Modal Logic of Relativity Theories

Preliminary draft of the PhD dissertation

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Chapter 1

Introduction

1.1 On logical foundations of relativity theories

Classical first-order logic is an expressive logic with strong logical properties. So it is an excellent framework to support axiomatizations of relativity theories. Our project lead by Hajnal Andréka and István Németi has numerous results from that, e.g.,

- Székely [2013] showed that the existence of superluminal particles is consistent with relativity (even in 4D) as long as those particles are not able to coordinatize their environment.
- It is provable from five simple axioms that no observer can move faster than light and that proof was checked by theorem provers as well, which shows another advantage of using formal logic in the foundations of physics. [Govindarajalulu, Bringsjord, and Taylor 2014]
- Using model-theoretical tools Madarász and Székely [2013] showed that many axiomatizations of special relativity can be done without assuming the structure of real numbers or their first-order theory, e.g., the main special relativistic effects (and even more) can be proved assuming only the field of rational numbers.
- With a first-order logic analysis it is possible to ask Why-type questions in physics: Which axioms are needed and which are superfluous in order to prove certain predictions of relativity theory. For instance, Andréka, Madarász, Németi, and Székely [2008] showed that the conservation of mass is not needed to prove the mass-increase theorem.

More on the foundational significance of that project can be found in a recent paper by Friend [2014].

1.2 Standard systems

The standard formal language of our research group has the advantage that it involves very natural primitives[1]: its basic predicates and relations are

[1] for details see Definition 48 and or [Andréka et al. 2012]

- $+$, $\cdot$, $\leq$, $=$ that refer to the standard mathematical terms
- Ph(b): "b is a light signal",
- Ob(b): "b is an observer/coordinate system",
- W(o, b, x, y, z, t):"According to the observer/coordinate system o, the body b occurs in the position (x, y, z, t)"

That language fits very well to the 'everyday language' of relativity theories where we use spacetime diagrams, and makes it possible to build up special and general relativity from very few but still logically (and conceptually) transparent axioms. The standard axiom systems that our group uses and in this report we will frequently will refer to are

- SpecRel, which can derive the basic predictions of special relativity.
- SpecRelComp is a complete and decidable extension of SpecRel: all of its models are elementary equivalent with Minkowski spacetimes where there are no accelerating observers.
- AccRel is an extension of SpecRel which allows the acceleration of coordinate systems.
- GenRel is reduct of AccRel which is complete w.r.t. differentiable manifolds and as such can be considered to be an axiomatization of general relativity.

Andréka, Madarász, Németi, and Székely [2012] gives a nice, short and precise introduction to SpecRel, AccRel and GenRel. The system SpecRelComp is discussed in [Andréka et al. 2007], but we will give a short summary of that in Chapter 5.5 too.

1.3 Operational definitions

The above language of spacetime diagrams, however, is not sufficient for those who are interested in an *operational foundation of physics,* i.e., in a foundation according to which every basic notion and axiom refers to (simple) *experiments.* We do not know that what does "to be a coordinate system" (Ob) or "coordinatizing" (W) mean in terms of experiments.

Ax [1978] gave an axiomatization of *special* relativity in a language whose primitives are

- aTp: "a transmits the signal p",
- aRp: "a receives the signal p",

These primitives refer to experiments, so any axiom system on that language can be considered to be operational. The only problem was that even if this axiom system is complete w.r.t. Minkowski spacetimes, no one knew how much can be expressed in that language. Since this language did not contained terms for numbers and mathematical operations and relations, it was questionable whether the basic paradigmatic effects of relativity such as length contraction, time dilation, etc. can be expressed in that language. Later Andréka and Németi [2014] showed that Ax's axiom system is surprisingly expressive: with the addition of some minor axioms (about selecting a meter rod) makes it *definitionally equivalent* with SpecRelComp. Definitional equivalence means that the two theory are about the same models, i.e., they are 'equi-expressive'. The main idea of the proof of this definitional equivalence is that though the language of Ax [1978] seems to be very primitive, numbers, mathematical operations and coordinate systems (the primitives of the language of SpecRelComp) are definable.

But it is still an open question whether similar results can be obtained with accelerating observers or w.r.t general relativity. As far as our group see, the ideas of [Andréka and Németi 2014] are not transferrable to general relativity. Such a result, if there is at all, must be achieved in a radically different way. The main problem in that is that inertiality of observers does not seem to be definable in that language.

That is where we are now and that is the point where our report steps into the picture. A *corollary* of our main result is a framework that can reproduce the same results (decidability, and completeness w.r.t. Minkowski spacetimes, definitional equivalence with SpecRelComp) such that it can be still considered to be operational. Its primitives are

- $+$, $\cdot$, $\leq$, that refer to the standard mathematical terms
- $e \prec e'$: "e is in the causal past of e'",
- $\mathrm{P}(e, a, x)$: "in the event e, a observes that its clock shows the time x"

Here the reference to the causality relation and events can be reduced the notion of 'change' using a *modal* framework, but we will discuss this in Section 1.4 below. Contrary to results of Ax and Andréka–Németi this operational attempt does not involve the definition of mathematics in terms of experiments, only the definition of those terms that refer to non-mathematical/physical phenomena. This price was not paid in vane: the definition of inertial observers is possible (and simple) in that language, and by that, the road to acceleration is paved, and the research for an operational axiomatization of *general* relativity is started.

But, as we mentioned, that system is only a corollary of a bigger result. The main result of that report is that we did this in a way that we linked the remarkable modal research of the literature into our research.

1.4 Modal perspective

Modal logic, especially *temporal logic* in the foundation of relativity theories are to provide a local perspective of relativistic time and to make the information about spacetime available *in the spacetime itself.*

Modal and temporal logics are, however, usually stay in the propositional level, i.e., no variable bounding quantifiers are used. Instead of these, the common primitive connectives in modal logic are $\Box$ and its dual, $\Diamond$, that stands for 'change': in the semantics, it changes the 'state' or 'model', i.e., the truth of some formulas. According to the relativistic interpretation, 'states' are events and 'change' is the change along the causal evolution of spacetime.

Temporal logics are modal logics where $\Box$ and $\Diamond$ are replaced with a **G** and a **F**, and there is an other connective that makes room for 'memory' in the form of ‘backward change’. The relativistic interpretation of these connectives are then

- $\mathbf{G}\varphi$: “φ is always going to be true in the causal future”,
- $\mathbf{F}\varphi$: “φ will be true in the causal future”,
- $\mathbf{H}\varphi$: “φ has always been the case in the causal past”,
- $\mathbf{P}\varphi$: “φ was the case in the causal past”,

If the temporal logic in question is propositional, or in other words, a zero-order logic, then this means that its primitive sentences do not bound variables, i.e., has no inner structure; they are just p-s and q-s, that can be true or false but they do not express a relation in the state. They are, however, freely interpretable – that is why we use the expression ‘temporal *logic*’ instead of ‘temporal *theories*’.

To enrich the expressive power of that language, we will use first-order temporal logics instead with the following (familiar) special primitives:

- $+\ \cdot$ and $\leq$: the standard mathematical terms
- $a{:}\tau$: “a observes that its clock shows the time τ”

Note that the resemblance with the first-order classical language of the previous section; there is no explicit reference to the events, while the primitive predicate is still operational. Here, a refers to clocks, that x refers to numbers. For a start, a clock is something that – contrary to a number – can change its denotation while the spacetime evolve / the state changes / the time elapse / we shift from one event to another along causality. In that language, a quantification will be local; we can quantify over only those clocks that are actually available:

- $\exists a\varphi$: “There is a clock a here such that φ is true”

We will assume that numbers will always going to be available in every event.

Note that in this language, the clock-relativized temporal operators, and as such the experienced past is immediately definable:

- $\mathbf{P}_a\varphi \overset{\text{def}}{\Leftrightarrow} \mathbf{P}(\exists x\, a{:}x \wedge \varphi)$: “somewhere in the causal past *where a occurred*, φ is true”

Therefore, this system can be considered as a multi-agent system, i.e., a system in which every agent (the clock) has its own modal operator. Since we have that local quantification over clocks as well, these agent can talk about each other, they can share information about their past – that is how the exploration of spacetime is look like in that language.

1.5 Main Result

A main result of the current research is an axiom system in the above outlined *first-order temporal logic* that has the following properties:

1. **Strong Expressive Power:** It can express the basic paradigmatic relativistic effects of kinematics such as time dilation, length contraction, twin paradox, etc.

2. **Operationality:** The coordinatization itself is definable using *(metric) tense operators* with *signalling procedures*. These operators refer to inertial agents drifting in space and conducting signalling experiments to discover the spacetime they live in.

3. **Completeness and Decidability:** It axiomatizes the true formulas of 4D Minkowski spacetime and as such it is decidable.

4. **Formally compared to SpecRel:** A (first-order modal variant of a) definitional equivalence can be proved with the axiom system SpecRel∪Comp of Andréka, Madarász, and Németi [2007].

1.6 Structure of this report

Chapter 2 This chapter contains the basics of the first-order temporal logic we are going to use: the language, the models, and a logical calculus that is complete w.r.t. the corresponding semantics. The main distinctive features of that logic are the usage of intensional objects/nonrigid designators to represent clocks, the presence of a rigid mathematical sort and that we use at most 0-ary predicate letters (propositional variables) that are non-mathematical. The completeness proof of that logic can be considered as a modification of the method of Goldblatt [2011], though we modified it in so many ways (for a summary, see p.34) that we provided a self-contained completeness proof in that chapter too.

Chapter 3 The main idea of our results is that pointing statements "a observes that now its clock shows x" can be used to tag and track events in the spacetime. This makes the expressive power of our logic so strong that it can be compared to the expressive power of hybrid clock logics. (The most important notion of our report, the so-called *hybrid sort definition* is about that on p.50.) To show this, we introduce first-order hybrid (clock) logics in Chapter 3. Strong hybrid logics (like those to ours correspond) have the property that they have the same expressive power as the corresponding classical language, so we introduce these notions also in that chapter: the notion of standard translation and hybrid translation, the two-way bridge between these two systems can be found here.

Chapter 4 Here we start to extend our basic clock logic into stronger and stronger logics to approach the clock logic of Minkowski spacetime in a way that the completeness theorem of Chapter 2 can be preserved via canonicity arguments. Instead of presenting the strongest system, we presented an expanding chain of systems, because that makes it possible to come back later and start a new research into a different direction, see Chapter 6.

Chapter 5.2 To compare our clock logic to other axiomatic sytems of ours (mainly to SpecRel of [Andréka et al. 2007]), we have to give an account on coordinate systems. In this chapter we show how to build a coordinate system in our operational language. That is also the place where the most physical content can be found: we define standard notions of physics, like inertiality, distance, speed, etc. That is also the place where we will give a finite scheme axiom system that will be complete for 4D Minkowski spacetimes, though the proof of this completeness will be given in Chapter 5.5.

Chapter 5.5 This is the place where we define the main language of our research group that focuses on spacetime-diagrams and as such it yields very simple, transparent and easily manageable (but not operational) axiomatization of relativity theories. We present the complete and decidable axiom system SpecRelComp of Minkowski spacetimes in that language and a proof that formally establishes the fact that this axiomatization and the axiom system SClTh introduced in Chapter 5.2 are *definitionally equivalent*. Though the proof is quite long, it shows clearly how can these two theories describe the very same models in their own language. This chapter is also the place where we show that every translation of the axioms of the standard axiom system SpecRelComp can be proved in the clock axiom system SClTh and vice versa. Two main corollaries of that is the completeness of SClTh w.r.t. 4D Minkowski spacetimes, and that SClTh is decidable.

Chapter 6 Now that we worked so hard on proving these results, we illustrate how can these results can be a base for numerous other researches. We are going to summarize in this chapter how these results can be used in the axiomatization of branching spacetimes, defining mass and giving an operational axiomatization of general relativity.

1.7 A guide to our basic notations

Notation 1 (Vector-notation, projections)**.** If $\vec{x} = \langle x_1, \ldots, x_n \rangle$, then we denote the ith member of $\vec{x}$ by $\vec{x}_i$ or $(\vec{x})_i$.

If f is a function with a codomain of some set of n-tuples, then for any $1 \leq k \leq n$,

$$f_k(\vec{x}) \stackrel{\text{def}}{=} (f(\vec{x}))_k$$

We will use the following abbreviations as well: If $i \leq j \leq n$, then for any n-tuple $\vec{x}$,

$$f_{i-j}(\vec{x}) \stackrel{\text{def}}{=} \langle v_i(\vec{x}), v_{i+1}(\vec{x}), \ldots, v_j(\vec{x}) \rangle$$

$$f_{i_1,i_2,\ldots,i_n}(\vec{x}) \stackrel{\text{def}}{=} \langle v_{i_1}(\vec{x}), v_{i_2}(\vec{x}), \ldots, v_{i_n}(\vec{x}) \rangle$$

We also use the vector-notation in syntax; if P is an n-ary predicate then

$$P(\langle x_1, \ldots, x_n \rangle) \stackrel{\text{def}}{=} P(x_1, \ldots, x_n)$$

Notation 2 (bounded quantification)**.** If a formula φ has open variables $v_1, v_2, \ldots, v_n$, then we usually (but not always) denote this fact by listing these variables in parentheses: $\varphi(v_1, v_2, \ldots, v_n)$.

We use the $\in$ symbol for bounded quantification. If v is free in φ, then

$$(\forall v \in \varphi)\psi \stackrel{\text{def}}{\Leftrightarrow} \forall v(\varphi(v) \rightarrow \psi)$$

Sequences of quantifications are abbreviated with commas:

$$\forall v_1, \ldots v_n \varphi \stackrel{\text{def}}{\Leftrightarrow} \forall v_1 \forall v_2 \cdots \forall v_n \varphi$$

And we do the very same with bounded quantifications as well:

$$(\forall v_1, v_2, \ldots, v_n \in \varphi)\psi \overset{\text{def}}{\Leftrightarrow} \forall v_1, \ldots, v_n((\varphi(v_1) \wedge \varphi(v_2) \wedge \cdots \wedge \varphi(v_n)) \rightarrow \psi)$$

If a tuple of variables occur in a bounded quantification instead of a list form, then we refer to the following generalization of bounded quantification:

$$(\forall \langle v_1, v_2, \ldots, v_n \rangle \in \varphi)\psi \overset{\text{def}}{\Leftrightarrow} \forall v_1, \ldots, v_n(\varphi(v_1, v_2, \ldots, v_n) \rightarrow \psi)$$

Note that here φ, contrary to the previous examples, has at least n free variables!

If φ is a binary relation, we use the infix notation instead of the prefix:

$$v_1 \varphi v_2 \overset{\text{def}}{\Leftrightarrow} \varphi(v_1, v_2)$$

We can use binary relations for bounds instead of $\in$, like it is standard for $<$:

$$(\forall v_2 \reflectbox{φ} v_1)\psi \overset{\text{def}}{\Leftrightarrow} \forall v_2(v_1 \varphi v_2 \rightarrow \psi)$$

Note that in this notation we always reflect the symbol of φ (like $>$ for $<$ and $\succ$ for $\prec$)

In Chapter 5.2, we will frequently define functions in the object language, but most of the time these functions will be partial. The following notational conventions will make the life easier there.

Notation 3 (Functions, partial functions)**.** Let v an arbitrary variable, and $\vec{v}$ is an n-tuple of arbitrary variables. A formula $F(\vec{v}, v')$ is a *function* in the system Γ, iff

$$\Gamma \vdash \exists y(F(\vec{v}, v_1) \wedge \forall z(F(\vec{v}, v_2) \rightarrow v_1 = v_2)),$$

We call $\mathrm{F}(\vec{w}, \vec{a}, \vec{x}, y)$ a *partial function* in Γ, if

$$\Gamma \vdash \forall y, z(\mathrm{F}(\vec{w}, \vec{a}, \vec{x}, y) \wedge \mathrm{F}(\vec{w}, \vec{a}, \vec{x}, z) \rightarrow y = z).$$

We refer to the only v' which satisfy $\varphi(\vec{v}, v')$ with the lower case, one-argumentless $f(\vec{v})$. Formally:

$$\varphi(f(\vec{v})) \overset{\text{def}}{\Leftrightarrow} \exists y(F(\vec{v}, v') \wedge \varphi(v'))$$

So if $F(\vec{w}, \vec{a}, \vec{x}, y)$ is only a partial function, then the truth of $\varphi(\mathrm{f}(\vec{w}, \vec{a}, \vec{x}))$ implies that $f(\vec{w}, \vec{a}, \vec{x})$ *is defined*, and has the property φ. Roughly speaking, using this notation, we will never have to excuse ourselves using partial functions.

Chapter 2

BCL: Basic Clock Logic

If the Reader is familiar with the admissible semantics of Goldblatt [2011], we suggest to check Section 2.4 first.

2.1 Language and models

2.1.1 Language

Definition 4. The language of BCL is given by the following syntax:

- Symbols:
  - Propositional variables: $p, q, \ldots$ $\qquad PrVar \stackrel{\text{def}}{=} \{p_i : i \in \omega\}$
  - Pointer variables: $a, b, c, \ldots$ $\qquad CVar \stackrel{\text{def}}{=} \{a_i : i \in \omega\}$
  - Mathematical variables: $x, y, z, \ldots$ $\qquad MVar \stackrel{\text{def}}{=} \{x_i : i \in \omega\}$
  - Mathematical constants: $\mathsf{r}_1, \mathsf{r}_2, \ldots$
  - Pointer constants: $\mathsf{c}_1, \mathsf{c}_2, \ldots$
  - Mathematical function symbols: $+, \cdot$
  - Mathematical predicate symbols: $\leq$
  - Logical symbols: $\neg, \wedge, \mathbf{P}, \mathbf{F}, =, \exists$
  - other: (,)

  We use the abbreviation $Var \stackrel{\text{def}}{=} PrVar \cup CVar \cup MVar$.

- Mathematical terms:

$$\tau ::= x \mid \mathsf{r} \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$$

- Pointer terms:

$$\pi ::= a \mid \mathsf{c}$$

- Formulas:

$$\varphi ::= \tau \leq \tau' \mid \tau = \tau' \mid \pi{:}\tau \mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{P}\varphi \mid \mathbf{F}\varphi \mid \exists x \varphi \mid \exists a \varphi$$

Intuitive readings are:

p	p happened in that world
$\tau + \tau'$	the sum of the numbers τ and τ' (in every world)
$\tau \cdot \tau'$	the product of the numbers τ and τ' (in every world)
$\tau \leq \tau'$	number τ_1 is less than or equal to the number τ' (in every world)
$\tau = \tau'$	the numbers τ and τ' are equal (in every world)
$\pi{:}\tau$	Pointer π points to τ in the world in which we are right now.
$\varphi \wedge \psi$	φ and ψ are true in that world.
$\neg\varphi$	φ is not true in that world.
$\exists x\varphi$	There is a number x such that φ.
$\exists a\varphi$	There is a pointer that is defined/points to a number in that world x such that φ.
$\mathbf{F}\varphi$	There is an alternative world of that world, where φ is true. (**F** stands for Future)
$\mathbf{P}\varphi$	This is an alternative world of a world, where φ is true. (**P** stands for Past)

Notation 5.

$\mathcal{E}\pi$	$\overset{\text{def}}{\Leftrightarrow}$	$\exists x\, \pi : x$	π exists in that world.
$\mathbf{H}\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\neg\mathbf{P}\neg\varphi$	It has always been that case that φ
$\mathbf{G}\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\neg\mathbf{F}\neg\varphi$	It is always going to be the case that φ
$\underline{\mathbf{H}}\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{H}\varphi \wedge \varphi$	It is now and always going to be the case that φ
$\underline{\mathbf{G}}\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{G}\varphi \wedge \varphi$	It is now has always been the case that φ
$\underline{\mathbf{P}}\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{P}\varphi \vee \varphi$	It is now or will be the case that φ
$\underline{\mathbf{F}}\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{F}\varphi \vee \varphi$	It is now or was the case that φ
$\mathbf{P}_\pi\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{P}(\mathcal{E}\pi \wedge \varphi)$	It was the case on the worldline of π that φ
$\mathbf{F}_\pi\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{F}(\mathcal{E}\pi \wedge \varphi)$	It will be the case on the worldline of π that φ
$\mathbf{H}_\pi\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{H}(\mathcal{E}\pi \rightarrow \varphi)$	It has always been that case on the worldline of π that φ
$\mathbf{G}_\pi\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{G}(\mathcal{E}\pi \rightarrow \varphi)$	It is always going to be the case on the worldline of π that φ
$\underline{\mathbf{H}}_\pi\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{H}_\pi\varphi \wedge \varphi$	It is now and always going to be the case on the worldline of π that φ
$\underline{\mathbf{G}}_\pi\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{G}_\pi\varphi \wedge \varphi$	It is now has always been the case on the worldline of π that φ
$\underline{\mathbf{P}}_\pi\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{P}_\pi\varphi \vee \varphi$	It is now or will be the case on the worldline of π that φ
$\underline{\mathbf{F}}_\pi\varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{F}_\pi\varphi \vee \varphi$	It is now or was the case on the worldline of π that φ

Notation 6 (Representatives of pointers)**.** If φ is built up only from mathematical equations, inequalities and pointing statements,

$$\varphi(a)_x \overset{\text{def}}{\Leftrightarrow} \exists x(a : x \wedge \varphi(x)),$$

where x is a variable not occurring in φ, called the representative of π in φ. E.g.: $a = 3 + x \rightsquigarrow \exists y(a : y \wedge y = 3 + x)$

If it is not important, we omit the parameter x.

If φ contains more connectives, we extend this notion in an almost homomorphic way:

$(\neg\varphi(a))_x$	$\overset{\text{def}}{\Leftrightarrow}$	$\neg\varphi(a)_x$	
$(\varphi \wedge \psi(a))_x$	$\overset{\text{def}}{\Leftrightarrow}$	$\varphi(a)_x \wedge \psi(a)_x$	
$(\mathbf{P}\varphi(a))_x$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{P}\varphi(a)_x$	
$(\mathbf{F}\varphi(a))_x$	$\overset{\text{def}}{\Leftrightarrow}$	$\mathbf{F}\varphi(a)_x$	
$(\exists y\varphi(a))_x$	$\overset{\text{def}}{\Leftrightarrow}$	$\exists y\varphi(a)_x$	if y is not x
$(\exists x\varphi(a))_x$	$\overset{\text{def}}{\Leftrightarrow}$	$\exists x\varphi(a)_y$	where y is not x and is not free in φ

2.1.2 Model structures

Remark 1. For the purpose of *Prop* below, the set of admissible propositions and their role in handling incompleteness, see [Goldblatt 2011].

A model structure is

$$\mathcal{S} \stackrel{\text{def}}{=} (W, \succ, \prec, Prop, U, \Theta, \mathbb{C})$$

- $W \neq \varnothing$, the set of events/possible worlds
- $\succ \subseteq W^2, \prec \subseteq W^2$, past and future relation
  - $x \succ y \iff y \prec x$
- $Prop \subseteq \wp(W)$, admissible propositions
  - $X \in Prop \Rightarrow W - X \in Prop$, (form a BAO)
  - $X, Y \in Prop \Rightarrow X \cap Y \in Prop$,
  - $X \in Prop \Rightarrow [\succ]X \in Prop$,
  - $X \in Prop \Rightarrow [\prec]X \in Prop$, where

$$[\succ]X \stackrel{\text{def}}{=} \{w : (\forall w)' w \succ w' \Rightarrow w' \in X\}$$

$$[\prec]X \stackrel{\text{def}}{=} \{w : (\forall w)' w \prec w' \Rightarrow w' \in X\}$$

- $U \neq \varnothing$, numbers
- $\Theta \notin U$, semantic value gap
- $\mathbb{C} \subseteq \{\alpha : W \to U \cup \{\Theta\}\}$. possible pointers

The notation

$$\mathrm{D}_w \stackrel{\text{def}}{=} \{\alpha \in \mathbb{C} \,:\, \alpha(w) \neq \Theta\}$$

will refer to the set of pointers existing (pointing to sg's) in w.

