B(abc) \vee B(bca) \vee B(cab))$ (Upper n-dimension)

10_2. $B(abc) \vee B(bca) \vee B(cab) \vee \exists x(ax \equiv bx \wedge ax \equiv cx)$ (Circumscribed tr.)

11. $\exists a \forall x, y(\alpha \wedge \beta \to B(axy)) \to \exists b \forall x, y(\alpha \wedge \beta \to B(aby))$ (Continuity scheme)

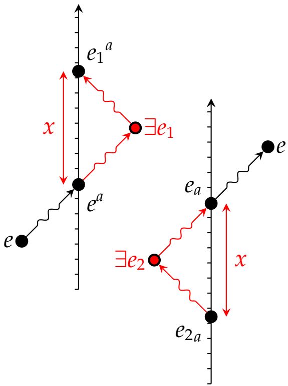
where α and β are first-order formulas, the first of which does not contain any free occurrences of a, b and y and the second any free occurrences of a, b, x.

AxRays

For every inertial observer, for any positive x and every direction (given by a light signal) there are lightlike separated events in the past and the future whose distances are exactly x.

$$(\forall x > 0)(\forall a \in \mathrm{In})\forall e \exists e_1 \exists e_2 (\exists e^a, e_a \in \mathrm{wline}_a)$$

$$\overrightarrow{e_2 e_a e} \wedge \delta^i(a, e_2) = x \wedge \overrightarrow{e e^a e_1} \wedge \delta^i(a, e_1) = x$$



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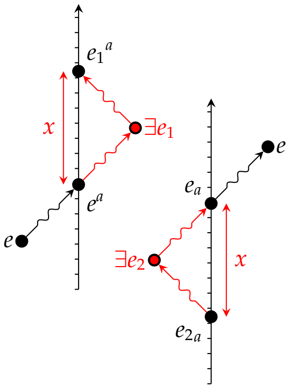


COROLLARIES:

- Segment construction
- Tarski's continuity axiom:

AxRays projects AxContinuity axiom to the spacetime, and AxLocExp provides the points.

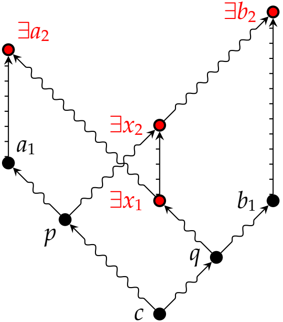
So the spaces and the spacetime is continuous because the clocks are.

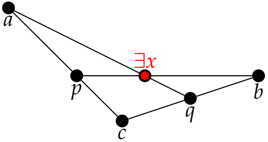
AxPasch

Pasch-axiom for light signals.

$$\big(a \overset{\mathrm{i}}{\uparrow\uparrow} b \wedge (\exists a_1 \in \mathrm{wline}_a)(\exists b_1 \in \mathrm{wline}_b)(\overrightarrow{cpa_1} \wedge \overrightarrow{cqb_1})\big) \to$$

$$\to (\exists x \overset{\mathrm{i}}{\uparrow\uparrow} a)(\exists x_1, x_2 \in \mathrm{wline}_x)(\exists a_2 \in \mathrm{wline}_a)(\exists b_2 \in \mathrm{wline}_b)(\overrightarrow{px_2b_2} \wedge \overrightarrow{qx_1a_2})$$

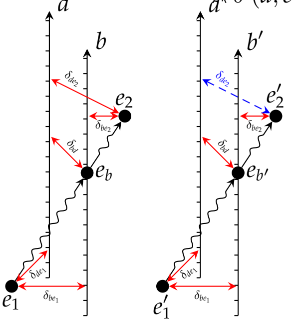


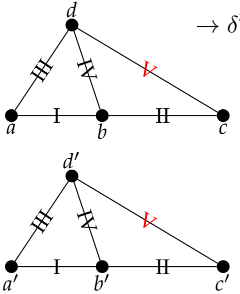


Ax5Segment

If two pairs of observers b,d and b',d' measures two pair of lightlike separated events e_1, e_2 and e_1', e_2' to the same distances, respectively, and the lightline crosses the worldlines of b and b', respectively, then the distances b-d and b'-d' are the same (for all of them).