2.1.3 Premodels

A premodel will be

$$\mathfrak{M} \stackrel{\text{def}}{=} \left(\mathcal{S}, [\![+]\!]^{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}}, [\![\leq]\!]^{\mathfrak{M}}, [\![\mathsf{r}_i]\!]^{\mathfrak{M}}, [\![\mathsf{c}_j]\!]^{\mathfrak{M}}\right)_{i\in I, j\in J}$$

- $\mathcal{S}$ is a pointer model structure,
- $[\![\mathsf{r}_i]\!]^{\mathfrak{M}} \in U$,
- $[\![\mathsf{c}_j]\!]^{\mathfrak{M}} \in \mathbb{C}$,
- $[\![+]\!]^{\mathfrak{M}} : U^2 \to U$,
- $[\![\cdot]\!]^{\mathfrak{M}} : U^2 \to U$,
- $[\![\leq]\!]^{\mathfrak{M}} \subseteq U^2$.

2.1.4 Assignments, meaning of terms

Definition 7 (Assignments)**.** We take composite assignments instead of tracking 3 or more different types of assignment functions during the evaluation of formulas.

$$\eta : \begin{cases} p \mapsto X \in Prop & \text{propositional evaluation} \\ x \mapsto u \in U & \text{mathematical assignment} \\ a \mapsto (w \mapsto u) \in \mathbb{C} & \text{pointer assignment} \end{cases}$$

We refer to the components of η in the following way:

$$\begin{aligned} \eta^m &\stackrel{\text{def}}{=} \eta\restriction MVar \\ \eta^c &\stackrel{\text{def}}{=} \eta\restriction CVar \\ \eta^p &\stackrel{\text{def}}{=} \eta\restriction PrVar \\ \eta^- &\stackrel{\text{def}}{=} \eta\restriction (MVar \cup CVar) \end{aligned}$$

$\eta[x \mapsto u]$ is the assignment which differs from η only with respect to x for which it maps u. Similarly for $\eta^c[a \mapsto \alpha]$ and $\eta^p[p \mapsto X]$. The set of all BCL-assignments on a model structure $\mathcal{S}$ or model $\mathfrak{M}$ will be denoted by $\mathrm{H}_{\mathcal{S}}$ or $\mathrm{H}_{\mathfrak{M}}$, respectively.

Definition 8 (Terms)**.** Mathematical terms:

$$\begin{aligned} [\![\mathsf{r}]\!]^{\mathfrak{M}}_{\eta} &\stackrel{\text{def}}{=} [\![\mathsf{r}]\!]^{\mathfrak{M}}, \\ [\![x]\!]^{\mathfrak{M}}_{\eta} &\stackrel{\text{def}}{=} \eta(x), \\ [\![\tau_1 + \tau_2]\!]^{\mathfrak{M}}_{\eta} &\stackrel{\text{def}}{=} [\![\tau_1]\!]^{\mathfrak{M}}_{\eta}\ [\![+]\!]^{\mathfrak{M}}\ [\![\tau_2]\!]^{\mathfrak{M}}_{\eta}, \\ [\![\tau_1 \cdot \tau_2]\!]^{\mathfrak{M}}_{\eta} &\stackrel{\text{def}}{=} [\![\tau_1]\!]^{\mathfrak{M}}_{\eta}\ [\![\cdot]\!]^{\mathfrak{M}}\ [\![\tau_2]\!]^{\mathfrak{M}}_{\eta}. \end{aligned}$$

Pointer terms:

$$\begin{aligned} [\![\mathsf{c}]\!]^{\mathfrak{M}}_{\eta} &\stackrel{\text{def}}{=} [\![\mathsf{c}]\!]^{\mathfrak{M}}, \\ [\![a]\!]^{\mathfrak{M}}_{\eta} &\stackrel{\text{def}}{=} \eta(a). \end{aligned}$$

2.1.5 Truth, validity, intension, consequence

Local truth:

$$\begin{aligned} \mathfrak{M}, \eta, w \models p &\stackrel{\text{def}}{\Leftrightarrow} w \in \eta(p), \\ \mathfrak{M}, \eta, w \models \tau_1 \leq \tau_2 &\stackrel{\text{def}}{\Leftrightarrow} \left\langle [\![\tau_1]\!]^{\mathfrak{M}}_{\eta}, [\![\tau_2]\!]^{\mathfrak{M}}_{\eta} \right\rangle \in [\![\leq]\!]^{\mathfrak{M}}, \\ \mathfrak{M}, \eta, w \models \tau_1 = \tau_2 &\stackrel{\text{def}}{\Leftrightarrow} [\![\tau_1]\!]^{\mathfrak{M}}_{\eta} = [\![\tau_2]\!]^{\mathfrak{M}}_{\eta}, \\ \mathfrak{M}, \eta, w \models \pi{:}\tau &\stackrel{\text{def}}{\Leftrightarrow} [\![\pi]\!]^{\mathfrak{M}}_{\eta}(w) = [\![\tau]\!]^{\mathfrak{M}}_{\eta}, \\ \mathfrak{M}, \eta, w \models \neg\varphi &\stackrel{\text{def}}{\Leftrightarrow} \mathfrak{M}, \eta, w \not\models \varphi, \\ \mathfrak{M}, \eta, w \models \varphi \wedge \psi &\stackrel{\text{def}}{\Leftrightarrow} \mathfrak{M}, \eta, w \models \varphi \text{ and } \mathfrak{M}, \eta, w \models \psi, \\ \mathfrak{M}, \eta, w \models \mathbf{P}\varphi &\stackrel{\text{def}}{\Leftrightarrow} (\exists w' \prec w)\ \mathfrak{M}, \eta, w' \models \varphi, \\ \mathfrak{M}, \eta, w \models \mathbf{F}\varphi &\stackrel{\text{def}}{\Leftrightarrow} (\exists w' \succ w)\ \mathfrak{M}, \eta, w' \models \varphi, \\ \mathfrak{M}, \eta, w \models \exists x\varphi &\stackrel{\text{def}}{\Leftrightarrow} (\exists u \in U)\ \mathfrak{M}, \eta[x \mapsto u], w \models \varphi, \\ \mathfrak{M}, \eta, w \models \exists a\varphi &\stackrel{\text{def}}{\Leftrightarrow} (\exists \alpha \in \mathrm{D}_w)\ \mathfrak{M}, \eta[a \mapsto \alpha], w \models \varphi. \end{aligned}$$

Global truth	$\mathfrak{M}, \eta \models \varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$(\forall w)\, \mathfrak{M}, \eta, w \models \varphi$
Weak validity	$\mathfrak{M}, \eta^p \models \varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$(\forall \eta \supset \eta^p)(\forall w)\, \mathfrak{M}, \eta, w \models \varphi$
Strong validity	$\mathfrak{M} \models \varphi$	$\overset{\text{def}}{\Leftrightarrow}$	$(\forall \eta, w)\, \mathfrak{M}, \eta, w \models \varphi$
Intension	$[\![\varphi]\!]_\eta^\mathfrak{M}$	$\overset{\text{def}}{=}$	$\{w : \mathfrak{M}, \eta, w \models \varphi\}$

(Worldwise) consequence

$$\Gamma \models \varphi \overset{\text{def}}{\Leftrightarrow} (\forall \eta, w, \mathfrak{M})\big[(\forall \psi \in \Gamma)\, \mathfrak{M}, \eta, w \models \psi \quad \Longrightarrow \quad \mathfrak{M}, \eta, w \models \varphi\big]$$

The Reader could think of the weak validity as a model-oriented perspective and to the strong validity as the frame-oriented perspective.

If we work with weak validity, we work with concrete meanings of the propositional variables. In this case we could follow the terminology of the model-oriented semantics, where propositional variables are called *atomic sentences*, and they represents concrete sentences of the model, such as $p \equiv$ "Lightning flashed" or $q \equiv$ "John loves Mary".

Strong validity corresponds to the frame-oriented semantics in standard propositional modal logic. Here a propositional variables represents a *freely interpretable* (within the boundaries of admissibility of course) sentence. A formula containing propositional variables thus says something about the behaviour of propositional sentences – in the relativistic temporal interpretatation, it says something about how the atomic sentences with pointing statements and causal tenses behave in the relativistic reality.

We will never consider any *real* frame or model-structure-oriented notion of validity, i.e., (W, R) or $(W, R, Prop, U, \Theta, \mathbb{C})$-oriented notion of validity, since such a notion would involve free interpretations of mathematics in the same way as strong validity involves free interpretations of $PrVar$. Mathematics now is a part of the *invariant* structure on which we interpret our language. In a Tarskian sense, mathematical functions $+$, $\cdot$ and the predicate $\leq$ are now considered to be *logical* functions and symbols, such as $=$ in the standard first-order classical logic; they are invariant under the interpretation. Roughly speaking, we will never be curious about what if $+$, $\cdot$ and $\leq$ means something completely different – we don't see the point of any such investigation.

2.1.6 Models

Definition 9. A *model* is a premodel in which all intensions are in $Prop$:

$$[\![\varphi]\!]_\eta^\mathfrak{M} \in Prop \quad \text{for all formula } \varphi.$$

Theorem 2. *A premodel $\mathfrak{M}$ is a model iff for all φ, π, τ*

$$[\![\pi{:}\tau]\!]_\eta^\mathfrak{M} \in Prop$$

$$[\![\varphi]\!]_\eta^\mathfrak{M} \in Prop \quad \Longrightarrow \quad [\![\forall x \varphi]\!]_\eta^\mathfrak{M} \in Prop$$

$$[\![\varphi]\!]_\eta^\mathfrak{M} \in Prop \quad \Longrightarrow \quad [\![\forall a \varphi]\!]_\eta^\mathfrak{M} \in Prop$$

2.2 Axiomatization

2.2.1 Templates

Let $\mathbf{L}$ be a placeholder for which we can substitute either $\mathbf{G}$ or $\mathbf{H}$, and $\mathbf{M}$ be the same for $\mathbf{F}$ or $\mathbf{P}$.

We will call the iterated use of $\varphi \to \mathbf{L}(-)$ on ψ and $\chi \to \psi$ *templates*. For example,

$$\varphi_1 \to \mathbf{G}(\varphi_2 \to \mathbf{H}(\varphi_3 \to \mathbf{H}(\varphi_4 \to \mathbf{H}(\varphi_5 \to \mathbf{G}(\varphi_6 \to \varphi_7)))))$$

is a template. *Co-templates* are the duals of these formulas, e.g.,

$$\varphi_1 \wedge \mathbf{F}(\varphi_2 \wedge \mathbf{P}(\varphi_3 \wedge \mathbf{P}(\varphi_4 \wedge \mathbf{P}(\varphi_5 \wedge \mathbf{F}(\varphi_6 \wedge \varphi_7)))))$$

The co-templates can be considered as a syntactical representation of a scanning process in the neighbour worlds, while the templates can be considered as an answer to the question "how can I find φ_7?". The subject of this scanning is always the last formula, in these examples, φ_7. To emphasize this, we will use the following notation for the template and co-template above:

$$[\langle \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 \rangle; \langle \mathbf{G}, \mathbf{H}, \mathbf{H}, \mathbf{H}, \mathbf{G} \rangle]\, \varphi_7,$$

and

$$\langle \langle \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 \rangle; \langle \mathbf{F}, \mathbf{P}, \mathbf{P}, \mathbf{P}, \mathbf{F} \rangle \rangle\, \varphi_7,$$

or more shortly,

$$\left[\vec{\varphi}; \vec{\mathbf{L}}\right] \varphi_7, \text{ and } \left\langle \vec{\varphi}; \vec{\mathbf{M}} \right\rangle \varphi_7$$

where $\vec{\varphi} = \langle \varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6 \rangle$ and $\vec{\mathbf{L}} = \langle \mathbf{G}, \mathbf{H}, \mathbf{H}, \mathbf{H}, \mathbf{G} \rangle$ and $\vec{\mathbf{M}} = \langle \mathbf{F}, \mathbf{P}, \mathbf{P}, \mathbf{P}, \mathbf{F} \rangle$

The motivation of this notion is that it is closed under conditionalization and necessitation, and this property is needed in the existence lemma of the canonical model construction (more precisely, in the lemmas (L^-) and (FE)).

Definition 10. Templates are given by the following recursive definitions:

$$\begin{aligned}
[\varnothing; \varnothing]\, \psi &\overset{\text{def}}{\Leftrightarrow} \psi \\
[\varphi; \varnothing]\, \psi &\overset{\text{def}}{\Leftrightarrow} (\varphi \to \psi) \\
\left[\langle \varphi', \vec{\varphi} \rangle; \langle \mathbf{G}, \vec{\mathbf{L}} \rangle\right] \psi &\overset{\text{def}}{\Leftrightarrow} \varphi' \to \mathbf{G} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] \psi \\
\left[\langle \varphi', \vec{\varphi} \rangle; \langle \mathbf{H}, \vec{\mathbf{L}} \rangle\right] \psi &\overset{\text{def}}{\Leftrightarrow} \varphi' \to \mathbf{H} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] \psi
\end{aligned}$$

And the co-templates by

$$\begin{aligned}
\langle \varnothing; \varnothing \rangle\, \psi &\overset{\text{def}}{\Leftrightarrow} \psi \\
\langle \varphi; \varnothing \rangle\, \psi &\overset{\text{def}}{\Leftrightarrow} (\varphi \wedge \psi) \\
\left\langle \langle \varphi', \vec{\varphi} \rangle; \langle \mathbf{F}, \vec{\mathbf{M}} \rangle \right\rangle \psi &\overset{\text{def}}{\Leftrightarrow} \varphi' \to \mathbf{F} \left\langle \vec{\varphi}; \vec{\mathbf{M}} \right\rangle \psi \\
\left\langle \langle \varphi', \vec{\varphi} \rangle; \langle \mathbf{P}, \vec{\mathbf{L}} \rangle \right\rangle \psi &\overset{\text{def}}{\Leftrightarrow} \varphi' \to \mathbf{P} \left\langle \vec{\varphi}; \vec{\mathbf{L}} \right\rangle \psi
\end{aligned}$$

where $\vec{\varphi}$ stands for an n-or $n+1$-tuple of formulas, $\vec{\mathbf{L}}$ is an n-tuple of universal modalities and $\vec{\mathbf{M}}$ is an n-tuple of existential modalities.

Remark 3. We will focus on templates, but co-templates are also used in the literature. E.g., in [Gabbay, Hodkinson, and Reynolds 1994]: Gabbay's celebrated IRR-style construction of irreflexive canonical models use IRR-theories that are maximal sets having the additional property

$$\frac{\left\langle \vec{\varphi}; \vec{\mathbf{M}} \right\rangle \top \in \Gamma}{(\exists p \in \mathrm{PropVar}) \left\langle \vec{\varphi}; \vec{\mathbf{M}} \right\rangle (p \wedge \mathbf{H}\neg p) \in \Gamma} \qquad \text{for all } \vec{\varphi} \text{ and } \vec{\mathbf{M}} \in \{\mathbf{F}, \mathbf{P}\}^*$$

Using the scanning analogy: IRR theories are those that have an evidence (p) that shows that they do not see themselves (check the template $\langle \varnothing; \varnothing \rangle$), and are in a neighbourhood of theories that are similar to them in this respect. We also note that using templates instead of co-templates in [Gabbay et al. 1994] makes it possible to decompose the completeness proofs to small, compact and reusable lemmas like ($\mathbf{L}^-$), (FE) and (L^+).

Lemma 4 (Template-lemmas)**.** *The following rules are consequences of the definition of templates and basic rules of classical propositional logic:*

- *Templates are closed to conditionalization*

$$\frac{\chi \to \left[\langle \varphi_1, \ldots, \varphi_n \rangle; \vec{\mathbf{L}}\right] \psi}{\left[\langle \chi_1 \wedge \varphi_1, \ldots, \varphi_n \rangle; \vec{\mathbf{L}}\right] \psi}$$

- *Templates are closed to necessitation*

$$\frac{\mathbf{G}\left[\vec{\varphi}; \vec{\mathbf{L}}\right] \psi}{\left[\langle \top, \vec{\varphi} \rangle; \langle \mathbf{G}, \vec{\mathbf{L}} \rangle\right] \psi} \qquad \frac{\mathbf{H}\left[\vec{\varphi}; \vec{\mathbf{L}}\right] \psi}{\left[\langle \top, \vec{\varphi} \rangle; \langle \mathbf{H}, \vec{\mathbf{L}} \rangle\right] \psi}$$

2.2.2 Axioms

Definition 11. A set of formulas L is a BCL if it contains all instances of the following axioms and is closed under the following rules:

PC1	$\varphi \to \psi \to \varphi$		
PC2	$(\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi$		
PC3	$\varphi \to \psi \to \varphi$	MP	$\dfrac{\varphi, \varphi \to \psi}{\psi}$
K	$\mathbf{L}(\varphi \to \psi) \to (\mathbf{L}\varphi \to \mathbf{L}\psi)$		
Bid	$(\varphi \to \mathbf{GP}\varphi) \wedge (\varphi \to \mathbf{HF}\varphi)$		
UI	$\forall x\varphi \to \varphi(\tau/x)$ τ is free for x in φ	N	$\dfrac{\varphi}{\mathbf{L}\varphi}$
EI	$\forall a\varphi \to \mathcal{E}\pi \to \varphi(\pi/a)$ π is free for a in φ		
BF	$\forall x\mathbf{L}\varphi \to \mathbf{L}\forall x\varphi$	$\forall$-Intro	$\dfrac{\varphi \to \psi}{\varphi \to \forall x\psi}$ where x is not free in φ
R	$\tau = \tau$		
SI	$\tau = \tau' \to \varphi \to \varphi(\tau'//\tau)$		
: F	$c{:}\tau \to c{:}\tau' \to \tau = \tau'$		
NNI	$\tau \neq \tau' \to \mathbf{L}\tau \neq \tau'$	$T\forall$-Intro	$\dfrac{\left[\vec{\varphi}; \vec{\mathbf{L}}\right] (\mathcal{E}a \to \psi)}{\left[\vec{\varphi}; \vec{\mathbf{L}}\right] \forall a\psi}$ where x is not free in $\vec{\varphi}$
NO	$\tau \leq \tau' \to \mathbf{L}\tau \leq \tau'$		
NNO	$\neg\tau \leq \tau' \to \mathbf{L}\neg\tau \leq \tau'$		

Remark 5. Every pointer logic contains the K-tautologies and the FOL+= tautologies restricted to the mathematical sort.

Proposition 6.

EI$'$ $\quad \vdash \mathcal{E}\pi \to \forall x\varphi \to \varphi(\pi/x)$ where π is a pointer term whose representative in φ is y (and as such it is free for x in φ)

Proof.

$\{\mathcal{E}\pi, \forall x\varphi, \forall y(\neg\pi : y \vee \neg\varphi(y/x))\} \vdash_{\mathrm{L}} \neg\pi : y \vee \neg\varphi(y/x)$	UI+MP
$\{\mathcal{E}\pi, \forall x\varphi, \forall y(\neg\pi : y \vee \neg\varphi(y/x))\} \vdash_{\mathrm{L}} \varphi(y/x)$	UI+MP
$\{\mathcal{E}\pi, \forall x\varphi, \forall y(\neg\pi : y \vee \neg\varphi(y/x))\} \vdash_{\mathrm{L}} \neg\pi : y$	MP
$\{\mathcal{E}\pi, \forall x\varphi, \forall y(\neg\pi : y \vee \neg\varphi(y/x))\} \vdash_{\mathrm{L}} \forall y\neg\pi : y$	UG
$\{\mathcal{E}\pi, \forall x\varphi, \forall y(\neg\pi : y \vee \neg\varphi(y/x))\} \vdash_{\mathrm{L}} \neg\exists y\pi : y$	DeMorgan
$\{\mathcal{E}\pi, \forall x\varphi, \forall y(\neg\pi : y \vee \neg\varphi(y/x))\} \vdash_{\mathrm{L}} \neg\mathcal{E}\pi$	def. of $\mathcal{E}$
$\{\mathcal{E}\pi, \forall x\varphi, \forall y(\neg\pi : y \vee \neg\varphi(y/x))\} \vdash_{\mathrm{L}} \bot$	PC
$\{\mathcal{E}\pi, \forall x\varphi\} \vdash_{\mathrm{L}} \forall y(\neg\pi : y \vee \neg\varphi(y/x)) \to \bot$	ded.thm.
$\{\mathcal{E}\pi, \forall x\varphi\} \vdash_{\mathrm{L}} \forall y\neg(\pi : y \wedge \varphi(y/x)) \to \bot$	DeMorgan
$\{\mathcal{E}\pi, \forall x\varphi\} \vdash_{\mathrm{L}} \neg\forall y\neg(\pi : y \wedge \varphi(y/x))$	PC
$\{\mathcal{E}\pi, \forall x\varphi\} \vdash_{\mathrm{L}} \exists y(\pi : y \wedge \varphi(y/x))$	DeMorgan
$\{\mathcal{E}\pi, \forall x\varphi\} \vdash_{\mathrm{L}} \varphi(\pi/x)$	def. of $\varphi(\pi/x)$
$\vdash_{\mathrm{L}} \mathcal{E}\pi \to \forall x\varphi \to \varphi(\pi/x)$	q.e.d.

■

Proposition 7.

NI $\quad \vdash \tau = \tau' \to \mathbf{L}\tau = \tau'$

Proof.

$$
\begin{array}{rll}
 & \vdash_{\mathrm{L}} \tau = \tau' \to \mathbf{L}(\tau = \tau) \to (\mathbf{L}(\tau = \tau))(\tau // \tau') & \text{SI} \\
 & \vdash_{\mathrm{L}} \tau = \tau' \to \mathbf{L}(\tau = \tau) \to \mathbf{L}(\tau = \tau') & \\
\{\tau = \tau'\} & \vdash_{\mathrm{L}} \mathbf{L}(\tau = \tau) \to \mathbf{L}(\tau = \tau') & \\
 & \vdash_{\mathrm{L}} \tau = \tau & \text{R} \\
 & \vdash_{\mathrm{L}} \mathbf{L}\tau = \tau & \text{N} \\
\{\tau = \tau'\} & \vdash_{\mathrm{L}} \mathbf{L}(\tau = \tau') & \text{MP} \\
 & \vdash_{\mathrm{L}} \tau = \tau' \to \mathbf{L}(\tau = \tau') & \text{MP}
\end{array}
$$

■

2.3 Completeness

2.3.1 Canonical models

Why not one canonical model?

Since we work with *one universal domain*, the canonical construction won't give one model.