$$d \overset{i}{\uparrow\uparrow} d' \wedge b \overset{i}{\uparrow\uparrow} b' \wedge e_b \mathcal{E} b \wedge e_b' \mathcal{E} b' \wedge \overrightarrow{e_1 e_b e_2} \wedge \overrightarrow{e_1' e_b' e_2'} \wedge$$
$$\wedge\ \delta^i(b, e_1) = \delta^i(b', e_1') \wedge \delta^i(b, e_2) = \delta^i(b', e_2') \wedge$$
$$\wedge\ \delta^i(d, e_1) = \delta^i(d', e_1') \wedge \delta^i(b, d) = \delta^i(b', d')) \rightarrow$$
$$\rightarrow \delta^i(d, e_2) = \delta^i(d', e_2')$$

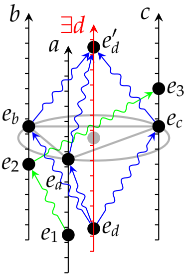




AxCircle

For every three non-collinear inertial observer there is a fourth one that measures them with the same distance.

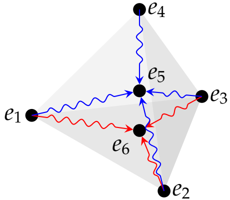
$$
\begin{aligned}
&(\forall a, b, c \in \mathrm{In})\Big(\big(a \overset{\mathrm{i}}{\uparrow\uparrow} b \overset{\mathrm{i}}{\uparrow\uparrow} c \wedge \\
&\quad \wedge \exists e_1, e_2, e_3 (e_1 \mathcal{E} a \wedge e_2 \mathcal{E} b \wedge e_3 \mathcal{E} c \wedge e_1 \rightsquigarrow e_2 \rightsquigarrow e_3 \wedge \neg e_1 \rightsquigarrow e_3)\big) \rightarrow \\
&\quad \rightarrow \exists d \exists e_a, e_b, e_c, e_d, e'_d (e_a \mathcal{E} a \wedge e_b \mathcal{E} b \wedge e_c \mathcal{E} c \wedge e_d \mathcal{E} d \wedge e'_d \mathcal{E} d \wedge \\
&\quad\quad \wedge e_d \rightsquigarrow e_a \rightsquigarrow e'_d \wedge e_d \rightsquigarrow e_b \rightsquigarrow e'_d \wedge e_d \rightsquigarrow e_c \rightsquigarrow e'_d)\Big)
\end{aligned}
$$



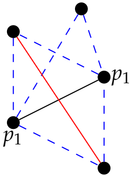
AxMinDim : n

The dimension of the spacetime is at least n. The formula says that $n-1$ lightcones never intersect in only one event.

$$\forall e_1, \ldots, e_n \left(\bigwedge_{i \leq n-1} e_i \rightsquigarrow e_n \rightarrow \exists e_{n+1} \left(\bigwedge_{i \leq n-1} e_i \rightsquigarrow e_{n+1} \wedge e_n \neq e_{n+1} \right) \right)$$



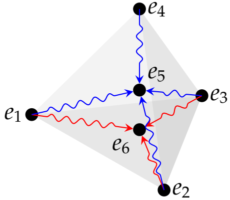
Tarski's lower n-dimensional axiom: Centers of circumscribed spheres around $n-1$ points cannot be covered with a line.



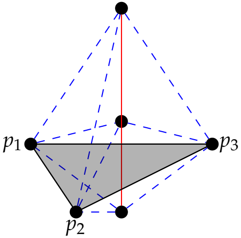
AxMaxDim : n

The dimension of the spacetime is at most n. The formula says that there are n lightcones that intersect at most in one event.

$$\exists e_1,\ldots,e_{n+1}\left(\bigwedge_{i\leq n} e_i \rightsquigarrow e_{n+1} \wedge \forall e_{n+2}\left(\bigwedge_{i\leq n} e_i \rightsquigarrow e_{n+2} \to e_{n+1} = e_{n+2}\right)\right)$$



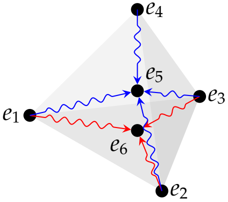
Tarski's upper n-dimensional axiom: Centers of circumscribed spheres around n points are on a line.