Let CMT be the set of closed mathematical terms.

$$
\begin{array}{lll}
\text{canonical objects} & [\![\tau]\!]_\Gamma \overset{\text{def}}{=} \{\tau' \in CMT \,:\, \tau = \tau' \in \Gamma\} \\
 & \qquad\qquad \text{This is an eq. class by (R), (SI)} \\
\text{canonical domain(s)} & U_{\mathrm{L}}^{\Gamma} \overset{\text{def}}{=} \{[\![\tau]\!]_\Gamma \,:\, \tau \in CMT\} \\
\text{canonical pointers} & [\![\mathsf{c}]\!]_\Gamma \overset{\text{def}}{=} \begin{cases} [\![\tau]\!]_\Gamma & \text{if } \mathsf{c}{:}\tau \in \Gamma \\ \varnothing & \text{otherwise} \end{cases} \\
 & \qquad\qquad \text{well-defined because of (:F)}
\end{array}
$$

If we had $U_{\mathrm{L}}^{\Gamma} = U_{\mathrm{L}}^{\Gamma'}$ for all Γ (one universal domain), then every world would share the same set of objects $[\![\tau]\!]_\Gamma$. But $[\![\tau]\!]_\Gamma$ strongly depends on Γ; Both $\nvdash_{\mathrm{L}} \mathsf{r}_1 = \mathsf{r}_2$ and $\nvdash_{\mathrm{L}} \mathsf{r}_1 \neq \mathsf{r}_2$ are true, so there will be canonical worlds $\Gamma^{=}$ and $\Gamma^{\neq}$ containing these formulas, respectively. But clearly, $[\![\mathsf{r}_1]\!]_{\Gamma^{=}} \neq [\![\mathsf{r}_1]\!]_{\Gamma^{\neq}}$, since $\mathsf{r}_2 \in [\![\mathsf{r}_1]\!]_{\Gamma^{=}}$ but $\mathsf{r}_2 \notin [\![\mathsf{r}_1]\!]_{\Gamma^{\neq}}$

But that's OK: we will see that the canonical construction will result a collection of models, and if sg is $\nvdash_{\mathrm{L}}$, then there will be a ("point-generated") model in the canonical collection which falsifies it.

key property

Usual notion of saturated/rich/inductive set of formulas won't be enough for the pointer sort, so we try something stronger:

Definition 12. Let us denote the set of pointer constants by CC. Γ is *rich-and-T-rich*, or shortly, rTr in L iff the following two propoerties hold:

- $\Gamma \vdash_{\mathrm{L}} \exists x\varphi \quad \Longrightarrow \quad (\exists \tau \in CMT)\ \Gamma \vdash_{\mathrm{L}} \varphi(\tau/x)$
- and a similar statement is true for $\exists a$ co-template-by-co-template:

$$
\begin{array}{l}
(\forall \vec{\varphi} \in \mathcal{L}^*)(\forall \vec{\mathbf{M}} \in \{\mathbf{F}, \mathbf{P}\}^*) \\
\qquad \Gamma \vdash_{\mathrm{L}} \left\langle \vec{\varphi}; \vec{\mathbf{M}} \right\rangle \exists a\varphi \quad \Longrightarrow \quad (\exists \mathsf{c} \in CC)\ \Gamma \vdash_{\mathrm{L}} \left\langle \vec{\varphi}; \vec{\mathbf{M}} \right\rangle (\mathcal{E}\mathsf{c} \wedge \varphi(\mathsf{c}/a))
\end{array}
$$

Similarly, Γ is *inductive-and-T-inductive*, or shortly, iTi in L, iff

- $\Gamma \vdash_{\mathrm{L}} \forall x\varphi \quad \Longleftarrow \quad (\forall \tau \in CMT)\ \Gamma \vdash_{\mathrm{L}} \varphi(\tau/x)$
- and a similar statement is true for $\forall a$ template-by-template:

$$(\forall \vec{\varphi} \in \mathcal{L}^*)(\forall \vec{\mathbf{L}} \in \{\mathbf{G},\mathbf{H}\}^*)$$
$$\Gamma \vdash_{\mathrm{L}} \left[\vec{\varphi};\vec{\mathbf{L}}\right] \forall a\varphi \quad \Longleftarrow \quad (\forall \mathsf{c} \in CC)\ \Gamma \vdash_{\mathrm{L}} \left[\vec{\varphi};\vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \varphi(\mathsf{c}/a))$$

Proposition 8.

$$\Gamma \text{ is L-maximal} \Longrightarrow \big(\Gamma \text{ is rTr in L} \iff \Gamma \text{ is iTi in L}\big)$$

$$\Gamma \text{ is } \mathit{rTr} \text{ in L} \Longrightarrow \Gamma \text{ is } \mathit{rich} \text{ for both sorts in L}$$

$$\Gamma \text{ is } \mathit{iTi} \text{ in L} \Longrightarrow \Gamma \text{ is } \mathit{inductive} \text{ for both sorts in L}$$

Canonical collection of pointer model structures

Let L be an arbitrary pointer logic. Then we define the canonical collection as the (non-model) structure:

$$\mathfrak{C}_{\mathrm{L}} \stackrel{\text{def}}{=} (W_{\mathrm{L}}, \succ_{\mathrm{L}}, \prec_{\mathrm{L}}, \mathit{Prop}_{\mathrm{L}}, U_{\mathrm{L}}, \mathbb{C}_{\mathrm{L}})$$

where

- $W_{\mathrm{L}} \stackrel{\text{def}}{=} \left\{\Gamma : \begin{array}{l}\Gamma \text{ is L-maximal}\\ \text{and iTi in L}\end{array}\right\}$
- $\Gamma \succ_{\mathrm{L}} \Gamma' \stackrel{\text{def}}{\Leftrightarrow} \mathbf{H}^-(\Gamma) \subseteq \Gamma'$
- $\Gamma \prec_{\mathrm{L}} \Gamma' \stackrel{\text{def}}{\Leftrightarrow} \mathbf{G}^-(\Gamma) \subseteq \Gamma'$
- $\mathit{Prop}_{\mathrm{L}} \stackrel{\text{def}}{=} \{[\![\varphi]\!]_{\mathrm{L}} : \varphi \in \mathcal{L}\}$,
  where $[\![\varphi]\!]_{\mathrm{L}} \stackrel{\text{def}}{=} \{\Gamma \in W_{\mathrm{L}} : \varphi \in \Gamma\}$.
- $U_{\mathrm{L}}^{\Gamma} \stackrel{\text{def}}{=} \{[\![\tau]\!]_{\Gamma} : \tau \in CMT\}$
- $\mathbb{C}_{\mathrm{L}} \stackrel{\text{def}}{=} \{[\![\mathsf{c}]\!] : \mathsf{c} \in CC\}$

here we used the notation

$$\mathbf{H}^-(\Gamma) \stackrel{\text{def}}{=} \{\varphi : \mathbf{H}\varphi \in \Gamma\} \qquad \text{and} \qquad \mathbf{G}^-(\Gamma) \stackrel{\text{def}}{=} \{\varphi : \mathbf{G}\varphi \in \Gamma\},$$

see Fig.2.1

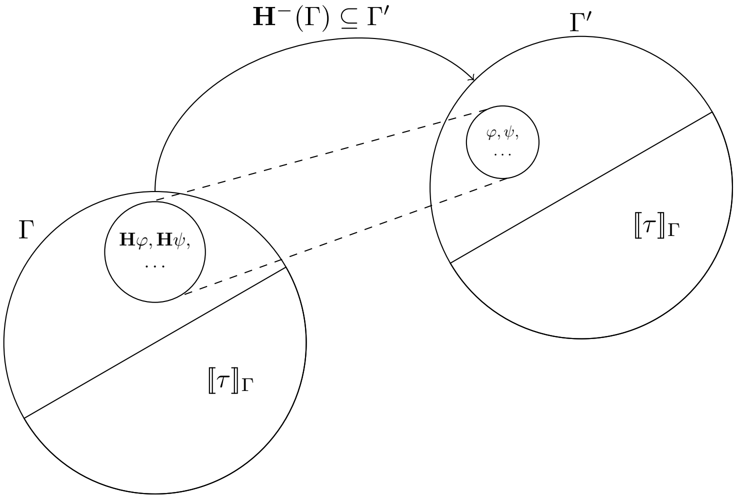
Now we will show that $\mathfrak{C}_{\mathrm{L}}$ is indeed a collection of pointer model structures. Let

$$\overline{R}_{\mathrm{L}} \stackrel{\text{def}}{=} \text{the smallest equivalence relation containing } R_{\mathrm{L}}.$$

where R is $\prec$ or $\succ$. We will show that

- the $\overline{R}_{\mathrm{L}}$-connected parts of $\mathfrak{C}_{\mathrm{L}}$ can be considered as a pointer model structure, and
- the $\overline{\prec}_{\mathrm{L}}$-connected parts are precisely the $\overline{\succ}_{\mathrm{L}}$-connected parts.

Figure 2.1: canonical alternative relation



To refer to these models we will use the contained worlds as names (like to refer to equivalence classes we use representatives).

We start with the latter:

Proposition 9.

$$\Gamma \succ_{\mathrm{L}} \Gamma' \iff \Gamma' \prec_{\mathrm{L}} \Gamma$$

Proof. Let us assume that $\Gamma \succ_{\mathrm{L}} \Gamma'$, i.e., $\varphi \in \Gamma'$ whenever $\mathbf{H}\varphi \in \Gamma$. Then we have to show that

$$\varphi \in \Gamma \quad \text{whenever} \quad \mathbf{G}\varphi \in \Gamma',$$

or, equivalently (K, RN, PC1-PC3, MP),

$$\mathbf{F}\varphi \in \Gamma' \quad \text{whenever} \quad \varphi \in \Gamma,$$

Let $\varphi \in \Gamma$. Then by (Bid), $\mathbf{HF}\varphi \in \Gamma$, and since $\Gamma \succ_{\mathrm{L}} \Gamma'$, $\mathbf{F}\varphi \in \Gamma'$.

The other direction can be done symmetrically. ■

Canonical model structures

Definition 13. The canonical pointer model structure of a canonical world Γ will be

$$\mathcal{S}_{\mathrm{L}}^{\Gamma} \overset{\text{def}}{=} \left(W_{\mathrm{L}}^{\Gamma}, \succ_{\mathrm{L}}^{\Gamma}, \prec_{\mathrm{L}}^{\Gamma}, \mathit{Prop}_{\mathrm{L}}^{\Gamma}, U_{\mathrm{L}}^{\Gamma}, \varnothing, \mathbb{C}_{\mathrm{L}}^{\Gamma}\right)$$

where

- $W_{\mathrm{L}}^{\Gamma} \overset{\text{def}}{=} \{\Gamma' \in W_{\mathrm{L}} : \Gamma \overline{\succ}_{\mathrm{L}} \Gamma'\}$
- $\succ_{\mathrm{L}}^{\Gamma} \overset{\text{def}}{=} \succ_{\mathrm{L}} \restriction W_{\mathrm{L}}^{\Gamma}$
- $\prec_{\mathrm{L}}^{\Gamma} \overset{\text{def}}{=} \prec_{\mathrm{L}} \restriction W_{\mathrm{L}}^{\Gamma}$
- $\mathit{Prop}_{\mathrm{L}}^{\Gamma} \overset{\text{def}}{=} \{X \cap W_{\mathrm{L}}^{\Gamma} : X \in \mathit{Prop}_{\mathrm{L}}\}$

- $\mathbb{C}^{\Gamma}_{\mathrm{L}} \stackrel{\text{def}}{=} \{[\![\mathsf{c}]\!] \upharpoonright W^{\Gamma}_{\mathrm{L}} : \mathsf{c} \in CC\}$

Proposition 10. *$\mathcal{S}^{\Gamma}_{\mathrm{L}}$ is a pointer model structure*

Proof.

1. $W^{\Gamma}_{\mathrm{L}} \neq \varnothing$ since $\Gamma \in W^{\Gamma}_{\mathrm{L}}$.
2. $\succ^{\Gamma}_{\mathrm{L}}, \prec^{\Gamma}_{\mathrm{L}} \subseteq (W^{\Gamma}_{\mathrm{L}})^2$ by definition.
3. $(W^{\Gamma}_{\mathrm{L}}, \succ^{\Gamma}_{\mathrm{L}}, \prec^{\Gamma}_{\mathrm{L}}, Prop^{\Gamma}_{\mathrm{L}})$ is a general frame, since $Prop^{\Gamma}_{\mathrm{L}}$ is a Boolean subalgebra of $\wp(W_{\mathrm{L}})$ and is closed to $\succ^{\Gamma}_{\mathrm{L}}$ and $\prec^{\Gamma}_{\mathrm{L}}$ by definition, so it is closed to $[\succ^{\Gamma}_{\mathrm{L}}]$ and $[\prec^{\Gamma}_{\mathrm{L}}]$ as well.
4. U^{Γ}_{L} is not empty since the set of rigid constants is not empty.
5. $\varnothing \notin U^{\Gamma}_{\mathrm{L}}$ since all the elements of U^{Γ}_{L} are equivalence classes of rigid constants, so each of these class are not empty.
6. $\mathbb{C}^{\Gamma}_{\mathrm{L}}$ is a set of functions defined on W^{Γ}_{L} by definition, and the values of these functions are either $\varnothing$ (again, by def.) or elements of U^{Γ}_{L}; the latter the only one which needs a detailed proof.

Lemma 11. *All $\Gamma' \in W^{\Gamma}_{\mathrm{L}}$ share the same domain $U^{\Gamma'}_{\mathrm{L}} = U^{\Gamma}_{\mathrm{L}}$*

Proof. W^{Γ}_{L} is connected, i.e., every world of it can be reached in finitely many back or forth step on $\succ^{\Gamma}_{\mathrm{L}}$. So it is enough to show that if $\Gamma_1 R^{\Gamma}_{\mathrm{L}} \Gamma_2$, i.e., $\mathbf{H}^{-}(\Gamma_1) \subseteq \Gamma_2$, then $[\![\tau]\!]_{\Gamma_1} = [\![\tau]\!]_{\Gamma_2}$:

(e) $[\![\tau]\!]_{\Gamma_1} \subseteq [\![\tau]\!]_{\Gamma_2}$: Since NI $\in \Gamma_1$, for all $\tau' \in [\![\tau]\!]_{\Gamma_1}$ it is true that $\mathbf{H}\tau = \tau' \in \Gamma_1$, so $\tau = \tau' \in \mathbf{H}^{-}(\Gamma_1) \subseteq \Gamma_2$, therefore $\tau' \in [\![\tau]\!]_{\Gamma_2}$. $\dashv$

(c) $[\![\tau]\!]_{\Gamma_1} \supseteq [\![\tau]\!]_{\Gamma_2}$: Suppose that there is a $\tau' \in [\![\tau]\!]_{\Gamma_2}$ s.t. $\tau' \notin [\![\tau]\!]_{\Gamma_1}$. The latter means that $\tau' \neq \tau \in \Gamma_1$. Then by NNI $\in \Gamma_1$, $\mathbf{H}\tau' \neq \tau \in \Gamma_1$, therefore $\tau' \neq \tau \in \mathbf{H}^{-}(\Gamma_1) \subseteq \Gamma_2$, therefore $\tau' \in [\![\tau]\!]_{\Gamma_2}$ which contradicts to our assumption. $\dashv$

■

Corollary 12. *For all $\Gamma' \in W^{\Gamma}_{\mathrm{L}}$, if $[\![\mathsf{c}]\!]^{\mathfrak{M}_{\mathrm{L}}}_{\Gamma'} \neq \varnothing$, then $[\![\mathsf{c}]\!]^{\mathfrak{M}_{\mathrm{L}}}_{\Gamma'} \in U^{\Gamma'}_{\mathrm{L}} = U^{\Gamma}_{\mathrm{L}}$.*

Corollary 13. $\Gamma \overline{\succ}_{\mathrm{L}} \Gamma' \Longrightarrow \mathcal{S}^{\Gamma}_{\mathrm{L}} = \mathcal{S}^{\Gamma'}_{\mathrm{L}}$.

Proof. By the lemma above we have $U^{\Gamma'}_{\mathrm{L}} = U^{\Gamma}_{\mathrm{L}}$. $\Gamma \overline{\succ}_{\mathrm{L}} \Gamma'$ implies $W^{\Gamma}_{\mathrm{L}} = W^{\Gamma'}_{\mathrm{L}}$, and every remaining component of the model structure are defined as relativizations/restrictions by $W^{\Gamma'}_{\mathrm{L}}$. ■

Hereby we proved Proposition 10. ■

Canonical pointer models

Definition 14. The canonical pointer model of a canonical world Γ will be

$$\mathfrak{M}^{\Gamma}_{\mathrm{L}} \stackrel{\mathrm{def}}{=} \left(\mathcal{S}^{\Gamma}_{\mathrm{L}}, [\![\;]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}\right)$$

where

- $[\![\mathsf{r}]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} \stackrel{\mathrm{def}}{=} [\![\mathsf{r}]\!]_{\Gamma}$
- $[\![\mathsf{c}]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} \stackrel{\mathrm{def}}{=} [\![\mathsf{c}]\!]^{\mathfrak{C}_{\mathrm{L}}} \restriction W^{\Gamma}_{\mathrm{L}}$
- $[\![\tau]\!]_{\Gamma}[\![+]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}[\![\tau']\!]_{\Gamma} \stackrel{\mathrm{def}}{=} [\![\tau+\tau']\!]_{\Gamma}$
- $[\![\tau]\!]_{\Gamma}[\![\cdot]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}[\![\tau']\!]_{\Gamma} \stackrel{\mathrm{def}}{=} [\![\tau\cdot\tau']\!]_{\Gamma}$
- $[\![\leq]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} \stackrel{\mathrm{def}}{=} \{\langle [\![\tau]\!]_{\Gamma}, [\![\tau']\!]_{\Gamma}\rangle \in (U^{\Gamma}_{\mathrm{L}})^2 \,:\, \tau \leq \tau' \in \Gamma\}$

Corollary 14 (CMM)**.** *Every canonical model of a canonical world is a model.*

Proof. This follows from Proposition 10 and from the fact that the objects defined above are elements of U^{Γ}_{L}, $\mathbb{C}^{\Gamma}_{\mathrm{L}}$, U^{Γ}_{L}, U^{Γ}_{L} and $\wp((U^{\Gamma}_{\mathrm{L}})^2)$, respectively. ■

Definition 15. Every substitution η which maps CMT-s to $MVar$-s and CC-s to $CVar$-s, gives an assignment on the mathematical and pointer variables on $\mathcal{S}^{\Gamma}_{\mathrm{L}}$:

$$\eta^{\Gamma}_{\mathrm{L}}(v) \stackrel{\mathrm{def}}{=} \begin{cases} [\![\eta(v)]\!]_{\Gamma} & \text{if } v \in MVar \\ [\![\eta(v)]\!]^{\mathfrak{C}_{\mathrm{L}}} \restriction W^{\Gamma}_{\mathrm{L}} & \text{if } v \in CVar \\ \{\Gamma' \in W^{\Gamma}_{\mathrm{L}} \,:\, v \in \Gamma\} & \text{if } v \in PrVar \end{cases}$$

Here the Reader should keep in my that our purpose is to prove the Truth Lemma, which now we are able to articulate:

$$\mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi \iff \varphi^{\eta} \in \Gamma$$

where η is an arbitrary substitutions described in the previous definition, and on the left side we have the assignment determined by this. On the right side, φ^{η} stands for the formula which can be gained by executing the substitution η.

Proposition 15. *Connected maximal sets name the same canonical model, i.e.,*

$$\Gamma \succeq_{\mathrm{L}} \Gamma' \Longrightarrow \mathfrak{M}^{\Gamma}_{\mathrm{L}} = \mathfrak{M}^{\Gamma'}_{\mathrm{L}}$$

Proof. Suppose that $\Gamma \succeq_{\mathrm{L}} \Gamma'$. We showed before that $\mathcal{S}^{\Gamma}_{\mathrm{L}} = \mathcal{S}^{\Gamma'}_{\mathrm{L}}$.

- $[\![\mathsf{c}]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} = [\![\mathsf{c}]\!]^{\mathfrak{C}_{\mathrm{L}}} \restriction W^{\Gamma}_{\mathrm{L}} = [\![\mathsf{c}]\!]^{\mathfrak{C}_{\mathrm{L}}} \restriction W^{\Gamma'}_{\mathrm{L}} = [\![\mathsf{c}]\!]^{\mathfrak{M}^{\Gamma'}_{\mathrm{L}}}$.
- For mathematical terms we have $[\![\tau]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} = [\![\tau]\!]_{\Gamma} \overset{\overset{\text{(e)}}{\downarrow}}{=}, (c)[\![\tau]\!]_{\Gamma'} = [\![\tau]\!]^{\mathfrak{M}^{\Gamma'}_{\mathrm{L}}}$.

Finally, to show that $[\![\leq]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} = [\![\leq]\!]^{\mathfrak{M}^{\Gamma'}_{\mathrm{L}}}$, we use the observation that NO and NNO are canonical for this property:

(e$_\leq$) $[\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma}} \subseteq [\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma}}$: By def. $\langle [\![\tau]\!]_\Gamma, [\![\tau']\!]_\Gamma\rangle \in [\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma'}}$ iff $\tau \leq \tau' \in \Gamma$. Since NO $\in \Gamma$, by UG and UI we have that $\mathbf{H}\tau \leq \tau' \in \Gamma$, so $\tau \leq \tau' \in \mathbf{H}^{-}(\Gamma) \subseteq \Gamma'$, so by def. $\langle [\![\tau]\!]_\Gamma, [\![\tau']\!]_\Gamma\rangle \in [\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma'}}$. $\dashv$

(c$_\leq$) $[\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma}} \supseteq [\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma'}}$: Suppose that there is a $\langle [\![\tau]\!]_\Gamma, [\![\tau']\!]_\Gamma\rangle \in [\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma'}}$ s.t. $\langle [\![\tau]\!]_\Gamma, [\![\tau']\!]_\Gamma\rangle \notin [\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma}}$. The latter means that $\neg\tau \leq \tau' \in \Gamma$. Then by NNO $\in \Gamma$, $\mathbf{H}\neg\tau \leq \tau' \in \Gamma$, therefore $\neg\tau' \leq \tau \in \mathbf{H}^{-}(\Gamma) \subseteq \Gamma'$, therefore $\langle [\![\tau]\!]_\Gamma, [\![\tau']\!]_\Gamma\rangle \in [\![\leq]\!]^{\mathfrak{M}_{\mathrm{L}}^{\Gamma}}$ which contradicts to our assumption. $\dashv$

■

And now here is an important corollary that we will need in the proof of the Truth Lemma:

Corollary 16. *If* $\Gamma \succeq_{\mathrm{L}} \Gamma'$,

$$\mathfrak{M}_{\mathrm{L}}^{\Gamma}, \eta_{\mathrm{L}}^{\Gamma}, \Sigma \models \varphi \iff \mathfrak{M}_{\mathrm{L}}^{\Gamma'}, \eta_{\mathrm{L}}^{\Gamma'}, \Sigma \models \varphi$$

2.3.2 Plan

Now we will show that

$$\Gamma \nvdash_{\mathrm{L}^0} \varphi \qquad \Longrightarrow \qquad \Gamma \not\models \varphi$$

The steps can be found in Table 2.1. Some steps are named and these names can be found in parentheses. We encourage the Reader to check Table 2.1 before starting to read a new lemma or theorem.

Remark 17. Note that we will able to strengthen such a completeness result for completeness w.r.t. *connected models*, since every canonical model is point-generated, hence connected. This will be extremely important later, when we will consider theories that can define their global hybrid operators whenever the model is connected.

2.3.3 (CEL): conservative extension lemma

Lemma 18 (CEL). *Let* $\mathcal{L} \stackrel{\text{def}}{=} \mathcal{L}^0 \cup C$ *where* C *is a set of new constants, and let* L *be the smallest set of* $\mathcal{L}$ *formulas that forms a logic including* L^0. *Then the logic* L *and* L^0 *agree on* L^0*-theorems, i.e.,*

$$\vdash_{\mathrm{L}^0} \varphi \quad \iff \quad \vdash_{\mathrm{L}} \varphi \qquad \text{for all } \varphi \in \mathcal{L}^0$$

Proof. See Section 1.8 of [Goldblatt 2011] ■

Proposition 19. *The following rule is admissible in* BCL*:*

$$\text{GC}^*: \quad \frac{\varphi(\mathsf{r}_1/x_1, \ldots, \mathsf{r}_k/x_k, \mathsf{c}_1/a_1, \ldots, \mathsf{c}_n/a_n)}{\varphi} \quad \begin{array}{c}\text{if the constants are distinct}\\ \text{and do not occur in } \varphi\end{array}$$

Proof. See the proof of Lemma 1.2.3. in [Goldblatt 2011]. ■

Table 2.1: Steps of the Completeness Proof

1. Suppose indirectly that $\Gamma \nvdash_{\mathrm{L}^0} \varphi$.
2. Take a logic L which extends L^0 with
   - infinitely many new mathematical constants MC^{new}, and
   - infinitely many new pointer constants CC^{new}.

   So $MC^{\mathrm{new}} \cap MC_{\mathcal{L}^0} = \varnothing$ and $CC^{\mathrm{new}} \cap CC_{\mathcal{L}^0} = \varnothing$.
   $CMT^{\mathrm{new}} \stackrel{\mathrm{def}}{=} \bigcap\{H \subseteq CMT_{\mathcal{L}} : MC^{\mathrm{new}} \subseteq H\}$.
3. Take a set of constants
   - $C_m \subseteq CMT^{\mathrm{new}}$, s.t. $C_m \geq \omega$, $CMT^{\mathrm{new}} - C_m \geq \omega$, and
   - $C_c \subseteq CC^{\mathrm{new}}$, s.t. $C_c \geq \omega$, $CC^{\mathrm{new}} - C_c \geq \omega$.
4. Take an arbitrary $\eta : \begin{array}{l} MVar \rightarrowtail C_m \\ CVar \rightarrowtail C_c \end{array}$.
5. (CEL) $\Gamma^\eta \nvdash_{\mathrm{L}} \varphi^\eta$.
6. $\Gamma^\eta \cup \{\neg\varphi^\eta\}$ is L-consistent.
7. (I) $\Gamma^\eta \cup \{\neg\varphi^\eta\}$ is iTi in L.
8. (L^+) $\Gamma^\eta \cup \{\neg\varphi^\eta\}$ is contained in a canonical world Γ^+
9. (Truth) $\mathfrak{M}^\Gamma_{\mathrm{L}}, \eta^\Gamma_{\mathrm{L}}, \Gamma^+ \models \Gamma$ but $\mathfrak{M}^\Gamma_{\mathrm{L}}, \eta^\Gamma_{\mathrm{L}}, \Gamma^+ \not\models \varphi$
10. (CMM) Since the canonical model is a model indeed, $\Gamma \not\models \varphi$.