Ax4Dim

The dimension of the spacetime is exactly 4; 3 lightcones never intersect in only one event and there are 4 lightcones intersect in at most one event.

$$\mathsf{AxMinDim} : 4 \wedge \mathsf{AxMaxDim} : 4$$



SPACE $D \geq 3$ FOLLOWS FROM SPACETIME $D \geq 4$

Space $D \geq 3$ follows from spacetime $D \geq 4$

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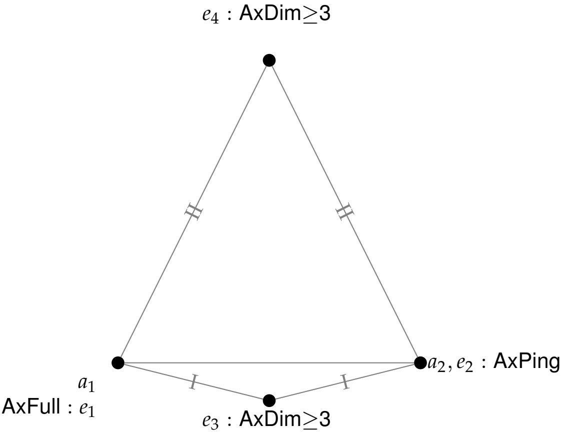

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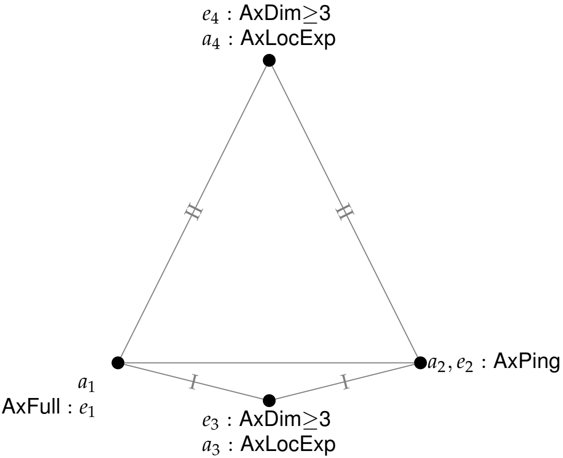

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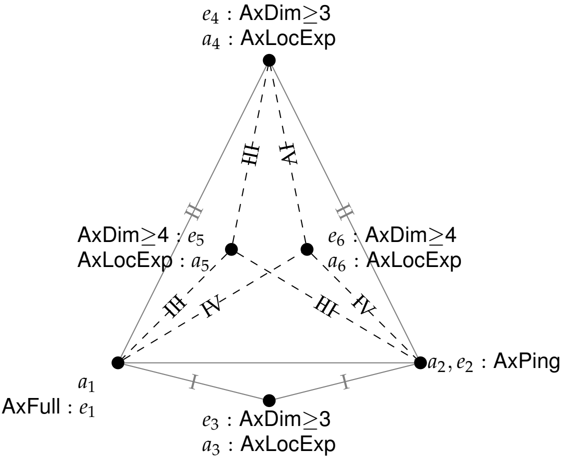

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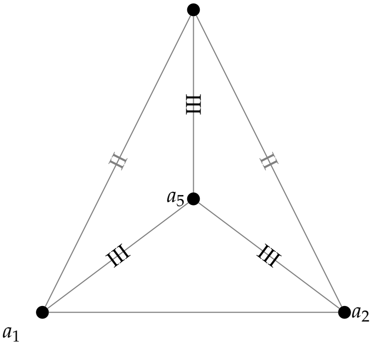

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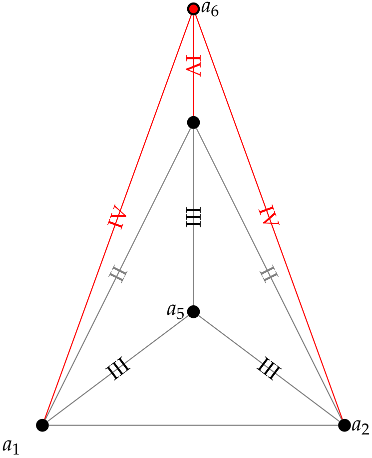

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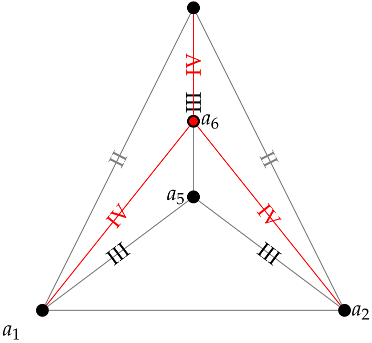

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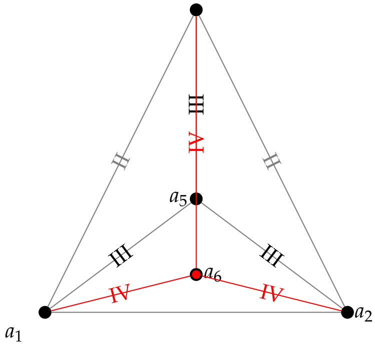

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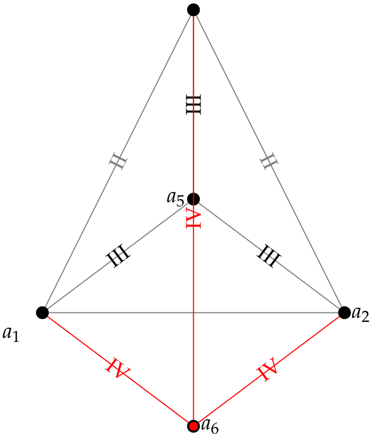
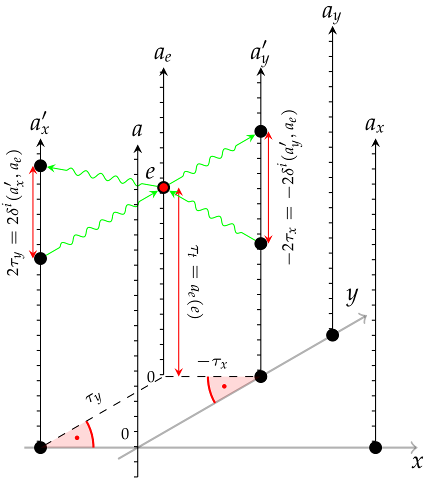


Coord.

EVERY EVENT IS COORDINATIZED WITH A 4-TUPLE. (TOTALITY)

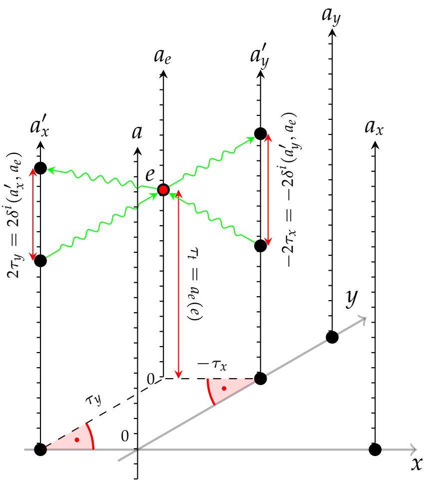
Let e be an arbitrary event. Since there is exactly one iscm there, we have a synchronized comover a_e of a in e. Then by definition, $a_e(e)$ will be the time coordinate. We can use Tarski's axioms to conclude that there are (unique) a'_x, a'_y and a'_z that are projections of the point a_e to the lines (a, a_x), (a, a_y) and (a, a_z), respectively. By AxPing, these projections can ping a_e, i.e., they can measure the spatial distance between them and a_e (and e), and thus we will have the spatial coordinates of e as well.

COROLLARY: No event has two different coordinates. (Functionality)



EVERY 4-TUPLE IS A COORDINATE OF AN EVENT. (SURJECTIVITY)

Let (t, x, y, z) be an arbitrary 4-tuple. It follows from Tarski's axioms that there are inertial comovers a'_x, a'_y and a'_z of a on the axes (a, a_x), (a, a_y) and (a, a_z), respectively, such that $\delta^i(a, a_x) = x$, $\delta^i(a, a_y) = y$ and $\delta^i(a, a_t) = t$. For all $i \in \{x, y, z\}$ Let P_i denote the plane that contains a'_i and is orthogonal to the line (a, a_i). Then by Tarski's axioms, these planes has one unique intersection, a_e. By the definition of the Coord, any event of wline$_{a_e}$ are coordinatized on the spatial coordinates (x, y, z). Now we know from Ax-Full that there is an event e of wline$_{a_e}$ such that $a(e) = t$.



COROLLARY: No 4-tuple is a coordinatization of two different events. (Injectivity)

Axioms

AXIOM SYSTEM

for Euclidean 3D coordinatesystems

$$\text{SClTh} \stackrel{\text{def}}{=} \left\{ \begin{array}{lll} \text{AxEField} & \text{AxFull} & \text{AxPasch} \\ \text{AxContinuity} & \text{AxSecant} & \text{Ax5Seg} \\ \text{AxExt} & \text{AxLocExp} & \text{AxCircle} \\ \text{AxChronology} & \text{AxRay} & \text{AxDim}\geq 4 \\ \text{AxPing} & \text{AxRound} & \text{AxDim}\leq 4 \end{array} \right\}$$

CONJECTURES:

Axiom system

for Euclidean 3D coordinatesystems

$$\mathrm{SClTh} \overset{\mathrm{def}}{=} \left\{ \begin{array}{lll} \mathsf{AxEField} & \mathsf{AxFull} & \mathsf{AxPasch} \\ \mathsf{AxContinuity} & \mathsf{AxSecant} & \mathsf{Ax5Seg} \\ \mathsf{AxExt} & \mathsf{AxLocExp} & \mathsf{AxCircle} \\ \mathsf{AxChronology} & \mathsf{AxRay} & \mathsf{AxDim}{\geq 4} \\ \mathsf{AxPing} & \mathsf{AxRound} & \mathsf{AxDim}{\leq 4} \end{array} \right\}$$

Conjectures:

1. $\mathsf{Ax4Dim} \Rightarrow \mathsf{AxCircle}$ above SClTh.