Proposition 20. *Let $\eta : MVar \to C_m \cup CVar \to C_c$ be **injective**.*

$$\Gamma \nvdash_{\mathrm{L}^0} \varphi \quad \Longrightarrow \quad \Gamma^\eta \nvdash_{\mathrm{L}} \varphi^\eta$$

Proof.

$$\begin{array}{rll}
 & \Gamma \nvdash_{\mathrm{L}^0} \varphi & \text{assumption} \\
(\forall \gamma_1, \dots, \gamma_n \in \Gamma) & \nvdash_{\mathrm{L}^0} \gamma_1 \to \cdots \gamma_n \to \varphi & \text{synt. compactness} \\
(\forall \gamma_1, \dots, \gamma_n \in \Gamma) & \nvdash_{\mathrm{L}} \gamma_1 \to \cdots \gamma_n \to \varphi & \text{CEL} \\
(\forall \gamma_1, \dots, \gamma_n \in \Gamma) & \nvdash_{\mathrm{L}} (\gamma_1 \to \cdots \gamma_n \to \varphi)^\eta & \text{GC}^* \\
(\forall \gamma_1, \dots, \gamma_n \in \Gamma) & \nvdash_{\mathrm{L}} \gamma_1^\eta \to \cdots \gamma_n^\eta \to \varphi^\eta & \\
 & \Gamma^\eta \nvdash_{\mathrm{L}} \varphi^\eta & \text{q.e.d.}
\end{array}$$

■

2.3.4 (I): Making it iTi

Proposition 21 (The Rule of Generalization on Constants)**.** *The following rule is admissible:*

$$(\mathrm{GC}) \qquad \frac{\varphi(\mathsf{c}/a)}{\varphi} \qquad \frac{\varphi(\mathsf{r}/x)}{\varphi}$$

Proof. See [Goldblatt 2011] p.10. and p.35. ■

Proposition 22 ($\mathrm{I_m}$)**.** *Σ is inductive in L if infinitely many mathematical constants do not occur in it, i.e., there is a set of mathematical constants C'_m s.t. $C'_m \geq \omega$, and $C'_m \cap \mathrm{Const}(\Sigma) = \varnothing$.*

Proof.

$$\begin{array}{ll}
\Sigma \vdash_{\mathrm{L}} \varphi(c/x) & \text{for all } c \in CMT\text{; assumption} \\
\quad \vdash_{\mathrm{L}} \underbrace{\sigma_1 \to \cdots \to \sigma_j \to \varphi(c/x)}_{\mathrm{Prf}} & \text{assumption} \\
\qquad \text{let } \{\mathsf{r}_1, \dots, \mathsf{r}_n\} = C'_m \cap \mathrm{Const}(\mathrm{Prf}) - \{c\}, & C'_m \geq \omega \\
\qquad \text{let } y_1, \dots, y_n, y \notin \mathrm{MVar}(\mathrm{Prf}), & \\
\qquad \text{let } \mathbf{s} := (y/c)(y_n/c_n) \cdots (y_1/c_1). & \\
\quad \vdash_{\mathrm{L}} (\sigma_1 \to \cdots \to \sigma_j \to \varphi(c/x))\mathbf{s} & \text{GC} \\
\quad \vdash_{\mathrm{L}} (\sigma_1)\mathbf{s} \to \cdots \to (\sigma_j)\mathbf{s} \to (\varphi(c/x))\mathbf{s} & \\
\quad \vdash_{\mathrm{L}} \sigma_1 \to \cdots \to \sigma_j \to (\varphi(c/x))\mathbf{s} & C'_m \cap \mathrm{Const}(\Sigma) = \varnothing \\
\quad \vdash_{\mathrm{L}} \forall y_1 \underbrace{\cdots \forall y_n \forall y[\sigma_1 \to \cdots \to \sigma_j \to (\varphi(c/x))\mathbf{s}]}_{F} & \text{UG} \\
\quad \vdash_{\mathrm{L}} \forall y_1 F \to F(c_1/y_1) & \text{UI} \\
\quad \vdash_{\mathrm{L}} \forall y_2 \cdots \forall y_n \forall y[\sigma_1 \to \cdots \to \sigma_j \to (\varphi(c/x))\mathbf{s}](c_1/y_1) & \text{MP} \\
\quad \vdots & \text{eliminate the remained} \\
\quad \vdash_{\mathrm{L}} \forall y[\sigma_1 \to \cdots \to \sigma_j \to (\varphi(c/x))\mathbf{s}](c_1/y_1) \cdots (c_n/y_n) & \text{indexed } \forall y_i\text{-s in the same way} \\
\quad \vdash_{\mathrm{L}} \forall y[\sigma_1 \to \cdots \to \sigma_j \to (\varphi(c/x))\mathbf{s}(c_1/y_1) \cdots (c_n/y_n)] & y_1, \dots, y_n \notin \mathrm{Var}(\Sigma) \\
\quad \vdash_{\mathrm{L}} \forall y[\sigma_1 \to \cdots \to \sigma_j \to \varphi(c/x)(y/c)] & y_1, \dots, y_n \notin \mathrm{Var}(\varphi) \\
\quad \vdash_{\mathrm{L}} \forall y[\sigma_1 \to \cdots \to \sigma_j \to \varphi(y/x)] & c \notin \mathrm{Const}(\varphi) \\
\quad \vdash_{\mathrm{L}} \sigma_1 \to \cdots \to \sigma_j \to \forall y \varphi(y/x) & y \notin \mathrm{Var}(\Sigma) \\
\quad \vdash_{\mathrm{L}} \sigma_1 \to \cdots \to \sigma_j \to \forall x \varphi & \\
\Sigma \vdash_{\mathrm{L}} \forall x \varphi & \text{q.e.d.}
\end{array}$$

■

Proposition 23 (Term Instantiation Rule)**.** *The following rule is admissible:*

$$\text{(TI)} \qquad \frac{\varphi}{\varphi(\pi/a)}$$

Proof. See [Goldblatt 2011] p.10. and p.34. ∎

Proposition 24 ($\mathrm{I_c}$)**.** *Σ is T-inductive in L if infinitely many pointer constants do not occur in it, i.e., there is a set of pointer constants C'_c s.t. $C'_c \geq \omega$, $C'_c \cap \mathrm{Const}(\Sigma) = \varnothing$*

Proof.

$(\forall \mathsf{c} \in CC)$	$\Sigma \vdash_{\mathrm{L}} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a))$	assumption
$(\forall \mathsf{c} \in CC)$	$\vdash_{\mathrm{L}} \sigma_1 \to \cdots \to \sigma_n \to \left[\vec{\varphi}; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a))$	synt.comp.
$(\forall \mathsf{c} \in CC)$	$\vdash_{\mathrm{L}} \left[\langle \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi_1, \vec{\varphi}_{2-n}\rangle; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a))$	template lemma
	$\vdash_{\mathrm{L}} \left[\langle \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi_1, \vec{\varphi}_{2-n}\rangle; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c}_i \to \psi(\mathsf{c}_i/a))$	where $\mathsf{c}_i \in C'_c - \mathrm{Const}(\vec{\varphi} \cup \{\psi\}$ (there is such a c_i since $C'_c \geq \omega$
	$\vdash_{\mathrm{L}} \left(\left[\langle \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi_1, \vec{\varphi}_{2-n}\rangle; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c}_i \to \psi(\mathsf{c}_i/a))\right)(\mathsf{c}_j/a)$	(TI) with (c_j/a) where $\mathsf{c}_j \in C'_c - (\mathrm{Const}(\vec{\varphi} \cup \{\psi\}) \cup \{\mathsf{c}$
	$\vdash_{\mathrm{L}} \left[\langle \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi_1, \vec{\varphi}_{2-n}\rangle(\mathsf{c}_j/a); \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c}_i \to \psi(\mathsf{c}_i/a))$	a does not occur in $(\mathcal{E}\mathsf{c}_i \to \psi(\mathsf{c}_i/a))$
	$\vdash_{\mathrm{L}} \left[\langle \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi_1, \vec{\varphi}_{2-n}\rangle(\mathsf{c}_j/a); \vec{\mathbf{L}}\right] (\mathcal{E}a \to \psi)$	GC: $\mathsf{c}_i \mapsto a$
	$\vdash_{\mathrm{L}} \left[\langle \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi_1, \vec{\varphi}_{2-n}\rangle(\mathsf{c}_j/a); \vec{\mathbf{L}}\right] \forall a\psi$	T$\forall$-Intro (a is not free in the template)
	$\vdash_{\mathrm{L}} \left(\left[\langle \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi_1, \vec{\varphi}_{2-n}\rangle; \vec{\mathbf{L}}\right] \forall a\psi\right)(\mathsf{c}_j/a)$	since a is not free in $\forall a\psi$
	$\vdash_{\mathrm{L}} \left[\langle \sigma_1 \wedge \cdots \wedge \sigma_n \wedge \varphi_1, \vec{\varphi}_{2-n}\rangle; \vec{\mathbf{L}}\right] \forall a\psi$	GC: $\mathsf{c}_j \mapsto a$
	$\vdash_{\mathrm{L}} \sigma_1 \to \cdots \to \sigma_n \to \left[\vec{\varphi}; \vec{\mathbf{L}}\right] \forall a\psi$	template lemma
	$\Sigma \vdash_{\mathrm{L}} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] \forall a\psi$	q.e.d.

∎

The use of (I)

Proposition 25. *$\Gamma^\eta \cup \{\neg\varphi^\eta\}$ is iTi in L.*

Proof. Since $\eta^m : MVar \rightarrowtail C_m$, we have that

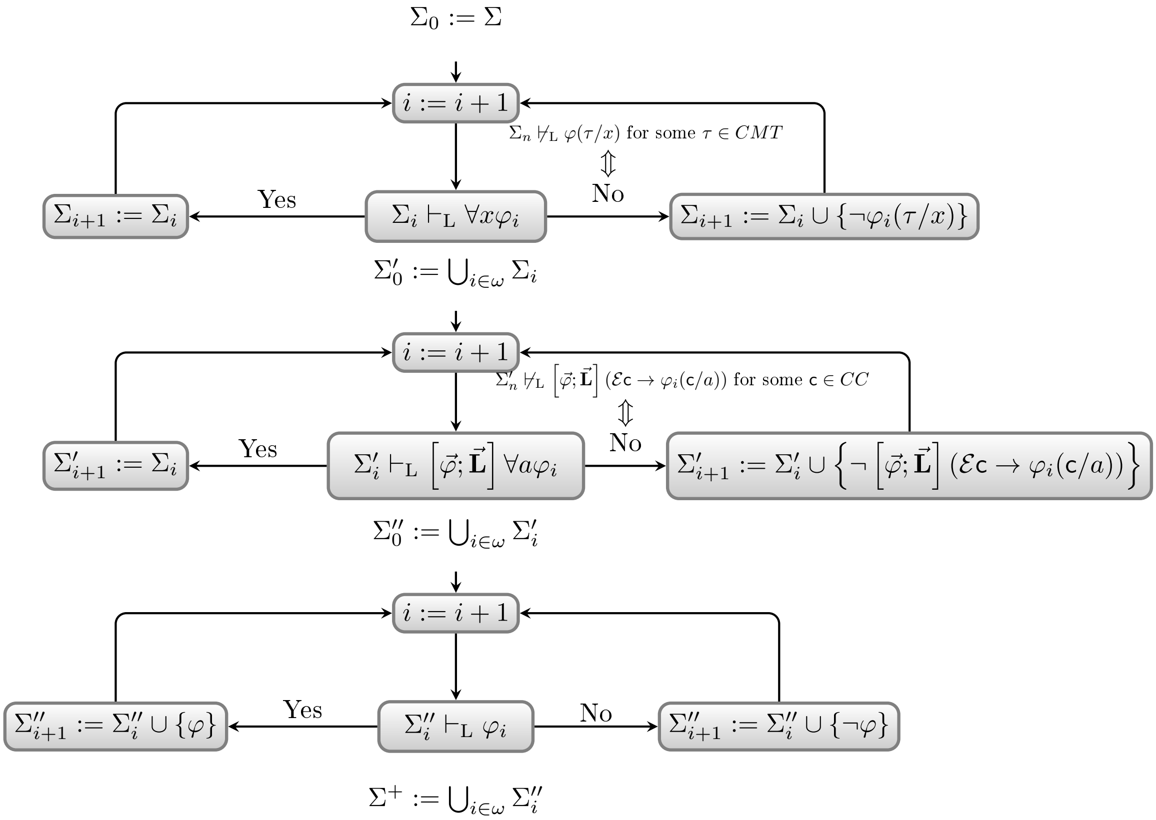
$$\mathrm{MConst}(\Gamma^\eta \cup \{\neg\varphi^\eta\}) \subseteq MC_{\mathcal{L}_0} \cup C_m$$

Let $C'_m := CMT^{\mathrm{new}} - C_m$.
Since $C'_m \geq \omega$ and $C'_m \cap \mathrm{MConst}(\Gamma^\eta \cup \{\neg\varphi^\eta\}) = \varnothing$ by the def. of C_m and η, by ($\mathrm{I_m}$), $\Gamma^\eta \cup \{\neg\varphi^\eta\}$ is inductive in L.

Since $\eta^c : CVar \rightarrowtail C_c$, we have that

$$\mathrm{CConst}(\Gamma^\eta \cup \{\neg\varphi^\eta\}) \subseteq CC_{\mathcal{L}_0} \cup C_c$$

Figure 2.2: Enrichment processes of the modified Lindenbaum lemma.



Let $C'_c := CC^{\mathrm{new}} - C_c$.
Since $C'_c \geq \omega$ and $C'_c \cap \mathrm{CConst}(\Gamma^\eta \cup \{\neg\varphi^\eta\}) = \varnothing$ by the def. of C_c and η, by ($\mathrm{I_c}$), $\Gamma^\eta \cup \{\neg\varphi^\eta\}$ is T-inductive in L. ∎

2.3.5 (L^+): The iTi-modified Lindbaum-lemma

Lemma 26. *Every iTi* L*-consistent set* Σ *is extendable to an* L*-maximal iTi* Σ^+ *one.*

Proof. We will program a process that enrich formula-by-formula our iTi L-consistent set Σ into an L-maximal iTi Σ^+ one. The standard Lindenbaum process is not enough since we have to be careful with the iTi property. So we first focus on enriching our Σ in a way that it will remain iTi no matter which consistent extension we choose later in the standard Lindenbaum process. For the precise definition of the processes see Figure 2.2. The processes will result in an iTi set by the following argumentations:

Suppose that $\Sigma^+ \nvdash_{\mathrm{L}} \forall x\varphi$. Then $(\forall\tau \in CMT)\Sigma^+ \vdash_{\mathrm{L}} \varphi(\tau/x)$. Then the subset $\Sigma_i \nvdash_{\mathrm{L}} \forall x\varphi$ either, where i is the step where $\forall\varphi_i = \forall x\varphi$. So by the definition of the process,

$$\begin{aligned}\Sigma_{i+1} &= \Sigma_i \cup \{\neg\varphi(\tau/x)\} \\ &\subseteq \Sigma^+ \vdash_{\mathrm{L}} \neg\varphi(\tau/x).\end{aligned}$$

Suppose that $\Sigma^+ \nvdash_L \left[\vec{\varphi};\vec{\mathbf{L}}\right]\forall a\varphi$. But for all $\mathsf{c} \in CC$ we have $\Sigma^+ \vdash_L \left[\vec{\varphi};\vec{\mathbf{L}}\right](\mathcal{E}\mathsf{c} \to \varphi(\mathsf{c}/a))$. Then the subset $\Sigma_i \nvdash_L \left[\vec{\varphi};\vec{\mathbf{L}}\right]\forall a\varphi$ either, so

$$\begin{aligned}\Sigma_{i+1} &= \Sigma_i \cup \left\{\neg\left[\vec{\varphi};\vec{\mathbf{L}}\right](\mathcal{E}\mathsf{c} \to \varphi(\mathsf{c}/a))\right\}\\ &\subseteq \Sigma^+ \vdash_L \neg\left[\vec{\varphi};\vec{\mathbf{L}}\right](\mathcal{E}\mathsf{c} \to \varphi(\mathsf{c}/a))\end{aligned}$$

■

Remark 27. So now we have that our original $\Gamma^\eta \cup \{\neg\varphi^\eta\}$ is contained in a canonical world Γ^+

2.3.6 Plan for the Truth Lemma

1. MALT: Membership acts like truth
   - Existence Lemma (membership of $\mathbf{P}\varphi$ and $\mathbf{F}\varphi$)
     - ($\mathbf{L}^-$)
     - (FE)
2. SALA: Closed term substitutions act like assignments
3. Proof by induction on the complexity of formulas

2.3.7 (MALT): Membership acts like truth

If Γ is L-maximal, then we have the following statements immediately:

Proposition 28 (MALT)**.**

$$\begin{array}{r|ccl} \text{negation complete} & \neg\varphi \in \Gamma & \iff & \varphi \notin \Gamma \\ \text{closed under } \vdash_L & \varphi \wedge \psi \in \Gamma & \iff & \varphi \in \Gamma \text{ and } \psi \in \Gamma \\ \succ_L & \mathbf{P}\varphi \in \Gamma & \Leftarrow & (\exists \Gamma' \prec_L \Gamma)\ \varphi \in \Gamma' \\ \prec_L & \mathbf{F}\varphi \in \Gamma & \Leftarrow & (\exists \Gamma' \succ_L \Gamma)\ \varphi \in \Gamma' \\ \text{inductive, UI} & \forall x\varphi \in \Gamma & \iff & (\forall \tau \in CMT)\ \varphi(\tau/x) \in \Gamma \\ \text{T-inductive, EI} & \forall a\varphi \in \Gamma & \iff & (\forall \mathsf{c} \in CC)\ \mathcal{E}\mathsf{c} \to \varphi(\mathsf{c}/a) \in \Gamma \end{array}$$

Proof. The last case is the special case of T-inductivity, when the template is just $\varnothing$. For the $\mathbf{P} \Leftarrow$ direction suppose indirectly that $(\exists\Gamma' \supseteq \mathbf{H}^-(\Gamma))\ \varphi \in \Gamma'$. But then $\mathbf{P}\varphi \notin \Gamma \Leftrightarrow \neg\mathbf{P}\varphi \in \Gamma \Leftrightarrow \mathbf{H}\neg\varphi \in \Gamma \Leftrightarrow \neg\varphi \in \mathbf{H}^-(\Gamma) \Leftrightarrow \neg\varphi \in \Gamma' \Leftrightarrow \bot \in \Gamma'$. The proof for $\mathbf{F} \Leftarrow$ is similar. All the other cases are standard. ■

To the other direction, for the so-called Existence lemmas, we will need the lemmas ($\mathbf{L}^-$), (FE) and (L^+).

2.3.8 Existence Lemma

The plan for the Existence lemma

($\mathbf{L}^-$) $\mathbf{H}^-(\Gamma)$ inherits iTi from Γ.

(FE) $\mathbf{H}^-(\Gamma) \cup \{\varphi\}$ inherits iTi from $\mathbf{H}^-(\Gamma)$.

- $\mathbf{H}^{-}(\Gamma) \cup \{\varphi\}$ is consistent if $\mathbf{P}\varphi \in \Gamma$.

(L^{+}) There is a canonical world Γ' extending $\mathbf{H}^{-}(\Gamma) \cup \{\varphi\}$.

Here the $\bullet$ step is standard:

Proposition 29. *$\mathbf{H}^{-}(\Gamma) \cup \{\varphi\}$ is consistent if $\mathbf{P}\varphi \in \Gamma$.*

Proof. Suppose indirectly that

$$\begin{array}{rl}
\mathbf{H}^{-}(\Gamma) \cup \{\varphi\} & \vdash_{\mathrm{L}} \bot \\
\mathbf{H}^{-}(\Gamma) & \vdash_{\mathrm{L}} \neg\varphi \\
\text{synt. compactness} & \vdash_{\mathrm{L}} \chi_1 \to \cdots \to \chi_n \to \neg\varphi \\
\text{RN + K-s} & \vdash_{\mathrm{L}} \mathbf{H}\chi_1 \to \cdots \to \mathbf{H}\chi_n \to \mathbf{H}\neg\varphi \\
\mathbf{H}\chi_1, \cdots, \mathbf{H}\chi_n & \vdash_{\mathrm{L}} \mathbf{H}\neg\varphi \\
\Gamma & \vdash_{\mathrm{L}} \mathbf{H}\neg\varphi \\
\Gamma & \vdash_{\mathrm{L}} \neg\Diamond\varphi \\
\Gamma \cup \{\mathbf{P}\varphi\} & \vdash_{\mathrm{L}} \bot \\
\Gamma & \vdash_{\mathrm{L}} \bot
\end{array}$$

■

Lemma 30 ($\mathbf{H}^{-}$)**.** *If Γ is inductive in* L*, then so is $\mathbf{H}^{-}(\Gamma)$. Similarly, if Γ is T-inductive in* L*, then so is $\mathbf{H}^{-}(\Gamma)$:*

Proof. Inductivity:

$$\begin{array}{lrl}
(\forall \tau \in CMT)\, \mathbf{H}^{-}(\Gamma) \vdash_{\mathrm{L}} \varphi(\tau/x) & & \text{assumption} \\
(\forall \tau \in CMT) \quad \Gamma \vdash_{\mathrm{L}} \mathbf{H}\varphi(c/x) & & \text{def. of. } \mathbf{H}^{-} \\
\Gamma \vdash_{\mathrm{L}} \forall x \mathbf{H}\varphi & & \Gamma \text{ is inductive in L} \\
\Gamma \vdash_{\mathrm{L}} \mathbf{H}\forall x\varphi & & \text{BF} \\
\mathbf{H}^{-}(\Gamma) \vdash_{\mathrm{L}} \forall x\varphi & & \text{def.of } \mathbf{H}^{-}
\end{array}$$

T-inductivity:

$$\begin{array}{lll}
(\forall \mathsf{c} \in CC)\, \mathbf{H}^{-}(\Gamma) \vdash_{\mathrm{L}} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a)) & & \text{assumption} \\
(\forall \mathsf{c} \in CC) \quad \Gamma \vdash_{\mathrm{L}} \mathbf{H} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a)) & & \text{def. of. } \mathbf{H}^{-} \\
(\forall \mathsf{c} \in CC) \quad \Gamma \vdash_{\mathrm{L}} \left[\langle \top, \vec{\varphi} \rangle; \langle \mathbf{H}, \vec{\mathbf{L}} \rangle\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a)) & & \text{template lemma} \\
\Gamma \vdash_{\mathrm{L}} \left[\langle \top, \vec{\varphi} \rangle; \langle \mathbf{H}, \vec{\mathbf{L}} \rangle\right] \forall a\psi & & \Gamma \text{ is T-inductive in L} \\
\Gamma \vdash_{\mathrm{L}} \mathbf{H} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] \forall a\psi & & \text{template lemma} \\
\mathbf{H}^{-}(\Gamma) \vdash_{\mathrm{L}} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] \forall a\psi & & \text{def. of. } \mathbf{H}^{-}
\end{array}$$

■

Lemma 31 ((FE): Finite extension lemma)**.** *If Σ is inductive then $\Sigma \cup \{\varphi\}$ is inductive as well. Similarly, if Σ is T-inductive then $\Sigma \cup \{\varphi\}$ is T-inductive as well.*

Proof. Inductivity:

$$\begin{array}{lll}
(\forall \tau \in CMT)\, \Sigma \cup \{\varphi\} \vdash_{\mathrm{L}} \psi(\tau/x) & & \text{assumption} \\
(\forall \tau \in CMT) & \Sigma \vdash_{\mathrm{L}} \varphi \to \psi(\tau/x) & \\
(\forall \tau \in CMT) & \Sigma \vdash_{\mathrm{L}} [\varphi \to \psi(y/x)](\tau/y) & \text{where } y \text{ does not occur in } \varphi \text{ and } \psi \\
 & \Sigma \vdash_{\mathrm{L}} \forall y[\varphi \to \psi(y/x)] & \text{inductivity of } \Sigma \\
 & \Sigma \vdash_{\mathrm{L}} \forall y \varphi \to \forall y \psi(y/x) & \text{UD} \\
 & \Sigma \vdash_{\mathrm{L}} \varphi \to \forall y \varphi & \text{VQ} \\
 & \Sigma \vdash_{\mathrm{L}} \varphi \to \forall y \psi(y/x) & \text{chain rule} \\
 & \Sigma \vdash_{\mathrm{L}} \varphi \to \forall x \psi & \\
 \Sigma \cup \{\varphi\} \vdash_{\mathrm{L}} \forall x \psi & & \text{q.e.d.}
\end{array}$$

T-inductivity:

$$\begin{array}{lll}
(\forall \mathsf{c} \in CC)\, \Sigma \cup \{\varphi\} \vdash_{\mathrm{L}} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a)) & & \text{assumption} \\
(\forall \mathsf{c} \in CC) & \Sigma \vdash_{\mathrm{L}} \varphi \to \left[\vec{\varphi}; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a)) & \text{ded.thm.} \\
(\forall \mathsf{c} \in CC) & \Sigma \vdash_{\mathrm{L}} \left[\langle \varphi \wedge \varphi_1, \vec{\varphi}_{2-n} \rangle; \vec{\mathbf{L}}\right] (\mathcal{E}\mathsf{c} \to \psi(\mathsf{c}/a)) & \text{template lemma} \\
 & \Sigma \vdash_{\mathrm{L}} \left[\langle \varphi \wedge \varphi_1, \vec{\varphi}_{2-n} \rangle; \vec{\mathbf{L}}\right] \forall a \psi & \Sigma \text{ is T-inductive} \\
 & \Sigma \vdash_{\mathrm{L}} \varphi \to \left[\vec{\varphi}; \vec{\mathbf{L}}\right] \forall a \psi & \text{template lemma} \\
 \Sigma \cup \{\varphi\} \vdash_{\mathrm{L}} \left[\vec{\varphi}; \vec{\mathbf{L}}\right] \forall a \psi & & \text{q.e.d.}
\end{array}$$

■

2.3.9 (SALA): Closed term substitutions act like assignments

Notation 16. For any $\eta : MVar \to CMT \cup CVar \to CC$, if v be a variable from $MVar$ or $CVar$ and t is a mathematical or pointer term, respectively, then

$$\begin{array}{rcl}
\varphi^{\eta} & \stackrel{\text{def}}{=} & \varphi\,(\,\eta(v_1)/v_1, \ldots,\ \eta(v_n)/v_n) \\
\varphi^{\eta\backslash v_i} & \stackrel{\text{def}}{=} & \varphi\,(\,\eta(v_1)/v_1, \ldots, \eta(v_{i-i})/v_{i-i}, v_i/v_i, \eta(v_{i+i})/v_{i+i}, \ldots,\ \eta(v_n)/v_n) \\
\varphi^{\eta[t/v_i]} & \stackrel{\text{def}}{=} & \varphi\,(\,\eta(v_1)/v_1, \ldots, v_{i-i}/v_{i-i}, t/v_i, v_{i+i}/v_{i+i}, \ldots,\ \eta(v_n)/v_n)
\end{array}$$

Remark 32. Note that

- φ^{η} has no free variable,
- $\varphi^{\eta\backslash v_i}$ has at most v_i free,

Proposition 33.

1. *A substitution η induces an 'almost-homomorphism' on the terms and formulas:*

$$\begin{array}{rcl}
\eta(\tau + \tau') & \stackrel{\text{def}}{=} & \eta(\tau) + \eta(\tau') \\
\eta(\tau \cdot \tau') & \stackrel{\text{def}}{=} & \eta(\tau) \cdot \eta(\tau') \\
p^{\eta} & = & p \\
(P(t_1, \ldots, t_n))^{\eta} & = & P(\eta(t_1), \ldots, \eta(t_n)) \\
(\varphi \wedge \psi)^{\eta} & = & \varphi^{\eta} \wedge \psi^{\eta} \\
(\neg\varphi)^{\eta} & = & \neg\varphi^{\eta} \\
(\mathbf{H}\varphi)^{\eta} & = & \mathbf{H}\varphi^{\eta} \\
(\mathbf{G}\varphi)^{\eta} & = & \mathbf{G}\varphi^{\eta} \\
(\forall v \varphi)^{\eta} & = & \forall v \varphi^{\eta\backslash v}
\end{array}$$

2. *Closed term substitutions yield f-variations:*

$$(\varphi(t/v))^{\eta} \;=\; (\varphi(t/v))^{\eta\setminus v} \;\overset{\text{if } t \text{ is closed}}{\overset{\downarrow}{=}}\; \varphi^{\eta\setminus v}(t/v) = \varphi^{\eta[t/v]}$$

3. *If η is a closed term substitution,*

$$[\![\tau]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta^{\Gamma}_{\mathrm{L}}} = [\![\tau^{\eta}]\!]_{\Gamma} \qquad\qquad \eta^{\Gamma}_{\mathrm{L}}(a)(\Sigma) = [\![a^{\eta}]\!]_{\Sigma}$$

Remark 34. Remember to Def. 15:

$$\eta^{\Gamma}_{\mathrm{L}}(x) \overset{\text{def}}{=} [\![\eta(x)]\!]_{\Gamma} \qquad \eta^{\Gamma}_{\mathrm{L}}(a)(\Sigma) \overset{\text{def}}{=} [\![\eta(a)]\!]^{\mathfrak{C}_{\mathrm{L}}} \restriction W^{\Gamma}_{\mathrm{L}}(\Sigma) = [\![\eta(a)]\!]_{\Sigma}$$

Proof. We prove only the 3rd:

- $[\![\mathsf{r}]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta^{\Gamma}_{\mathrm{L}}} = [\![\mathsf{r}]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} = [\![\mathsf{r}]\!]_{\Gamma} = [\![\mathsf{r}^{\eta}]\!]_{\Gamma}$
- $[\![x]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta^{\Gamma}_{\mathrm{L}}} = \eta^{\Gamma}_{\mathrm{L}}(x) = [\![\eta(x)]\!]_{\Gamma} = [\![x^{\eta}]\!]_{\Gamma}$
- $[\![\tau\cdot\tau']\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta^{\Gamma}_{\mathrm{L}}} = [\![\tau]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta^{\Gamma}_{\mathrm{L}}}[\![\cdot]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}[\![\tau']\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta^{\Gamma}_{\mathrm{L}}} = [\![\tau^{\eta}]\!]_{\Gamma}[\![\cdot]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}[\![\tau'^{\eta}]\!]_{\Gamma} = [\![\tau^{\eta}\cdot\tau'^{\eta}]\!]_{\Gamma} = [\![(\tau\cdot\tau')^{\eta}]\!]_{\Gamma}$
- $[\![\tau+\tau']\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta^{\Gamma}_{\mathrm{L}}}$ is similar
- $\eta^{\Gamma}_{\mathrm{L}}(a)(\Sigma) = [\![\eta(a)]\!]_{\Sigma} = [\![a^{\eta}]\!]_{\Sigma}$

■

Proposition 35 (SALA). *If η is a closed term substitution,*

$$\eta^{\Gamma}_{\mathrm{L}}[x\mapsto[\![\tau]\!]_{\Gamma}] = (\eta[\tau/x])^{\Gamma}_{\mathrm{L}} \qquad\qquad \eta^{\Gamma}_{\mathrm{L}}[a\mapsto[\![\mathsf{c}]\!]] = (\eta[\mathsf{c}/a])^{\Gamma}_{\mathrm{L}}$$

Proof. We can distinguish two cases: when this function is applied to x and when it is not. Let y a variable that is different from x.

$$\begin{aligned}\eta^{\Gamma}_{\mathrm{L}}[x\mapsto[\![\tau]\!]_{\Gamma}](x) &= [\![\tau]\!]_{\Gamma}\\ &= [\![\eta[\tau/x](x)]\!]_{\Gamma}\\ &= (\eta[\tau/x])^{\Gamma}_{\mathrm{L}}(x)\end{aligned} \qquad \begin{aligned}\eta^{\Gamma}_{\mathrm{L}}[x\mapsto[\![\tau]\!]_{\Gamma}](y) &= \eta^{\Gamma}_{\mathrm{L}}(y)\\ &= [\![\eta(y)]\!]_{\Gamma}\\ &= [\![\eta[\tau/x](y)]\!]_{\Gamma}\\ &= (\eta[\tau/x])^{\Gamma}_{\mathrm{L}}(y)\end{aligned}$$

Similarly, let b a variable that is different from a.

$$\begin{aligned}\eta^{\Gamma}_{\mathrm{L}}[a\mapsto[\![\mathsf{c}]\!]](a)(\Sigma) &= [\![\mathsf{c}]\!]_{\Sigma}\\ &= [\![\eta[\mathsf{c}/a](a)]\!]_{\Sigma}\\ &= (\eta[\mathsf{c}/a])^{\Gamma}_{\mathrm{L}}(a)(\Sigma)\end{aligned} \qquad \begin{aligned}\eta^{\Gamma}_{\mathrm{L}}[a\mapsto[\![\mathsf{c}]\!]_{\Gamma}](b)(\Sigma) &= \eta^{\Gamma}_{\mathrm{L}}(b)(\Sigma)\\ &= [\![\eta(b)]\!]_{\Sigma}\\ &= [\![\eta[\mathsf{c}/a](b)]\!]_{\Sigma}\\ &= (\eta[\mathsf{c}/a])^{\Gamma}_{\mathrm{L}}(b)(\Sigma)\end{aligned}$$

■

2.3.10 Truth Lemma

Lemma 36 (Truth Lemma)**.**

$$\mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi \iff \varphi^{\eta} \in \Gamma$$

Proof.

- $\varphi \equiv \tau \leq \tau'$: inequalities

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \tau \leq \tau' & \iff & \left\langle [\![\tau]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta}, [\![\tau']\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta} \right\rangle \in [\![\leq]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} & \text{def.of } \models \\ & \iff & \langle [\![\tau^{\eta}]\!]_{\Gamma}, [\![\tau'^{\eta}]\!]_{\Gamma} \rangle \in [\![\leq]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} & \text{(SALA)} \\ & \iff & \tau^{\eta} \leq \tau'^{\eta} \in \Gamma & \text{def.of } [\![\leq]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}} \\ & \iff & (\tau \leq \tau')^{\eta} \in \Gamma & \end{array}$$

- $\varphi \equiv \tau = \tau'$: mathematical equalities

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \tau = \tau' & \iff & [\![\tau]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta} = [\![\tau']\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta} & \text{def.of } \models \\ & \iff & [\![\tau^{\eta}]\!]_{\Gamma} = [\![\tau'^{\eta}]\!]_{\Gamma} & \text{(SALA)} \\ & \iff & \tau^{\eta} = \tau'^{\eta} \in \Gamma & \text{def.of } [\![\tau]\!]_{\Gamma} \\ & \iff & (\tau = \tau')^{\eta} \in \Gamma & \end{array}$$

- $\varphi \equiv \mathsf{c}{:}\tau$: pointer constant pointing

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \mathsf{c}{:}\tau & \iff & [\![\mathsf{c}]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}(\Gamma) = [\![\tau]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta} & \text{def.of } \models \\ & \iff & [\![\mathsf{c}]\!]_{\Gamma} = [\![\tau^{\eta}]\!]_{\Gamma} & \text{def.of } [\![\mathsf{c}]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}(\Gamma), \text{ (SALA)} \\ & \iff & (\exists \tau' : \mathsf{c})\tau' = \tau^{\eta} \in \Gamma & \text{def.of } [\![\mathsf{c}]\!]_{\Gamma} \text{ and } [\![\tau]\!]_{\Gamma} \\ & \iff & \mathsf{c}{:}\tau^{\eta} \in \Gamma & \text{SI} \\ & \iff & (\mathsf{c}{:}\tau)^{\eta} \in \Gamma & \end{array}$$

- $\varphi \equiv a{:}\tau$: pointing statements

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models a{:}\tau & \iff & \eta(a)(\Gamma) = [\![\tau]\!]^{\mathfrak{M}^{\Gamma}_{\mathrm{L}}}_{\eta} & \text{def.of } \models \\ & \iff & [\![a^{\eta}]\!]_{\Gamma} = [\![\tau^{\eta}]\!]_{\Gamma} & \text{(SALA)} \\ & \iff & (\exists \tau' : a^{\eta})\tau' = \tau^{\eta} \in \Gamma & \text{def.of } [\![a^{\eta}]\!]_{\Gamma} \text{ and } [\![\tau]\!]_{\Gamma} \\ & \iff & a^{\eta}{:}\tau^{\eta} \in \Gamma & \text{SI} \\ & \iff & (a{:}\tau)^{\eta} \in \Gamma & \end{array}$$

- $\varphi \equiv p$: propositional variables

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models p & \iff & \Gamma \in \eta^{\Gamma}_{\mathrm{L}}(p) & \text{def.of } \models \\ & \iff & p \in \Gamma & \text{def.of } \eta^{\Gamma}_{\mathrm{L}} \\ & \iff & p^{\eta} \in \Gamma & \end{array}$$

- $\varphi \equiv \neg\varphi$: negation

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \neg\varphi & \iff & \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \not\models \varphi & \text{def.of} \models \\ & \iff & \varphi^{\eta} \notin \Gamma & \text{ind.hip.} \\ & \iff & \neg\varphi^{\eta} \in \Gamma & \text{neg. completeness} \\ & \iff & (\neg\varphi)^{\eta} \in \Gamma & \end{array}$$

- $\varphi \equiv \varphi \wedge \psi$: conjunction

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi \wedge \psi & \iff & \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi \text{ and} & \\ & & \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \psi & \text{def.of} \models \\ & \iff & \varphi^{\eta} \in \Gamma \text{ and } \psi^{\eta} \in \Gamma & \text{ind.hip.} \\ & \iff & \varphi^{\eta} \wedge \psi^{\eta} \in \Gamma & \text{deductively closed} \\ & \iff & (\varphi \wedge \psi)^{\eta} \in \Gamma & \end{array}$$

- $\varphi \equiv \mathbf{P}\varphi$: modality

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \mathbf{P}\varphi & \iff & (\exists \Gamma' \supseteq \mathbf{H}^{-}(\Gamma))\ \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma' \models \varphi & \text{def.of} \models, \succ^{\Gamma}_{\mathrm{L}} \\ & \iff & (\exists \Gamma' \supseteq \mathbf{H}^{-}(\Gamma))\ \mathfrak{M}^{\Gamma'}_{\mathrm{L}}, \eta^{\Gamma'}_{\mathrm{L}}, \Gamma' \models \varphi & \mathfrak{M}^{\Gamma}_{\mathrm{L}} = \mathfrak{M}^{\Gamma'}_{\mathrm{L}} \\ & \iff & (\exists \Gamma' \supseteq \mathbf{H}^{-}(\Gamma))\ \varphi^{\eta} \in \Gamma' & \text{ind.hip.} \\ & \iff & \Diamond\varphi^{\eta} \in \Gamma' & \text{Existence Lemma, (MALT)} \\ & \iff & (\Diamond\varphi)^{\eta} \in \Gamma' & \end{array}$$

- $\varphi \equiv \mathbf{F}\varphi$ is similar
- $\varphi \equiv \forall x\varphi$: mathematical quantification

$$\begin{array}{rcll} \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \forall x\varphi & \iff & (\forall [\![\tau]\!]_{L} \in U^{\Gamma}_{\mathrm{L}})\ \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi & \text{def.of} \models, U^{\Gamma}_{\mathrm{L}} \\ & \iff & (\forall [\![\tau]\!]_{L} \in U^{\Gamma}_{\mathrm{L}})\ \mathfrak{M}^{\Gamma}_{\mathrm{L}}, (\eta[\tau/x])^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi & \text{(SALA)} \\ & \iff & (\forall \tau \in CMT)\ \mathfrak{M}^{\Gamma}_{\mathrm{L}}, (\eta[\tau/x])^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi & \\ & \iff & (\forall \tau \in CMT)\ \varphi^{\eta[\tau/x]} \in \Gamma & \text{ind.hip.} \\ & \iff & (\forall \tau \in CMT)\ \varphi^{\eta\backslash x,\eta}(\tau/x) \in \Gamma & \text{(SALA)} \\ & \iff & \forall x\varphi^{\eta\backslash x,\eta} \in \Gamma & \text{inductivity} \\ & \iff & (\forall x\varphi)^{\eta} \in \Gamma & \text{the `almost'} \\ & & & \text{part of hom.} \end{array}$$

- $\varphi \equiv \forall a\varphi$: pointer quantification

$$\begin{array}{lll}
\multicolumn{3}{l}{\mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}, \Gamma \models \forall a\varphi \iff} \\
\iff & (\forall [\![\mathsf{c}]\!]_L \in \mathbb{C}^{\Gamma}_{\mathrm{L}\Gamma})\ \mathfrak{M}^{\Gamma}_{\mathrm{L}}, \eta^{\Gamma}_{\mathrm{L}}[a \mapsto [\![\mathsf{c}]\!]_L], \Gamma \models \varphi & \text{def.of } \models, \mathbb{C}^{\Gamma}_{\mathrm{L}\Gamma} \\
\iff & (\forall [\![\mathsf{c}]\!]_L \in \mathbb{C}^{\Gamma}_{\mathrm{L}\Gamma})\ \mathfrak{M}^{\Gamma}_{\mathrm{L}}, (\eta[\mathsf{c}/a])^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi & \text{(SALA)} \\
\iff & (\forall \mathsf{c} \in CC)((\exists \tau \in CMT) c{:}\tau \in \Gamma) \text{ implies} & \\
 & \qquad \mathfrak{M}^{\Gamma}_{\mathrm{L}}, (\eta[\mathsf{c}/a])^{\Gamma}_{\mathrm{L}}, \Gamma \models \varphi & \text{def.of } \mathbb{C}^{\Gamma}_{\mathrm{L}\Gamma} \\
\iff & (\forall \mathsf{c} \in CC)((\exists \tau \in CMT) c{:}\tau \in \Gamma) \Rightarrow \varphi^{\eta[\mathsf{c}/a]} \in \Gamma & \text{ind.hip.} \\
\iff & (\forall \mathsf{c} \in CC)((\exists \tau \in CMT) c{:}\tau \in \Gamma) \Rightarrow \varphi^{\eta\backslash a}(\mathsf{c}/a) \in \Gamma & \text{(SALA)} \\
\iff & (\forall \mathsf{c} \in CC)\exists x\, c : x \in \Gamma \Rightarrow\ \varphi^{\eta\backslash a}(\mathsf{c}/a) \in \Gamma & \text{UI} \\
\iff & (\forall \mathsf{c} \in CC)\mathcal{E}\mathsf{c} \in \Gamma \Rightarrow\ \varphi^{\eta\backslash a}(\mathsf{c}/a) \in \Gamma & \text{def.of } \mathcal{E}(\mathsf{c}) \\
\iff & (\forall \mathsf{c} \in CC)\mathcal{E}\mathsf{c} \to \varphi^{\eta\backslash a}(\mathsf{c}/a) \in \Gamma & \\
\iff & \forall a\varphi^{\eta\backslash a} \in \Gamma & \text{(T-)inductivity} \\
\iff & (\forall a\varphi)^{\eta} \in \Gamma & \text{the `almost'} \\
 & & \text{part of hom.}
\end{array}$$

■

2.3.11 Conclusion

Theorem 37. *The smallest pointer logic is strongly complete w.r.t. the class of all pointer models:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \Longleftarrow \Gamma \models \varphi$$

What is more, since our canonical models are all connected, we have the following corollary:

Corollary 38. *The smallest pointer logic is strongly complete w.r.t. the class of all* ***connected*** *pointer models:*

$$\Gamma \vdash_{\mathrm{L}} \varphi \Longleftarrow \Gamma \models^{c} \varphi$$

where

$$\Gamma \models^{c} \varphi \overset{\text{def}}{\Leftrightarrow} \mathfrak{M}, \eta, w \models \varphi \text{ whenever } \mathfrak{M}, \eta, w \models \Gamma \text{ for all connected } \mathfrak{M}$$

2.4 Connections with Goldblatt's admissible semantics

This logic and its completeness proof is a modification of [Goldblatt 2011]. The main modifications are the followings:

- We consider temporal languages.
- We focus on not only constant but universal domain structures.
- Mathematical functions and relations will be considered as logical functions (their interpretation are not allowed to vary when we state that a sentence is *valid*, so they are part of the model structure/frame part of the semantics.)

- We have complex (mathematical) terms in the language.

- Goldblatt had two sorts: rigid variables and not-necessarility rigid variables, and the former was a part of the latter. We treat rigid and intensional objects as two disjoint sorts.

- Goldblatt's $a = x$ ("intensional term a's denotation in that world is the same as the denotation of the rigid term x in that (= any) world") is denoted by $a{:}x$. We will define *intensional or global* identities $a = b$ later, that means that a and b *always* points to the same number. Goldblatt's extensional/local identities '$a = b$' can be regained as $\exists x(a{:}x \wedge b{:}x)$. This slight change will result in a strengthening of the completeness theorem, since we don't have to restrict the set of Kripkean formula sets to those that are object-rich. (The definition itself will ensure that property).

- If we use the notation $a = b$, then it will be stronger than Goldblatt's notation: It will mean not only local, but global identity (and as such an undefinable, primitive notion): $a = b$ means that they are defined in exactly the same worlds and they denote the same objects in all these worlds.

- The meaning of the predicates are not allowed to vary. (So the standard translation will be only a first-order language). Even when we will consider branching spacetimes, we will allow only propositional variables. (So the standard translation will never refer to a theory that is stronger than a monadic predicate logic.)

- We represent partial functions with total functions in the usual way: When a partial function is undefined, then its total representative will have a value Θ (representative of semantic value gap) outside of the universe.

Chapter 3

Global extensions and connections with classical logic

3.1 Global extension: HCL

Definition 17. The language of HCL is given by the following syntax:

- Symbols:
  - Propositional variables: $p, q, \ldots$ $\quad PrVar \stackrel{\text{def}}{=} \{p_i : i \in \omega\}$
  - Nominal variables: $e, e', e'', \ldots$ $\quad NVar \stackrel{\text{def}}{=} \{e_i : i \in \omega\}$
  - Pointer variables: $a, b, c, \ldots$ $\quad CVar \stackrel{\text{def}}{=} \{a_i : i \in \omega\}$
  - Mathematical variables: $x, y, z, \ldots$ $\quad MVar \stackrel{\text{def}}{=} \{x_i : i \in \omega\}$
  - Mathematical constants: $\mathsf{r}_1, \mathsf{r}_2, \ldots$
  - Pointer constants: $\mathsf{c}_1, \mathsf{c}_2, \ldots$
  - Mathematical function symbols: $+, \cdot$
  - Mathematical predicate symbols: $\leq$
  - Logical symbols: $\neg, \wedge, \mathbf{P}, \mathbf{F}, @, \mathsf{E}, \downarrow, =, \exists$
  - other: $(,)$

  We use the abbreviation $Var^+ \stackrel{\text{def}}{=} Var \cup NVar$.

- Mathematical terms:

$$\tau ::= x \mid \mathsf{r} \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$$

- Pointer terms:

$$\pi ::= a \mid \mathsf{c}$$

- Formulas:

$$\varphi ::= \tau \leq \tau' \mid \tau = \tau' \mid \pi = \pi' \mid \pi{:}\tau \mid$$
$$\mid p \mid \neg\varphi \mid \varphi \wedge \psi \mid \mathbf{P}\varphi \mid \mathbf{F}\varphi \mid \exists x\varphi \mid \exists a\varphi \mid$$
$$\mid @_e\varphi \mid \mathsf{E}\varphi \mid {\downarrow} e\varphi$$

Intuitive readings are:

e	This event is e.
$@_e\varphi$	In the event e it is the case that φ.
${\downarrow}e\varphi$	If we refer to this event with e, it is the case that φ.
$\mathsf{E}\varphi$	In some event e it is the case that φ.

Definition 18. Models are the same as before. The evaluation η, however, maps singletons to the nominal variables, i.e.,

$$\eta : \begin{cases} p \ \mapsto \ X \in Prop & \text{propositional evaluation} \\ e \ \mapsto \ \{w\} \in \wp(W) & \text{nominal evaluation} \\ x \ \mapsto \ u \in U & \text{mathematical assignment} \\ a \ \mapsto \ (w \mapsto u) \in \mathbb{C} & \text{pointer assignment} \end{cases}$$

We use the same defintions for restrictions as in Def 7 with the addition of $\eta^n \stackrel{\text{def}}{=} \eta\restriction NVar$. The set of all HCL-assignments on a model structure $\mathcal{S}$ and model $\mathfrak{M}$ will be denoted by $\mathrm{H}^+_{\mathcal{S}}$ and $\mathrm{H}^+_{\mathfrak{M}}$, respectively.