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CONJECTURES:

1. $\mathsf{Ax4Dim} \Rightarrow \mathsf{AxCircle}$ above SClTh.

2. $\mathsf{AxSome} : \forall a \exists x \exists e\ \mathrm{P}(e,a,x) \Rightarrow \mathsf{AxFull} : \forall a \forall x \exists e\ \mathrm{P}(e,a,x)$ above SClTh.
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CONJECTURES:

1. Ax4Dim $\Rightarrow$ AxCircle above SClTh.

2. AxSome : $\forall a \exists x \exists e \; \mathrm{P}(e, a, x) \Rightarrow$ AxFull : $\forall a \forall x \exists e \; \mathrm{P}(e, a, x)$ above SClTh. (AxRays and AxPing!)

3. Since we defined Inertials, but we never used that definition, it is expectable that some axioms follows from the defining property (AxRound?), or can be derived from new but way more weaker axioms (AxLocExp?).

SpecRel

SPATIAL DISTANCE

$$\mathrm{sd}_a(e, e') = \tau \overset{\text{def}}{\Leftrightarrow} (\exists a' \in \mathrm{Space}_a)(a \in \mathrm{D}_e \wedge \delta^i(a, e') = \tau)$$

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$$\mathrm{sd}_a(e,e') = \tau \iff (\exists \langle a_x, a_y, a_z\rangle \in \mathrm{CoordSys(a)})\exists \vec{x}\vec{y}$$
$$\mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \wedge \mathrm{Coord}_{a,a_x,a_y,a_z}(e') = \vec{y} \wedge \tau = |\vec{x}_{2-4} - \vec{y}_{2-4}|$$

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Pythagoras's theorem:

$$\delta^i(a_e, a_{e'})^2 = \delta^i(a_e, b)^2 + \delta^i(b, a_{e'})^2$$

where $b \in \mathrm{Space_a}$ is a clock with which

$$\mathrm{Ort}(a'_x, a, b) \wedge \mathrm{Ort}(a'_y, a, b) \wedge \mathrm{Ort}(a'_z, a, b)$$

where a'_x, a'_y, a'_z, are the projections of a_e to the axes of the coordinate system (see the figure of coordinatization).

Elapsed time

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ELAPSED TIME

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The clocks that measures the time in the events are the same in both definitions by Proposition ‘there are no two iscms in one event’, so practically, both formula refer to the same measurement.

SPEED

$$\mathrm{v}_a(e, e') \stackrel{\text{def}}{=} \frac{\mathrm{sd}_a(e, e')}{\mathrm{et}_a(e, e')}$$

SIMPLE SPECREL

Simple-AxSelf $\forall a(\forall e \in \text{wline}_a)(\forall\langle a_x, a_y, a_z\rangle \in \text{CoordSys(a)})$
$\exists t\ \text{Coord}_{a,a_x,a_y,a_z}(e) = (t, 0, 0, 0)$

Simple-AxPh $(\forall a \in \text{In})\forall e, e' \quad \text{v}_a(e, e') = 1 \leftrightarrow e \,\rightsquigarrow\, e'$

Simple-AxEv $\forall e(\forall\langle a, a_x, a_y, a_z\rangle, \langle a', a'_x, a'_y, a'_z\rangle \in \text{CoordSys})$
$\exists \vec{x}\ \text{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \rightarrow \exists \vec{y}\ \text{Coord}_{a',a'_x,a'_y,a'_z}(e) = \vec{y}$

Simple-AxSym $(\forall a, a' \in \text{In})\forall e, e'$
$\text{et}_a(e, e') = \text{et}_{a'}(e, e') = 0 \rightarrow \text{sd}_a(e, e') = \text{sd}_{a'}(e, e')$

Simple-AxThExp $\forall a \forall e, e' \quad \text{v}_a(e, e') < 1 \rightarrow (\exists a' \in \text{In})e, e' \in \text{wline}_{a'}$