To simplify some definitions, let us refer to the only element of a singleton S by $(S)^-$, i.e. $(\{w\})^- \stackrel{\text{def}}{=} w$.

The truth of BCL-formulas are the same as in BCL. The semantics of the new formulas are:

$$\mathfrak{M}, \eta, w \models e \quad \stackrel{\text{def}}{\Leftrightarrow} \quad w \in \eta(e) \iff (\eta(e))^- = w$$
$$\mathfrak{M}, \eta, w \models @_e\varphi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \mathfrak{M}, \eta, (\eta(e))^- \models \varphi$$
$$\mathfrak{M}, \eta, w \models {\downarrow}e\varphi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \mathfrak{M}, \eta[e \mapsto \{w\}], w \models \varphi$$
$$\mathfrak{M}, \eta, w \models \mathsf{E}\varphi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad (\exists v)\mathfrak{M}, \eta, v \models \varphi$$

We denote the dual of E by A: $\mathsf{A}\varphi \stackrel{\text{def}}{\Leftrightarrow} \neg\mathsf{E}\neg\varphi$.

Remark 39. The satisfaction operator is definable with the somewhere operator, since the equivalence $@_e\varphi \leftrightarrow \mathsf{E}(e \wedge \varphi)$ is valid.

Remark 40. The equality of clocks is definable using the everywhere operator:

$$\pi = \pi' \stackrel{\text{def}}{\Leftrightarrow} \mathsf{A}\forall x(\pi{:}x \leftrightarrow \pi'{:}x)$$

$$\mathfrak{M}, \eta, w \models \pi = \pi' \iff [\![\pi]\!]^{\mathfrak{M}} = [\![\pi']\!]^{\mathfrak{M}}$$

Remark 41. The satisfaction operator is definable with the somewhere operator, since the equivalence $@_e\varphi \leftrightarrow \mathsf{E}(e \wedge \varphi)$ is valid.

Definition 19 (Hybrid sort definition)**.** Let L be a BCL. We say that *in* L *a hybrid sort is definable* iff there is a translation $\mathrm{HSD}_\xi : \mathcal{L}_{\mathsf{HCL}} \to \mathcal{L}_{\mathsf{BCL}}$ a systematic assignment transformation[1] $\mathrm{tr} : \mathfrak{M} \mapsto (\mathrm{H}^+_{\mathfrak{M}} \to \mathrm{H}_{\mathfrak{M}})$ such that the following properties hold:

[1] Note that tr is not necessarily a function in the set-theoretic sense, since it is not guaranteed that the class of models forms a set.

1. For every $\mathfrak{M}$, w, HCL-assignment η and HCL-formula φ the following equivalence holds:

$$\mathfrak{M}, \eta, w \models \varphi \iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(\varphi)$$

2. For any $C \in \{\neg, \mathbf{F}, \mathbf{P}, \exists x, \exists a : x \in MVar, a \in CVar\}$ translation HSD_{ξ} satisfies the following equations :

$$\begin{array}{rcl} \mathrm{HSD}_{\xi}(\varphi \wedge \psi) & = & \mathrm{HSD}_{\xi}(\varphi) \wedge \mathrm{HSD}_{\xi}(\psi) \\ \mathrm{HSD}_{\xi}(C\varphi) & = & C\mathrm{HSD}_{\xi}(\varphi) \end{array}$$

where ξ is an injective function $\xi : Var^{+} \rightarrowtail Var$ such that for some set of BCL-formulas F

$$\xi : \begin{cases} PrVar & \to & PrVar \\ CVar & \to & CVar \\ MVar & \to & MVar \\ NVar & \to & F \end{cases}$$

HSD_{ξ} is called a *hybrid sort definition based on* ξ.

Definition 20 (Hybrid operator definition)**.** Let L be a BCL. We say that in L a binder Bw (like $\downarrow w$ or $@_w$) is definable iff there is a HSD_{ξ} hybrid sort definition that satisfies the equation

$$\mathrm{HSD}_{\xi}(Bw\varphi) \quad = \quad \delta(\xi(w), \mathrm{HSD}_{\xi}(\varphi))$$

where δ is an explicit definition in L.

Remark 42. A hybrid sort definition is always possible by using propositional variables in extensions of BCL that have the somewhere operator :

$$\xi : \begin{cases} p_i & \mapsto & p_{2i} \\ a_i & \mapsto & a_i \\ x_i & \mapsto & x_i \\ e_i & \mapsto & p_{2i+1} \end{cases}$$

$$\begin{array}{rcl} \mathrm{HSD}_{\xi}(\varphi) & \stackrel{\mathrm{def}}{=} & \xi(\varphi) \text{ for any } \varphi \in NVar \cup PrVar \\ \mathrm{HSD}_{\xi}(\varphi) & \stackrel{\mathrm{def}}{=} & \varphi \text{ for any } \varphi \in At - (NVar \cup PrVar) \\ \mathrm{HSD}_{\xi}(\varphi \wedge \psi) & \stackrel{\mathrm{def}}{=} & \mathrm{HSD}_{\xi}(\varphi) \wedge \mathrm{HSD}_{\xi}(\psi) \\ \mathrm{HSD}_{\xi}(C\varphi) & \stackrel{\mathrm{def}}{=} & C\mathrm{HSD}_{\xi}(\varphi) \\ & & \text{for any } C \in \{\neg, \mathbf{F}, \mathbf{P}, \exists x, \exists a : x \in MVar, a \in CVar\} \end{array}$$

and stating the usual axioms for nominals for the set $\{p_{2i+1} : i \in \omega\}$. Def 19 will become important when we define the hybrid sort *without* relying on the propositional variables, e.g. in Thm. 60, because in this way we can construct nominals in propositional variable-free fragments as well.

3.2 Corresponding classical language $\mathcal{L}_P^{CFOL}$

3.2.1 Language

$\mathcal{L}_P^{CFOL}$ is defined to be the following classical first-order language:

- Symbols:
  - Predicate variables: $P, P', P'', \ldots$ $\quad PrVar \stackrel{\text{def}}{=} \{P_i : i \in \omega\}$
  - Pointer variables: $a, b, c, \ldots$ $\quad ClVar \stackrel{\text{def}}{=} \{a_i : i \in \omega\}$
  - Mathematical variables: $x, y, z, \ldots$ $\quad MVar \stackrel{\text{def}}{=} \{x_i : i \in \omega\}$
  - Event variables: $e, e', e'', \ldots$ $\quad NVar \stackrel{\text{def}}{=} \{e_i : i \in \omega\}$
  - Mathematical constants: $\mathsf{r}_1, \mathsf{r}_2, \ldots$
  - Pointer constants: $\mathsf{c}_1, \mathsf{c}_2, \ldots$
  - Mathematical function symbols: $+, \cdot$
  - Mathematical predicate symbol: $\leq$
  - Event predicate: $\prec$
  - Intersort predicate: P
  - Logical symbols: $\neg, \wedge, =, \exists$
- Terms:

$$\pi ::= a \mid \mathsf{c} \qquad\qquad \tau ::= x \mid \mathsf{r} \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$$

- Formulas:

$$\varphi ::= \pi = \pi' \mid \tau = \tau' \mid \tau \leq \tau' \mid e = e' \mid e \succ e' \mid \mathrm{P}(e, \pi, \tau) \mid \\ \neg\varphi \mid \varphi \wedge \psi \mid \exists x \varphi \mid \exists a \varphi \mid \exists e \varphi$$

Notation 21. We will use the following abbreviations:

$$\begin{aligned} a(e) = \tau &\stackrel{\text{def}}{\Leftrightarrow} \mathrm{P}(\pi, e, \tau) \\ a[\tau] = e &\stackrel{\text{def}}{\Leftrightarrow} \mathrm{P}(\pi, e, \tau) \\ \pi \in \mathrm{D}_e &\stackrel{\text{def}}{\Leftrightarrow} \exists x\, \mathrm{P}(\pi, e, x) \\ e \in \mathrm{wline}_\pi &\stackrel{\text{def}}{\Leftrightarrow} \exists x\, \mathrm{P}(\pi, e, x) \end{aligned}$$

When we use the first notation, we will always assume that P is a partial function, i.e., we take the *unique pointing axiom*

(UP) $\qquad (\mathrm{P}(\pi, e, \tau) \wedge \mathrm{P}(\pi, e, \tau')) \to \tau = \tau',$

so the equation-notation will always going to be justified.

3.2.2 Axioms

Derivation, $\vdash$, is defined in the standard classical first-order way.

Definition 22. A first-order theory is called a *pointer system*, iff it derives the standard FOL axioms and (UP).

3.2.3 Models

$$\mathfrak{M} = \big(W, U, C, P, \prec^{\mathfrak{M}}, \mathrm{P}^{\mathfrak{M}}, +^{\mathfrak{M}}, \cdot^{\mathfrak{M}}, \leq^{\mathfrak{M}}, \mathsf{r}_i^{\mathfrak{M}}, \mathsf{c}_j^{\mathfrak{M}}\big)_{i \in I, j \in J}$$

where

- $W \neq \varnothing$, $U \neq \varnothing$, C is an arbitrary set, $P \neq \varnothing$,
- $\prec^{\mathfrak{M}} \subseteq W^2$,
- $\mathrm{P}^{\mathfrak{M}} \subseteq W \times C \times U$,
- $+^{\mathfrak{M}}, \cdot^{\mathfrak{M}} : U^2 \to U$.
- $\leq^{\mathfrak{M}} \subseteq U^2$,
- $\mathsf{r}_i^{\mathfrak{M}} \in U$.
- $\mathsf{c}_i^{\mathfrak{M}} \in C$.

3.3 Standard Translation

Now we can define the standard translation of the hybrid (and non-hybrid) language:

$$\begin{array}{rcll}
\mathrm{ST}_e(p) & \stackrel{\text{def}}{=} & P(e) & \\
\mathrm{ST}_e(\tau \leq \tau') & \stackrel{\text{def}}{=} & \tau \leq \tau' & \\
\mathrm{ST}_e(\tau = \tau') & \stackrel{\text{def}}{=} & \tau = \tau' & \\
\mathrm{ST}_e(\pi{:}\tau) & \stackrel{\text{def}}{=} & \mathrm{P}(\pi, e, \tau) & \\
\mathrm{ST}_e(\neg\varphi) & \stackrel{\text{def}}{=} & \neg\mathrm{ST}_e(\varphi) & \\
\mathrm{ST}_e(\varphi \wedge \psi) & \stackrel{\text{def}}{=} & \mathrm{ST}_e(\varphi) \wedge \mathrm{ST}_e(\psi) & \\
\mathrm{ST}_e(\mathbf{F}\varphi) & \stackrel{\text{def}}{=} & (\exists e' \succ e)\, \mathrm{ST}_{e'}(\varphi) & e' \text{ is a fresh variable} \\
\mathrm{ST}_e(\mathbf{P}\varphi) & \stackrel{\text{def}}{=} & (\exists e' \prec e)\, \mathrm{ST}_{e'}(\varphi) & e' \text{ is a fresh variable} \\
\mathrm{ST}_e(\exists x\varphi) & \stackrel{\text{def}}{=} & \exists x\, \mathrm{ST}_e(\varphi) & \\
\mathrm{ST}_e(\exists a\varphi) & \stackrel{\text{def}}{=} & (\exists a \in \mathrm{D}_e)\, \mathrm{ST}_e(\varphi) & \\
\hline
\mathrm{ST}_e(e') & \stackrel{\text{def}}{=} & e = e' & \\
\mathrm{ST}_e(@_{e'}\varphi) & \stackrel{\text{def}}{=} & \mathrm{ST}_{e'}(\varphi) & \\
\mathrm{ST}_e({\downarrow} e'\varphi) & \stackrel{\text{def}}{=} & \exists e'(e = e' \wedge \mathrm{ST}_e(\varphi)) & \\
\mathrm{ST}_e(\mathsf{E}\varphi) & \stackrel{\text{def}}{=} & \exists e'\, \mathrm{ST}_{e'}(\varphi) &
\end{array}$$

Theorem 43. *For every $\varphi \in \mathcal{L}_{\mathsf{HCL}}$ (including $\mathcal{L}_{\mathsf{BCL}}$)*

$$\begin{array}{rcl}
\mathfrak{M}, \eta, w \models \varphi & \iff & \mathfrak{M} \models \mathrm{ST}_e(\varphi) \quad [\eta[e \mapsto w]] \\
\mathfrak{M}, \eta \models \varphi & \iff & \mathfrak{M} \models \forall e\, \mathrm{ST}_e(\varphi) \\
\mathfrak{M}, \eta^- \models \varphi & \iff & \mathfrak{M} \models \forall P_1 \ldots \forall P_n \forall e\, \mathrm{ST}_e(\varphi)
\end{array}$$

UNDER CONSTRUCTION

3.4 Hybrid translation

Now we can define the hybrid translation of the classical language:

$$
\begin{array}{lll}
\mathrm{HT}(P(e)) & \stackrel{\text{def}}{=} & @_e p \\
\mathrm{HT}(e = e') & \stackrel{\text{def}}{=} & @_e e' \\
\mathrm{HT}(e \prec e') & \stackrel{\text{def}}{=} & @_e \mathbf{F} e' \\
\mathrm{HT}(\tau = \tau') & \stackrel{\text{def}}{=} & \tau = \tau' \\
\mathrm{HT}(\tau \leq \tau') & \stackrel{\text{def}}{=} & \tau \leq \tau' \\
\mathrm{HT}(a = a') & \stackrel{\text{def}}{=} & a = a' \stackrel{\text{def}}{\Leftrightarrow} \mathsf{A}\forall x(a{:}x \leftrightarrow a'{:}x) \\
\mathrm{HT}(\mathrm{P}(e, a, \tau)) & \stackrel{\text{def}}{=} & @_e a{:}\tau \\
\mathrm{HT}(\neg\varphi) & \stackrel{\text{def}}{=} & \neg\mathrm{HT}(\varphi) \\
\mathrm{HT}(\varphi \wedge \psi) & \stackrel{\text{def}}{=} & \mathrm{HT}(\varphi) \wedge \mathrm{HT}(\psi) \\
\mathrm{HT}(\exists e\varphi) & \stackrel{\text{def}}{=} & \mathsf{E}{\downarrow}e\mathrm{HT}(\varphi) \\
\mathrm{HT}(\exists a\varphi) & \stackrel{\text{def}}{=} & \mathsf{E}\exists a\mathrm{HT}(\varphi) \\
\mathrm{HT}(\exists x\varphi) & \stackrel{\text{def}}{=} & \exists x\mathrm{HT}(\varphi)
\end{array}
$$

Theorem 44. *For every* $\varphi \in \mathcal{L}_{\mathsf{STCL}}$*,*

$$\mathfrak{M}, \eta, w \models {\downarrow}e\,\mathrm{HT}(\varphi) \iff \mathfrak{M} \models \varphi \quad [\eta[e \mapsto w]]$$

UNDER CONSTRUCTION

Chapter 4

Forming Clocks from Pointers

Definition 23 (Euclidean and Minkowski norms)**.** Let $w \in \mathbb{R}^n$.
The Euclidean norm of w, $|w|$ is $|w| \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} w_i^2}$.
The Minkowski norm of w, $\mu(w)$ is $\mu(w) \stackrel{\text{def}}{=} w_1^2 - \sum_{i=2}^{n} w_i^2$.

Note that $\mu(w) = |w_1|^2 - |w_{2-n}|^2$.

4.1 Intended models: Minkowski models

In this section, we consider two intended models. In one of them, $\mathfrak{Mink}_{\mathrm{A}}$, clocks can accelerate, but in the other, $\mathfrak{Mink}_{\mathrm{I}}$, they cannot. We start with the accelerating one:

Definition 24 (Intended models)**.**

$$\mathfrak{Mink}_{\mathrm{A}} = \Big(W, \succ, \prec, \wp(W), U, \Theta, \mathbb{C}, [\![+]\!]^{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}}, [\![\leq]\!]^{\mathfrak{M}}\Big)$$

- $\Big(U, [\![+]\!]^{\mathfrak{M}}, [\![\cdot]\!]^{\mathfrak{M}}, [\![\leq]\!]^{\mathfrak{M}}\Big) \stackrel{\text{def}}{=} \mathbb{R}$ is the field of reals.
- $W = \mathbb{R}^4$
- $w \prec w'$ iff $\mu(w - w') \leq 0$ and $w_1 < w_1'$.
- $\mathbb{C} = \{\alpha : \alpha^{-1}$ is a timelike curve $\}$ s.t. all α use the measure system of **R**, i.e., the set of all those partial function $\alpha : W \to U$, for which
  - α is an injective and surjective function.
  - α^{-1} is a continuously differentiable function w.r.t. euclidean metric, i.e.,

$$(\forall x \in U)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in U)$$
$$|x - y| \leq \delta \quad \Rightarrow \quad \frac{|\alpha^{-1}(x) - \alpha^{-1}(y)|}{|x - y|} \leq \varepsilon,$$

and the existing and unique derivative of $(\alpha^{-1})'$ granted by the previous formula is continuous, i.e.,

$$(\forall x \in U)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in U)$$
$$|x - y| < \delta \quad \Rightarrow \quad \big|(\alpha^{-1})'(x) - (\alpha^{-1})'(y)\big| < \varepsilon$$

– the tangent vector of α^{-1} is always timelike, i.e.,

$$(\forall x \in U) \qquad \mu\left(\left(\alpha^{-1}\right)'(x)\right) < 0$$

The non-accelerating intended model $\mathfrak{Mink}_{\mathrm{I}}$ differs only in the set of pointers, which is the set of timelike *lines*, i.e.,

$$\mathbb{C}_{\mathrm{I}} \stackrel{\text{def}}{=} \{\alpha \in \mathbb{C} : (\exists x, y \in U)(\forall z \in U)(\exists \lambda \in U)$$
$$\alpha^{-1}(z) = \alpha^{-1}(x) + \lambda \cdot (\alpha^{-1}(x) - \alpha^{-1}(y))\}$$

Remark 45. The property that the tangent vector of α^{-1} is always timelike can be written in the form

$$(\forall x \in U) \qquad \big|\alpha_1^{-1}(x)\big| > \big|\alpha_{2-4}^{-1}(x)\big|$$

The alternative relation $\prec$ can be defined without μ:

$$w \prec w' \iff |w_1 - w'_1| > |w_{2-4} - w'_{2-4}| \text{ and } w_1 < w'_1$$

This is sometime referred as the 'after'-relation α in the literature, see [Goldblatt 1980; Shapirovsky and Shehtman 2005].

4.2 Canonicity

4.2.1 Basics

Definition 25. Γ is *canonical for* K iff

- Γ is valid on K, i.e.,
$$(\forall \mathfrak{M})\ \mathfrak{M} \in \mathsf{K} \Rightarrow \mathfrak{M} \models \Gamma.$$
- If $\mathrm{L}' \supseteq \Gamma$ is a BCL,
then $\mathfrak{M}^{\Sigma}_{\mathrm{L}'} \in \mathsf{K}$ for all canonical model $\mathfrak{M}^{\Sigma}_{\mathrm{L}'}$.

Definition 26. Γ is *canonical for* K *above* Γ' iff

- Γ is valid on K within the models of Γ', i.e.,
$$(\forall \mathfrak{M} \models \Gamma')\ \mathfrak{M} \in \mathsf{K} \Rightarrow \mathfrak{M} \models \Gamma.$$
- If $\mathrm{L}' \supseteq \Gamma \cup \Gamma'$ is a BCL,
then $\mathfrak{M}^{\Sigma}_{\mathrm{L}'} \in \mathsf{K}$ for all canonical model $\mathfrak{M}^{\Sigma}_{\mathrm{L}'}$.

Proposition 46. *If a bidirectional pointer logic* L *is canonical for* K *then* L *is sound and strongly complete w.r.t.* K.

Proof. The soundness part follows from the assumption that L is canonical for K, that is, all theorems of L is valid on all $\mathfrak{M} \in \mathsf{K}$.

For strong completeness, we have to prove that whenever $\Gamma \nvdash_{\mathrm{L}} \varphi$, then there is a counter model $\mathfrak{M} \in \mathsf{K}$. We know from the completeness theorems of pointer logics, that if Σ is a canonical world extending $\Gamma \cup \{\neg\varphi\}$, then the canonical model $\mathfrak{M}_{\mathrm{L}}^{\Sigma}$ will be a counter model. Then our job would be to show that $\mathfrak{M}_{\mathrm{L}}^{\Sigma} \in \mathsf{K}$. But this is true by the assumption that L is canonical for K. ■

Corollary 47. *If a pointer logic* L *is canonical for* K *above* L'*, and* L' *is canonical for some* $\mathsf{K}' \supseteq \mathsf{K}$*, then* $\mathrm{L} \cup \mathrm{L}'$ *is sound and strongly complete w.r.t.* K.

4.2.2 Classical Theories

Notation 27. Let $\mathcal{L}^m$ be the set of mathematical formulas (formulas not containing pointer variables or modalities). If K^c is a class of *classical* first-order models, then $\mathrm{Th}(\mathsf{K}^c) \stackrel{\text{def}}{=} \{\varphi \in \mathcal{L}^m : \mathsf{K} \models \varphi\}$ where $\models$ is now the standard classical satisfaction relation.

Proposition 48. *Any classical theory* $\mathrm{Th}(\mathsf{K}^c)$ *is canonical for the class (of pointer models)*

$$\left\{\mathfrak{M} : \left(U, [\![+]\!]^{\mathfrak{M}}, [\![\times]\!]^{\mathfrak{M}}, [\![\leq]\!]^{\mathfrak{M}}\right) \text{ is elementary eq. with } \mathsf{K}^c.\right\}$$

Proof. Since we deal with formulas that are valid on a class of classical models (i.e., true in all model with all assignments), for every opened formula $\varphi(\vec{x}) \in \mathrm{Th}(\mathsf{K}^c)$ we have $\varphi(\vec{x}) \leftrightarrow \forall x \varphi(\vec{x}) \in \mathrm{Th}(\mathsf{K}^c)$. So to prove that every formula is true in every world of the canonical models, it is enough to check the closed formulas.

Let $L' \supseteq \mathrm{Th}(\mathsf{K}^c)$ be a pointer logic. Since every canonical world $\Sigma \in W_{\mathrm{L}'}$ is an L'-consistent set and $\vdash_{\mathrm{L}'} \mathrm{Th}(\mathsf{K}^c)$, we have $\mathrm{Th}(\mathsf{K}^c) \subseteq \Sigma$. Now if $\varphi \in \mathrm{Th}(\mathsf{K}^c)$ is a closed formula, then for an arbitrary substitution η and η, $\varphi^{\eta} \in \Sigma$. Then by the Truth Lemma $\mathfrak{M}_{\mathrm{L}'}^{\Sigma}, \eta_{\mathrm{L}'}^{\Sigma}, \eta_{\mathrm{L}'}^{\Sigma}, \Sigma \models \varphi$. Since φ has no free mathematical or pointer variables and Σ was arbitrary, $\mathfrak{M}_{\mathrm{L}'}^{\Sigma} \models \varphi$ for all canonical model $\mathfrak{M}_{\mathrm{L}'}^{\Sigma}$. ■

Corollary 49. *The classical theory* $\mathrm{Th}(\mathbb{R})$ *is canonical for the class*

$$\left\{\mathfrak{M} : \left(U, [\![+]\!]^{\mathfrak{M}}, [\![\times]\!]^{\mathfrak{M}}, [\![\leq]\!]^{\mathfrak{M}}\right) \text{ is a real closed field}\right\},$$

and this class is axiomatizable by finitely many schemes.

4.2.3 Modal canonical formulas

Notation 28.

$$\begin{aligned}
\mathrm{wline}_{\alpha} &\stackrel{\text{def}}{=} \{w \in W : \alpha(w) \neq \Theta\} \\
w \succ_{\alpha} w' &\stackrel{\text{def}}{\Leftrightarrow} w \succ w' \text{ and } w' \in \mathrm{wline}_{\alpha} \\
w \succeq_{\alpha} w' &\stackrel{\text{def}}{\Leftrightarrow} w \succ_{\alpha} w' \text{ or } w = w'
\end{aligned}$$

Note that

$$(\exists w') w \succ_{\alpha} w' \iff w \in \langle \succ \rangle \mathrm{wline}_{\alpha}$$

From now on we skip all the components of $\mathfrak{M}, \eta, w \models \varphi$ that does not play any role, e.g.,

$$\eta, w \Vdash \varphi \stackrel{\text{def}}{\Leftrightarrow} \mathfrak{M}, \eta, w \models \varphi$$

But note that this does not mean that the skipped parts are quantified over; they are still parameters.

Proposition 50.

$$
\begin{array}{rrcl}
 & [\![\pi]\!]_\eta \in \mathrm{D}_w & \Longleftrightarrow & \eta, w \Vdash \mathcal{E}\pi \\
 & \mathrm{wline}_{[\![\pi]\!]_\eta} & = & [\![\mathcal{E}\pi]\!]_\eta \\
 & & & \\
(\forall w' \preceq w) & w' \Vdash \varphi & \Longleftrightarrow & w \Vdash \underline{\mathbf{H}}\varphi \\
(\forall w' \prec_{[\![\pi]\!]_\eta} w) & \eta, w' \Vdash \varphi & \Longleftrightarrow & \eta, w \Vdash \mathbf{H}_\pi\varphi \\
 & & & \\
(\exists w' \preceq w) & w' \Vdash \varphi & \Longleftrightarrow & w \Vdash \underline{\mathbf{P}}\varphi \\
(\exists w' \prec_{[\![\pi]\!]_\eta} w) & \eta, w' \Vdash \varphi & \Longleftrightarrow & \eta, w \Vdash \mathbf{P}_\pi\varphi \\
 & & & \\
(\forall w' \succeq w) & w' \Vdash \varphi & \Longleftrightarrow & w \Vdash \underline{\mathbf{G}}\varphi \\
(\forall w' \succ_{[\![\pi]\!]_\eta} w) & \eta, w' \Vdash \varphi & \Longleftrightarrow & \eta, w \Vdash \mathbf{G}_\pi\varphi \\
 & & & \\
(\exists w' \succ w) & w' \Vdash \varphi & \Longleftrightarrow & w \Vdash \underline{\mathbf{F}}\varphi \\
(\exists w' \succ_{[\![\pi]\!]_\eta} w) & \eta, w' \Vdash \varphi & \Longleftrightarrow & \eta, w \Vdash \mathbf{F}_\pi\varphi
\end{array}
$$

transitivity

Proposition 51. *The formula*

$$(\mathrm{H4}) \stackrel{\mathrm{def}}{=} \mathbf{H}\varphi \to \mathbf{HH}\varphi$$

is canonical for the transitivity of causality, i.e., for

$$w \succ w' \succ w'' \Rightarrow w \succ w''$$

Proof. <u>VALIDITY:</u> If the formula $\mathbf{H}\varphi \to \mathbf{HH}\varphi$ is false in a world with a formula φ, then there is a world w in which $w \Vdash \mathbf{H}\varphi$ but $w \Vdash \mathbf{PP}\neg\varphi$ therefore there are worlds $w \succ w' \succ w''$ s.t. $w \Vdash \neg\varphi$, but since the relation is transitive, $w \Vdash \varphi$.

<u>CANONICITY:</u> Let L be a pointer logic that contains the scheme (H4) and let Γ, Γ', Γ'' be arbitrary canonical worlds s.t. $\Gamma \succ_{\mathrm{L}} \Gamma' \succ_{\mathrm{L}} \Gamma''$. We have to prove that $\mathbf{H}^{-}(\Gamma) \subseteq \Gamma''$. Take a $\mathbf{H}\varphi \in \Gamma$. Then by (H4), $\mathbf{HH}\varphi \in \Gamma$, therefore $\varphi \in \mathbf{H}^{2^-}(\Gamma) \subseteq \Gamma''$. ∎

Proposition 52. *The formula*

$$(\mathrm{C.3}) \stackrel{\mathrm{def}}{=} \mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\varphi \to \psi) \vee \mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\psi \to \varphi)$$

is canonical for the property saying that clocks are not branching in the past, i.e.,

$$(w \succ_\alpha w_1 \text{ and } w \succ_\alpha w_2) \Rightarrow (w_1 \succ_\alpha w_2 \text{ or } w_1 = w_2 \text{ or } w_2 \succ_\alpha w_1),$$

Proof. <u>VALIDITY:</u> Take a model $\mathfrak{M}$ in which $(w \succ_\alpha w_1$ and $w \succ_\alpha w_2) \Rightarrow (w_1 \succ_\alpha w_2$ or $w_1 = w_2$ or $w_2 \succ_\alpha w_1)$ for all $w, w_1, w_2 \in W$ and $\alpha \in \mathbb{C}$. Assume that $\mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\varphi \to \psi) \vee \mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\psi \to \varphi)$ is false in a world w with a η pointer-assignment, so $w \Vdash \mathbf{P}_\pi(\underline{\mathbf{H}}_\pi\varphi \wedge \neg\psi)$ and $w \Vdash \mathbf{P}_\pi(\underline{\mathbf{H}}_\pi\psi \wedge \neg\varphi)$. By these formulas there are worlds $w_1, w_2 \in \mathrm{wline}_{[\![\pi]\!]_\eta}$ that are at the same time $\succ$-related to w, s.t. $w_1 \Vdash \underline{\mathbf{H}}_\pi\varphi\wedge\neg\psi$ and $w_2 \Vdash \underline{\mathbf{H}}_\pi\psi\wedge\neg\varphi$. The truth of these formulas forces w_1 and w_2 to be different, and since at least one of them $\succ$-related to the other (since there is no branching in $\mathrm{wline}_{[\![\pi]\!]_\eta}$), we have the desired contradiction.

<u>Canonicity:</u> Let L be a pointer logic containing the formula (C.3). Let Γ, $[\![\mathsf{c}]\!]$ be arbitrary but fixed. Let $\Gamma_1, \Gamma_2 \in \mathrm{wline}_{[\![\mathsf{c}]\!]}$ be arbitrary $\succ_{\mathrm{L}}$-neighbours of Γ.

If $\Gamma_1 = \Gamma_2$, then we are ready. If $\Gamma_1 \neq \Gamma_2$, then suppose indirectly that they are not related by $\succ_{\mathrm{L}}$ at all. That would mean that there is a formula $\mathbf{H}\varphi \in \Gamma_1$ for which $\varphi \notin \Gamma_2$, and similarly, that there is a formula $\mathbf{H}\psi \in \Gamma_2$ for which $\psi \notin \Gamma_1$. It comes from PC1, N and K that $\mathbf{H}\chi \to \mathbf{H}(\mathcal{E}\pi \to \chi)$, i.e., $\mathbf{H}\chi \to \mathbf{H}_\pi\chi$, so we have that $\mathbf{H}_{\mathsf{c}}\varphi, \neg\psi \in \Gamma_1$ and $\mathbf{H}_{\mathsf{c}}\psi, \neg\varphi \in \Gamma_2$.

In this case we would have that $\neg(\underline{\mathbf{H}}_{\mathsf{c}}\psi \to \varphi) \in \Gamma_1$ and $\neg(\underline{\mathbf{H}}_{\mathsf{c}}\varphi \to \psi) \in \Gamma_2$, therefore, since both of Γ_1 and Γ_2 are $\succ_{\mathrm{L}}$-related to Γ and $\Gamma_1, \Gamma_2 \in \mathrm{wline}_{[\![\mathsf{c}]\!]}$ we have that $\mathbf{P}_{\mathsf{c}}\neg(\underline{\mathbf{H}}_{\mathsf{c}}\psi \to \varphi) \in \Gamma$ and $\mathbf{P}_{\mathsf{c}}\neg(\underline{\mathbf{H}}_{\mathsf{c}}\varphi \to \psi) \in \Gamma$, i.e., even $\mathbf{P}_{\mathsf{c}}\neg(\underline{\mathbf{H}}_{\mathsf{c}}\psi \to \varphi) \wedge \mathbf{P}_{\mathsf{c}}\neg(\underline{\mathbf{H}}_{\mathsf{c}}\varphi \to \psi) \in \Gamma$, hence $\neg\mathbf{H}_{\mathsf{c}}(\underline{\mathbf{H}}_{\mathsf{c}}\psi \to \varphi) \wedge \neg\mathbf{H}_{\mathsf{c}}(\underline{\mathbf{H}}_{\mathsf{c}}\varphi \to \psi) \in \Gamma$ which makes Γ inconsistent. ■

Proposition 53. *The formula*

$$(\forall\mathrm{E}\exists\mathrm{CS}) \stackrel{\mathrm{def}}{=} \exists a\mathcal{E}a$$

is canonical for the property saying that there is a clock in every event, i.e.,

$$D(w) \neq \varnothing$$

Proof. <u>Validity:</u> Take a model $\mathfrak{M}$ in which $D(w) \neq \varnothing$ for all $w \in W$. Let w, η be arbitrary but fixed. Take an $\alpha \in D(w)$. Then $\eta[a \mapsto \alpha], w \Vdash \mathcal{E}a$, hence $\eta, w \Vdash \exists a\mathcal{E}a$.

<u>Canonicity:</u> Let L be a pointer logic containing the formula $\exists a\mathcal{E}a$. Let Γ be arbitrary but fixed. Then $\exists a\mathcal{E}a \in \Gamma$, and since Γ is iTi, there is a constant a s.t. $\mathcal{E}\mathsf{a} \in \Gamma$, i.e., $\exists x\ \mathsf{a}{:}x \in \Gamma$, and since again, Γ is iTi, there is a $\tau \in CMT$ s.t. $\mathsf{c}{:}\tau \in \Gamma$, and by that, $[\![\mathsf{a}]\!]_\Gamma = [\![\tau]\!]$. Therefore, $[\![\mathsf{a}]\!] \in D(\Gamma)$. ■

Proposition 54. *The formula*

$$(\mathrm{CN}) \stackrel{\mathrm{def}}{=} \pi{:}\tau \to \mathbf{H}_\pi\neg\pi{:}\tau$$

is canonical for the property saying that the state of the pointer is always new, i.e.,

$$\alpha(w) = u \Rightarrow (\forall w' \prec_\alpha w)\ \alpha(w') \neq u$$

Proof. <u>Validity:</u> Take a model $\mathfrak{M}$ in which $\alpha(w) = u \Rightarrow (\forall w' \prec_\alpha w)\ \alpha(w') \neq u$ for all $w \in W$, $\alpha \in \mathbb{C}$ and $u \in U$. Let w, η be arbitrary but fixed. Assume that $\eta, w \Vdash \pi{:}\tau$, i.e., $[\![\pi]\!]_\eta = [\![\tau]\!]_\eta$. Since in $\mathfrak{M}$ the state of the pointers must change, we have $(\forall w' \prec_{[\![\pi]\!]_\eta} w)[\![\pi]\!]_\eta(w') \neq [\![\tau]\!]_\eta$, i.e., $(\forall w' \prec_{[\![\pi]\!]_\eta} w) w' \Vdash \neg\pi{:}\tau$, hence $w \Vdash \mathbf{H}_\pi\neg\pi{:}x$.

<u>Canonicity:</u> Let L be a pointer logic containing the formula scheme (CN). Let Γ, $[\![\mathsf{c}]\!]$ and $[\![\tau]\!]$ (eq.class of some τ) be arbitrary but fixed. Suppose that $[\![\mathsf{c}]\!]_\Gamma = [\![\tau]\!]$. This means that $\mathsf{c}{:}\tau \in \Gamma$. Since L, thus Γ contains (CN), $\mathsf{c}{:}\tau \to \mathbf{H}_{\mathsf{c}}\neg\mathsf{c}{:}\tau \in \Gamma$, hence $\mathbf{H}_{\mathsf{c}}\neg\mathsf{c}{:}\tau \in \Gamma$. This is an abbreviation for $\mathbf{H}(\exists x\ \mathsf{c}{:}x \to \neg\mathsf{c}{:}\tau) \in \Gamma$.

Now let $\Gamma' {\prec_{\mathrm{L}[\![\mathsf{c}]\!]}} \Gamma$ be arbitrary. By definition, $\Gamma' \in \mathrm{wline}_{[\![\mathsf{c}]\!]}$. By the def of $\succ_{\mathrm{L}}$, $\mathbf{H}^-(\Gamma) \subseteq \Gamma'$, therefore $\exists x\ \mathsf{c}{:}x \to \neg\mathsf{c}{:}\tau \in \Gamma'$. But since $\Gamma' \in \mathrm{wline}_{[\![\mathsf{c}]\!]}$, we have that $\mathcal{E}\mathsf{c} \in \Gamma'$, i.e., $\exists x\ \mathsf{c}{:}x$. Therefore, $\neg\mathsf{c}{:}\tau \in \Gamma'$, $\mathsf{c}{:}\tau \notin \Gamma'$ so $[\![\mathsf{c}]\!]_{\Gamma'} \neq [\![\tau]\!]$. ■

Table 4.1: Summary of the terminology of the different types of nominals

(strong)	$\leftrightsquigarrow$	exactly one
weak	$\leftrightsquigarrow$	at most one but satisfiable
(global)	$\leftrightsquigarrow$	in the whole model
past	$\leftrightsquigarrow$	only in the causal past

4.3 Defining hybrid sorts and operators

Definition 29 (Causal Past)**.** Suppose that the alternative relation is transitive. The *causal past* of a world w is the (downward closed) set $\{v : v \prec w\}$

Definition 30 (Nominal types)**.**

- We say that φ is a *nominal of logic* L iff no matter what L-models $\mathfrak{M}$ and assignments η, η and η we take, φ is true in exactly one world, i.e.,

$$(\exists w) \quad [\![\varphi]\!]_{\eta}^{\mathfrak{M}} = \{w\}$$

- We say that φ is a *weak nominal of logic* L iff φ is satisfiable, and no matter what L-models $\mathfrak{M}$ assignments η, η and η we take, φ cannot be true in more than one worlds.

$$(\exists w) \quad [\![\varphi]\!]_{\eta}^{\mathfrak{M}} \subseteq \{w\}$$

  We say that φ is a *weak past-nominal of logic* L iff φ is satisfiable, and no matter what assignments η, η and η we take, the intension of φ is true in at most one world of any world's causal past.

$$(\forall v)(\exists w) \quad [\![\varphi]\!]_{\eta}^{\mathfrak{M}} \cap \{v' : v' \prec v\} \subseteq \{w\}$$

4.3.1 A BCL with weak past-nominals

Theorem 55. *In any* BCL *containing the axioms*

$$\begin{array}{rl} \text{(CH.3)} & \mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\varphi \to \psi) \vee \mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\psi \to \varphi) \\ \text{(CN)} & a{:}x \to \mathbf{H}_a \neg a{:}x \end{array}$$

the pointing statemens $\pi{:}\tau$ are weak past-nominals.

Proof. Let $\{w' : w' \prec w\}$, η and η be arbitrary but fixed. We have to prove that for any pointing statement $\pi{:}\tau$, the intension $[\![\pi{:}\tau]\!] \cap \{w' : w' \prec w\}$ is either a singleton or the empty set. Suppose that it has more elements than one. So we have two distinct elements of this set; let us call them v and u. Since $v, u \in \{w' : w' \prec w\}$, $w \succ v$ and $w \succ u$. By our assumption, π exists both in v and u, therefore by the non-branching provided by (CH.3) we have that either $v \succ u$ or $u \succ v$. Either case it is, $\pi{:}\tau$ and $\mathbf{P}\pi{:}\tau$ will be true in one of the worlds, which is prohibited by (CN). ■

Remark 56. This axiom system

- is in the basic modal fragment,
- tolerates branching,
- tolerates black holes.

Theorem 57. *The formulas*

$$\text{(CG)} \stackrel{\text{def}}{=} (\mathbf{P}\pi_1{:}\tau_1 \wedge \mathbf{P}\pi_2{:}\tau_2) \to \exists c \exists x\, \mathbf{H}\big((\pi_1{:}\tau_1 \vee \pi_2{:}\tau_2) \to \mathbf{P}c{:}x\big)$$

where c and x do not occur in π_1, π_2, τ_1 and τ_2, is canonical for for the chronological convergence, i.e.,

$$w \succ w_1 \text{ and } w \succ w_2 \Rightarrow (\exists w_3 \ll w) w_1 \succ w_3 \text{ and } w_2 \succ w_3$$

above (CH.3) + (CN) + (∀E∃CS).

Note that here we are still in the basic modal fragment. The proof is easy, but we give an extremely detailed proof because it shed a light on the process of how we use pointing statements to tag events.

Proof. <u>Validity:</u> Suppose that a model has the convergence property $w \succ w_1$ and $w \succ w_2 \Rightarrow (\exists w_3 \ll w) w_1 \succ w_3$ and $w_2 \succ w_3$, but there are v, η, η, w, τ_1, τ_2, π_1 and π_2,, s.t. c does not occur in π_1 and π_2 and

$$\begin{array}{lll} (1) & \eta, w & \Vdash \mathbf{P}\pi_1{:}\tau_1 \\ (2) & \eta, w & \Vdash \mathbf{P}\pi_2{:}\tau_2 \\ (3) & \eta, w & \Vdash \neg\exists c \exists x\, \mathbf{H}\big((\pi_1{:}\tau_1 \vee \pi_2{:}\tau_2) \to \mathbf{P}c{:}x\big) \end{array}$$

Then we have that

$$\eta, w_1 \Vdash \pi_1{:}\tau_1 \quad \text{where } w \succ w_1 \tag{4.1}$$

$$\eta, w_2 \Vdash \pi_2{:}\tau_2 \quad \text{where } w \succ w_2 \tag{4.2}$$

Since our model has the convergence property described above, there is a world w_3 s.t. $w_1 \succ w_3$, $w_2 \succ w_3$ and there is a clock $\alpha \in \mathrm{D}_w \cap \mathrm{D}_{w^3}$. Then we can reason in the following way:

$$\eta[c \mapsto \alpha] w_3 \Vdash \mathcal{E}c$$

$$\eta[c \mapsto \alpha] w_3 \Vdash \exists x\, c{:}x$$

$$\eta[c \mapsto \alpha, x \mapsto u], w_3 \Vdash c{:}x \tag{4.3}$$

$$\eta[c \mapsto \alpha], w \Vdash \mathcal{E}c$$

$$\eta, w \Vdash \forall c \forall x \mathbf{P}\big((\pi_1{:}\tau_1 \vee \pi_2{:}\tau_2) \wedge \mathbf{H}\neg c{:}x\big) \quad \text{negation of (3)}$$

$$\eta[c \mapsto \alpha, x \mapsto u], w \Vdash \mathbf{P}\big((\pi_1{:}\tau_1 \vee \pi_2{:}\tau_2) \wedge \mathbf{H}\neg c{:}x\big)$$

$$\eta[c \mapsto \alpha, x \mapsto u], w' \Vdash (\pi_1{:}\tau_1 \vee \pi_2{:}\tau_2) \wedge \mathbf{H}\neg c{:}x \qquad \text{where } w \succ w'$$

$$\eta[c \mapsto \alpha, x \mapsto u], w' \Vdash \pi_1{:}\tau_1 \vee \pi_2{:}\tau_2 \tag{4.4}$$

$$\eta[c \mapsto \alpha, x \mapsto u], w' \Vdash \mathbf{H}\neg c{:}x \tag{4.5}$$

From (4.4) we have $\eta[c \mapsto \alpha, x \mapsto u], w' \Vdash \pi_1{:}\tau_1$ or $\eta[c \mapsto \alpha, x \mapsto u], w' \Vdash \pi_2{:}\tau_2$. Since c and x does not occur in π_1, π_2, τ_1 and τ_2 we have $\eta, w' \Vdash \pi_1{:}\tau_1$

or $\eta, w' \Vdash \pi_2{:}\tau_2$. And since $\pi_1{:}\tau_1$ and $\pi_2{:}\tau_2$ are weak $\succ$-nominals of (CH.3) + (CN) + (∀E∃C), by (4.1) and (4.2) we have $w' = w_1$ or $w' = w_2$, so then (4.5) says

$$\eta[c \mapsto \alpha, x \mapsto u], w_1 \Vdash \mathbf{H}\neg c{:}x \qquad \text{or} \qquad \eta[c \mapsto \alpha, x \mapsto u], w_2 \Vdash \mathbf{H}\neg c{:}x$$

Either way, $\eta[c \mapsto \alpha, x \mapsto u], w_3 \Vdash \neg c{:}x$ which contradicts to (4.3).

Canonicity: Let L be a logic that extends (CH.3) + (CN) + (∀E − ∃C) and contains the formula scheme G'. We have to prove that in every canonical model $\mathfrak{M}^{\Gamma}_{\mathrm{L}}$,

$$\Gamma \succ \Gamma_1 \text{ and } \Gamma \succ \Gamma_2 \Rightarrow (\exists \Gamma_3) w_1 \succ w_3 \text{ and } w_2 \succ w_3 \text{ and } (\exists [\![\mathsf{c}]\!])\Gamma, \Gamma_3 \in \mathrm{wline}_{[\![\mathsf{c}]\!]}$$

Suppose that $\Gamma \succ \Gamma_1$ and $\Gamma \succ \Gamma_2$. The plan is

1. Using the formula we will show that there are closed terms c, τ and canonical worlds Γ_3, Γ'_3 s.t.

$$\mathsf{c}{:}\tau \in \Gamma_3 \supseteq \mathbf{H}^-(\Gamma_1) \qquad \text{and} \qquad \mathsf{c}{:}\tau \in \Gamma'_3 \supseteq \mathbf{H}^-(\Gamma_2)$$

2. By that we have that $\Gamma_1 \succ \Gamma_3$, $\Gamma_2 \succ \Gamma'_3$ and $(\exists [\![c]\!])\Gamma, \Gamma_3, \Gamma'_3 \in \mathrm{wline}_{[\![c]\!]}$.
3. Since the canonical model is a model of (CH.3) + (CN) + (∀E∃C), $\mathsf{c}{:}\tau$ is a weak $\succ$-nominal of that logic, so $\Gamma_3 = \Gamma'_3$, which completes the proof.

By ((∀E∃C)), we can take closed terms c_1, τ_1, c_2 and τ_1 for which $\mathsf{c}_1{:}\tau_1 \in \Gamma_1$ and $\mathsf{c}_2{:}\tau_2 \in \Gamma_2$. Then we have that

$$\begin{array}{rlll}
 & \Gamma_1 \vdash_{\mathrm{L}} & \mathsf{c}_1{:}\tau_1 & \\
 & \Gamma_2 \vdash_{\mathrm{L}} & \mathsf{c}_2{:}\tau_2 & \\
 & \Gamma \vdash_{\mathrm{L}} & \mathbf{P}\mathsf{c}_1{:}\tau_1 \wedge \mathbf{P}\mathsf{c}_2{:}\tau_2 & \mathbf{P}^+\text{-lemma} \\
 & \Gamma \vdash_{\mathrm{L}} & \exists c \exists x\, \mathbf{H}\big((\mathsf{c}_1{:}\tau_1 \vee \mathsf{c}_2{:}\tau_2) \to \mathbf{P}c{:}x\big) & \text{G'} \\
 & \Gamma \vdash_{\mathrm{L}} & \mathbf{H}\big((\mathsf{c}_1{:}\tau_1 \vee \mathsf{c}_2{:}\tau_2) \to \mathbf{P}\mathsf{c}{:}\tau\big) & \text{richness} \\
 & \Gamma_1 \vdash_{\mathrm{L}} & (\mathsf{c}_1{:}\tau_1 \vee \mathsf{c}_2{:}\tau_2) \to \mathbf{P}\mathsf{c}{:}\tau & \\
 & \Gamma_2 \vdash_{\mathrm{L}} & (\mathsf{c}_1{:}\tau_1 \vee \mathsf{c}_2{:}\tau_2) \to \mathbf{P}\mathsf{c}{:}\tau & \\
 & \Gamma_1 \vdash_{\mathrm{L}} & \mathbf{P}\mathsf{c}{:}\tau & \\
 & \Gamma_2 \vdash_{\mathrm{L}} & \mathbf{P}\mathsf{c}{:}\tau & \\
\exists \Gamma_3 \prec \Gamma_1 & \Gamma_3 \vdash_{\mathrm{L}} & \mathsf{c}{:}\tau & \text{Existence lemma} \\
\exists \Gamma'_3 \prec \Gamma_2 & \Gamma'_3 \vdash_{\mathrm{L}} & \mathsf{c}{:}\tau & \text{Existence lemma}
\end{array}$$

■

4.3.2 A BCL with weak nominals

Theorem 58. *In* BCL *that contains the axioms*

$$\begin{array}{ll}
\text{(CG)} & (\mathbf{P}\pi_1{:}\tau_1 \wedge \mathbf{P}\pi_2{:}\tau_2) \to \exists c \exists x\, \mathbf{H}\big((\pi_1{:}\tau_1 \vee \pi_2{:}\tau_2) \to \mathbf{P}c{:}x\big) \\
 & \qquad\qquad\qquad\qquad \scriptstyle\text{where } c \text{ and } x \text{ do not occur in } \pi_1,\ \pi_2,\ \tau_1 \text{ and } \tau_2 \\
\text{(CG.3)} & \mathbf{G}_\pi(\underline{\mathbf{G}}_\pi\varphi \to \psi) \vee \mathbf{G}_\pi(\underline{\mathbf{G}}_\pi\psi \to \varphi) \\
\text{(CH.3)} & \mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\varphi \to \psi) \vee \mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\psi \to \varphi) \\
\text{(CN)} & a{:}x \to \mathbf{H}_a\neg a{:}x \\
(\forall\text{E}\exists\text{C}) & \exists a \mathcal{E} a \\
\text{(H4)} & \mathbf{H}\varphi \to \mathbf{H}\mathbf{H}\varphi
\end{array}$$

the pointing statements are weak nominals, if we consider only connected models.

Proof. Let $\mathfrak{M}$ be a connected model, and let η and η be arbitrary but fixed. We have to prove that for any pointing statement $\pi{:}\tau$, the intension $[\![\pi{:}\tau]\!]$ is either a singleton or the empty set. Suppose that it has more elements than one. So we have two distinct elements of this set; let us call them w and v. Since in a connected model every element is accessible with any point in finitely many steps on the alternative relations $\succ$ and $\prec$, w has access to v in finitely many steps. Then since (H4) is canonical for transitivity, we can reduce these steps to finite zigzag-chains, like

$$wRw_1R^{-1}w_2R\dots w_nR^{-1}v$$

where $R \in \{\succ, \prec\}$. (Or with a different ending). By the presence of the axiom (CG), the model is convergent, and using this property and the transitivity again, we can reduce these chains to

$$wRuR^{-1}v$$

Then we can apply the previous argumentation as before, but now with (CG.3). ■

Remark 59. This system

- requires the whole temporal language,
- tolerates black holes,
- does not tolerate branching: In BST-s, pointing statements should be only weak $\succ$-nominals, since the point of branching space is that "The event in which $\pi{:}\tau$ and the result of the coin-flipping is head" is different than "The event in which $\pi{:}\tau$ when the result of the coin-flipping is tail", and both of them exists and none of them is in the causal past of the other.

4.3.3 A BCL with nominals and hybrid operators

Theorem 60. *In every* BCL *containing the axiom system* CL_0

(CG)	$(\mathbf{P}\pi_1{:}\tau_1 \wedge \mathbf{P}\pi_2{:}\tau_2) \to \exists c \exists x\, \mathbf{H}\big((\pi_1{:}\tau_1 \vee \pi_2{:}\tau_2) \to \mathbf{P}c{:}x\big)$ where c and x do not occur in π_1, π_2, τ_1 and τ_2
(CG.3)	$\mathbf{G}_\pi(\underline{\mathbf{G}}_\pi\varphi \to \psi) \vee \mathbf{G}_\pi(\underline{\mathbf{G}}_\pi\psi \to \varphi)$
(CH.3)	$\mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\varphi \to \psi) \vee \mathbf{H}_\pi(\underline{\mathbf{H}}_\pi\psi \to \varphi)$
(CN)	$a{:}x \to \mathbf{H}_a \neg a{:}x$
(∀E∃C)	$\exists a \mathcal{E} a$
(4)	$\mathbf{H}\varphi \to \mathbf{H}\mathbf{H}\varphi$
(∀C∃E)	$a{:}x \wedge x < y \to \mathbf{F}a{:}y$
	$a{:}x \wedge x > y \to \mathbf{P}a{:}y$
(ZZ)	$\mathbf{P}\mathbf{F}\mathcal{E}\pi$
(Trichotomy)	$x < y \vee x > y \vee x = y$

the following statements are true if we restrict ourselves to connected models*:*

1. *The 'somewhere' operator* E *is definable as* $\mathsf{E}\varphi \overset{\text{def}}{\Leftrightarrow} \mathbf{PF}\varphi$*, i.e.,*

$$\mathfrak{M}, \eta, w \models \mathsf{E}\varphi \iff (\exists v)\ \mathfrak{M}, \eta, v \models \varphi$$

(or also in the sense of Def. 20.)

2. *Pointing statements* $\pi{:}\tau$ *are nominals.*

3. *A hybrid sort with satisfaction and save operator is definable in the sense of Def. 19 and Def. 20.*

Remark 61.

- Point 2 means that all the intensions $[\![\pi{:}\tau]\!]^{\mathfrak{M}}_{\eta}$ are singletons.
- Point 3 implies that for all connected $\mathfrak{M}$, with any η, w, φ and $a{:}x$

$$\mathfrak{M}, \eta, w \models @_{a:x}\varphi \iff \mathfrak{M}, \eta, \left([\![a{:}x]\!]^{\mathfrak{M}}_{\eta}\right)^{-} \models \varphi$$

- Point 3 implies that for all connected $\mathfrak{M}$, with any η, w, φ and $a{:}x$ *if* a *and* x *only occur together in* φ*, i.e.,* a *or* x *occurs in a subformula* ψ *of* φ*, then* $a{:}x$ *is a subformula of* ψ*, then for an* $\alpha \in \mathbb{C}$*,* $u \in U$ *for which* $\alpha(w) = u$

$$\mathfrak{M}, \eta, w \models {\downarrow}a{:}x\ \varphi \iff \mathfrak{M}, \eta[a \mapsto \alpha, x \mapsto u], w \models \varphi$$

First we prove the following lemma:

Lemma 62. *The relation*

$$w \curlyvee\!\!\!\!\nearrow v \overset{\text{def}}{\Leftrightarrow} (\exists u)\ v \succ u \prec w$$

is universal on the connected model structures, i.e.,

$$(\forall w)(\forall v) \quad w \curlyvee\!\!\!\!\nearrow v$$

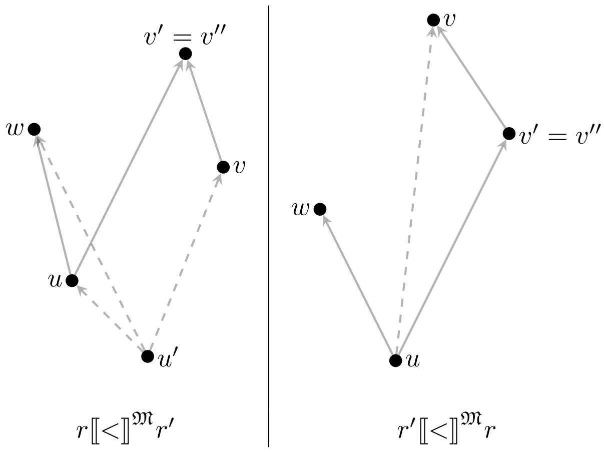
Proof. Let $\mathfrak{M}$, η, η, w and v be arbitrary but fixed. By ($\forall$E$\exists$C) we have that there is an $\alpha \in \mathbb{C}$ and an $r \in U$ s.t. $\eta[x \mapsto r, a \mapsto \alpha], v \Vdash a{:}x$. Now by (ZZ) we have that $\eta[x \mapsto r, a \mapsto \alpha], w \Vdash \mathbf{PF}\mathcal{E}a$, i.e., there is a world u, v' and a number $r' \in U$ s.t. $w \succ u \prec v'$ and

$$\begin{aligned} \eta[x \mapsto r', a \mapsto \alpha], v' \Vdash a{:}x \iff & \quad \eta[y \mapsto r', a \mapsto \alpha], v' \Vdash a{:}y \\ \iff & \quad \eta[x \mapsto r, y \mapsto r', a \mapsto \alpha], v' \Vdash a{:}y \end{aligned}$$

Now by (Trichotomy) we have that $r[\![<]\!]^{\mathfrak{M}}r'$ or $r'[\![<]\!]^{\mathfrak{M}}r$ or $r = r'$. These cases are illustrated in Fig. 4.1.

- If $r = r'$ then applying (:F) and Theorem 58 which states that pointing statements are weak nominals, we have that $v' = v$ and by that $w \succ u \prec v$.
- If $r[\![<]\!]^{\mathfrak{M}}r'$ then in v we have that $\eta[x \mapsto r, y \mapsto r', c \mapsto \alpha], v \Vdash c{:}x \wedge x < y$, therefore by ($\forall$C$\exists$E), $\eta[x \mapsto r, y \mapsto r', c \mapsto \alpha], v \Vdash \mathbf{F}c{:}y$, and by that, that there is a $v'' \succ v$ s.t. $\eta[x \mapsto r, y \mapsto r', c \mapsto \alpha], v'' \Vdash \mathbf{c}{:}y$. But by Theorem 58 pointing statements are weak nominals, $v'' = v'$, so $v \prec v' \succ u$. By the chronological convergence provided by (CH.3) our frame is convergent as well, so there is a u' s.t. $v \succ u' \prec u$. By $u' \prec u \prec w$ and the transitivity provided by (4) we have that $w \succ u' \prec v$.

Figure 4.1: Illustration of the proof of Lemma 62.



- If $r'[\![<]\!]^{\mathfrak{M}} r$ then in v we have that $\eta[x \mapsto r, y \mapsto r', c \mapsto \alpha], v' \Vdash c{:}y \wedge y < x$, therefore by (∀C∃E), $\eta[x \mapsto r, y \mapsto r', c \mapsto \alpha], v' \Vdash \mathbf{F}c{:}x$, and by that, that there is a $v'' \succ v'$ s.t. $\eta[x \mapsto r, y \mapsto r', c \mapsto \alpha], v'' \Vdash \mathbf{c}{:}y$. But by Theorem 58 pointing statements are weak nominals, $v'' = v$, so $u \prec v' \prec v \succ u$, and by the transitivity provided by (4) we have that $w \succ u \prec v$.

Hereby we proved Lemma 62. Note that we did not use the full power of (∀C∃E) – only one of them is enough to prove that lemma. ■

Now we are ready to prove Theorem 60.

Proof.

1. Point 1 is a consequence of Lemma 62.

2. To prove that pointing statements are nominals, we can use Theorem 58, that is, that pointing statements are weak nominals. So we only have to prove that every pointing statement is true in at least one world. Let $\mathfrak{M}$, η, w, a and x be arbitrary but fixed. We will show that $[\![a{:}x]\!]^{\mathfrak{M}}_{\eta} \neq \varnothing$. From (ZZ) we have that there is a v s.t. $w \curlyvee v$ and $v \in \mathrm{wline}_{\eta(a)}$. Now using (Trichotomy) we have that $\eta(x) < \eta(a)(v)$ or $\eta(x) > \eta(a)(v)$ or $\eta(x) = \eta(a)(v)$. In case of $\eta(x) = \eta(a)(v)$ we are ready, and for the two other cases we can use the two parts of (∀C∃E) to prove the existence of a world that satisfies $a{:}x$.

3. We will use pointing statements to represent the nominals. However, we cannot use all of them; in the light of Remark 61, we should separate the nominal-representative pointing statements and the other uses of pointing statements. To do so, we will use oddly indexed the pointing statements to represent nominals of HCL, i.e., the set

$$F \stackrel{\text{def}}{=} \{a_{2i+1}{:}x_{2i+1} \; : \; i \in \omega\},$$

and the evenly indexed pointing statements are reserved to be the representation of the pointing statements of HCL. Therefore, we use the following variable matching ξ:[1]

$$\xi : \begin{array}{rcl} p_n & \mapsto & p_n \\ x_n & \mapsto & x_{2i} \\ a_n & \mapsto & a_{2i} \\ e_n & \mapsto & \langle a_{2i+1}, x_{2i+1} \rangle \end{array}$$

Now the hybrid sort (and operator) definition will be the following:

$$\begin{array}{rcl} \mathrm{HSD}_\xi(e) & \stackrel{\text{def}}{=} & \xi_1(e) \colon \xi_2(e) \\ \mathrm{HSD}_\xi(\varphi) & \stackrel{\text{def}}{=} & \varphi[x_i/\xi(x_i), a_i/\xi(a_i)] \text{ for any } \varphi \in At - NVar \\ \mathrm{HSD}_\xi(\varphi \wedge \psi) & \stackrel{\text{def}}{=} & \mathrm{HSD}_\xi(\varphi) \wedge \mathrm{HSD}_\xi(\psi) \\ \mathrm{HSD}_\xi(C\varphi) & \stackrel{\text{def}}{=} & C\mathrm{HSD}_\xi(\varphi) \text{ for any } C \in \{\neg, \mathbf{F}, \mathbf{P}\} \\ \mathrm{HSD}_\xi(\exists v \varphi) & \stackrel{\text{def}}{=} & \exists \xi(v) \mathrm{HSD}_\xi(\varphi) \text{ for any } v \in MVar \cup CVar \\ \mathrm{HSD}_\xi(\mathsf{E}\varphi) & \stackrel{\text{def}}{=} & \mathbf{PF}\mathrm{HSD}_\xi(\varphi) \\ \mathrm{HSD}_\xi(@_e\varphi) & \stackrel{\text{def}}{=} & \mathbf{PF}\big(\xi_1(e) \colon \xi_2(e) \wedge \mathrm{HSD}_\xi(\varphi)\big) \\ \mathrm{HSD}_\xi(\downarrow e\varphi) & \stackrel{\text{def}}{=} & \exists \xi_1(e) \exists \xi_2(e) \big(\xi_1(e) \colon \xi_2(e) \wedge \mathrm{HSD}_\xi(\varphi)\big) \end{array}$$

where ξ_1 and ξ_2 are the compositions with the first and second projection functions with ξ. To define the a modell-dependent transformation tr : $\mathfrak{M} \mapsto (\mathrm{H}^+_{\mathfrak{M}} \to \mathrm{H}_{\mathfrak{M}})$ we have to set the denotation of variables a_{2i+1} and x_{2i+1} in a way that

$$\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models a_{2i+1} \colon x_{2i+1} \iff \mathfrak{M}, \eta, w \models e_i \tag{4.6}$$

To do so we have to find a clock $\alpha \in \mathbb{C}$ and a number $u \in U$ witnessing the event $\eta(e)$. The axiom (∀E∃C) ensures the existence of such α and u. By the axiom of choice, there is a choice function $c^\eta : NVar \to \mathbb{C} \times U$ for which

$$c^\eta(e_i) = \langle \alpha_i, u_i \rangle \iff \alpha_i((\eta(e_i))^-) = u_i.$$

As usual, we denote the first and second elements of $c(e_i)$ by $c^\eta_1(e_i)$ and $c^\eta_2(e_i)$, respectively. Using this notation we can reformulate the previous equivalence:

$$c^\eta_1(e_i)(w) = c^\eta_2(e_i) \iff \eta(e_i) = \{w\} \tag{4.7}$$

Now we can use that c^η to define tr:

$$\mathrm{tr}_{\mathfrak{M}}(\eta) : \begin{array}{rcl} p_n & \mapsto & \eta(p_n) \\ a_{2i} & \mapsto & \eta(a_i) \\ a_{2i+1} & \mapsto & c^\eta_1(e_i) \\ x_{2i} & \mapsto & \eta(x_i) \\ x_{2i+1} & \mapsto & c^\eta_2(e_i) \end{array}$$

[1] To follow the definition Def. 19, we could have defined not a variable matching but the function

$$\xi' : \begin{array}{rcl} p_n & \mapsto & p_n \\ x_n & \mapsto & x_{2i} \\ a_n & \mapsto & a_{2i} \\ w_n & \mapsto & a_{2i+1} \colon x_{2i+1} \in F \end{array}$$

Although Def. 19 is more general, we stick with the variable-oriented ξ since it follows more closely the standards of sort definition technics of [Andréka et al. 2001] and [Andréka and Németi 2014].

It is easy to show now that the equivalence (4.6) holds with that tr, but we will prove this also in detail in a moment (see the ‘Nominal variables’ case on p. 54) Now we prove the equivalence

$$\mathfrak{M}, \eta, w \models \varphi \iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(\varphi)$$

by induction:

- Propositional variables: $\mathfrak{M}, \eta, w \models p_i \iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(p_i)$

$$\begin{aligned}
\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(p_i) &\iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models p_i && \text{def.of } \mathrm{HSD}_{\xi} \\
&\iff \mathfrak{M}, \eta, w \models p_i && \eta{\restriction}PrVar = \mathrm{tr}_{\mathfrak{M}}(\eta){\restriction}PrVar
\end{aligned}$$

- Nominal variables: $\mathfrak{M}, \eta, w \models e_i \iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(e_i)$

$$\begin{aligned}
\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(e_i) &\iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models a_{2i+1}{:}x_{2i+1} && \text{def.of } \mathrm{HSD}_{\xi} \\
&\iff \mathrm{tr}_{\mathfrak{M}}(\eta)(a_{2i+1})(w) = \mathrm{tr}_{\mathfrak{M}}(\eta)(x_{2i+1}) && \text{def.of } \models \\
&\iff c_1^{\eta}(e_i)(w) = c_2^{\eta}(e_i) && \text{def.of } \mathrm{tr}_{\mathfrak{M}}(\eta) \\
&\iff \eta(e_i) = \{w\} && \text{by (4.7)} \\
&\iff \mathfrak{M}, \eta, w \models e_i
\end{aligned}$$

- Pointing statements: $\mathfrak{M}, \eta, w \models a_i{:}x_i \iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(a_i{:}x_i)$

$$\begin{aligned}
\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(a_i{:}x_i) &\iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models a_{2i}{:}x_{2i} && \text{def.of } \mathrm{HSD}_{\xi} \\
&\iff \mathrm{tr}_{\mathfrak{M}}(\eta)(a_{2i})(w) = \mathrm{tr}_{\mathfrak{M}}(\eta)(x_{2i}) && \text{def.of } \models \\
&\iff \eta(a_i)(w) = \eta(x_i) && \text{def.of } \mathrm{tr}_{\mathfrak{M}}(\eta) \\
&\iff \mathfrak{M}, \eta, w \models a_i{:}x_i
\end{aligned}$$

- The remaining atomic formulas are similar to the case of pointing statements.
- Conjunction: $\mathfrak{M}, \eta, w \models \varphi \wedge \psi \iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(\varphi \wedge \psi)$

$$\begin{aligned}
&\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(\varphi \wedge \psi) \\
\iff &\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(\varphi) \wedge \mathrm{HSD}_{\xi}(\psi) && \text{def.of } \mathrm{HSD}_{\xi} \\
\iff &\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(\varphi) \text{ and} \\
&\qquad \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(\psi) && \text{def.of } \models \\
\iff &\mathfrak{M}, \eta, w \models \varphi \text{ and } \mathfrak{M}, \eta, w \models \psi && \text{ind.hip.} \\
\iff &\mathfrak{M}, \eta, w \models \varphi \wedge \psi && \text{def.of } \models
\end{aligned}$$

- The remaining $\neg$, $\mathbf{P}$, $\mathbf{F}$ cases are similar to the case of conjunction.
- Quantification: Let v be a clock or mathematical variable, and O be $\mathbb{C}$ or U, respectively.

$$\begin{aligned}
&\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_{\xi}(\exists v_i \varphi) \\
\iff &\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \exists \xi(v_i) \mathrm{HSD}_{\xi}(\varphi) && \text{def.of } \mathrm{HSD}_{\xi} \\
\iff &(\exists o \in O) \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta)[\xi(v_i) \mapsto o], w \models \mathrm{HSD}_{\xi}(\varphi) && \text{def.of } \models \\
\iff &(\exists o \in O) \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta)[v_{2i} \mapsto o], w \models \mathrm{HSD}_{\xi}(\varphi) && \text{def.of } \xi \\
\iff &(\exists o \in O) \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta[v_i \mapsto o]), w \models \mathrm{HSD}_{\xi}(\varphi) && \text{def.of } \mathrm{tr}_{\mathfrak{M}}(\eta) \\
\iff &(\exists o \in O) \mathfrak{M}, \eta[v_i \mapsto o], w \models \varphi && \text{ind.hip.} \\
\iff &\mathfrak{M}, \eta, w \models \exists v_i \varphi && \text{def.of } \models
\end{aligned}$$

- The somewhere operation:

$$
\begin{array}{lll}
& \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_\xi(\mathsf{E}\varphi) & \\
\iff & \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathbf{PF}\mathrm{HSD}_\xi(\varphi) & \text{def.of HSD}_\xi \\
\iff & (\exists v)\mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), v \models \mathrm{HSD}_\xi(\varphi) & \text{Lemma 62} \\
\iff & (\exists v)\mathfrak{M}, \eta, v \models \varphi & \text{ind.hip.} \\
\iff & \mathfrak{M}, \eta, w \models \mathsf{E}\varphi & \text{def.of} \models
\end{array}
$$

- In the light of Remark 41 we left the case of the satisfaction operation @ to the Reader as well.

- Save operation: $\mathfrak{M}, \eta, w \models {\downarrow}e_i\, \varphi \iff \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_\xi({\downarrow}e_i\, \varphi)$
  In this case we follow the following abbreviations

$$
\begin{array}{rcl}
f(\alpha, u) & \stackrel{\text{def}}{=} & \mathrm{tr}_{\mathfrak{M}}(\eta)[a_{2i+1} \mapsto \alpha, x_{2i+1} \mapsto u] \\
f(e_i) & \stackrel{\text{def}}{=} & \mathrm{tr}_{\mathfrak{M}}(\eta)[a_{2i+1} \mapsto c_1^{\eta[e_i \mapsto \{w\}]}(e_i), x_{2i+1} \mapsto c_2^{\eta[e_i \mapsto \{w\}]}(e_i)]
\end{array}
$$

$$
\begin{array}{lll}
& \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \mathrm{HSD}_\xi({\downarrow}e_i\varphi) & \\
\iff & \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \exists\xi_1(e_i)\exists\xi_2(e_i)\big(\xi_1(e_i){:}\xi_2(e_i) \wedge \mathrm{HSD}_\xi(\varphi)\big) & \text{def.of HSD}_\xi \\
\iff & \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta), w \models \exists a_{2i+1}\exists x_{2i+1}\big(a_{2i+1}{:}x_{2i+1} \wedge \mathrm{HSD}_\xi(\varphi)\big) & \text{def.of } \xi \\
\iff & (\exists\alpha \in \mathbb{C})(\exists u \in U)\mathfrak{M}, f(\alpha, u), w \models a_{2i+1}{:}x_{2i+1} \wedge \mathrm{HSD}_\xi(\varphi) & \text{def.of} \models \\
\iff & (\exists\alpha \in \mathbb{C})(\exists u \in U)\ \alpha(w) = u \text{ and } \mathfrak{M}, f(\alpha, u), w \models \mathrm{HSD}_\xi(\varphi) & \text{def.of} \models \\
\Leftarrow & \mathfrak{M}, f(e_i), w \models \mathrm{HSD}_\xi(\varphi) & \text{def.of } c^{\eta[e_i \mapsto \{w\}]} \\
\iff & \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta[e_i \mapsto \{w\}]), w \models \mathrm{HSD}_\xi(\varphi) & \text{def.of } \mathrm{tr}_{\mathfrak{M}}(\eta) \text{ and } c^{\eta[e_i \mapsto \{w\}]} \\
\iff & \mathfrak{M}, \eta[e_i \mapsto \{w\}], w \models \varphi & \text{ind.hip} \\
\iff & \mathfrak{M}, \eta, w \models {\downarrow}e_i\varphi & \text{def.of} \models
\end{array}
$$

  The $\Rightarrow$ direction follows from the following

  Lemma 63. *Whenever we have an $\langle\alpha, u\rangle \in \mathbb{C} \times U$ such that*
  (1) $\alpha(w) = u$,
  (2) $\mathfrak{M}, f(\alpha, u), w \models \mathrm{HSD}_\xi(\varphi)$,
  then we can choose $\langle\alpha, u\rangle$ to be $c^{\eta[e_i \mapsto \{w\}]}(e_i)$, i.e., $f(\alpha, u)$ to be $f(e_i)$.

  Proof. Suppose that we have such an α and u. We prove by (a nested) induction on formulas. Notice that by the definition of HSD_ξ oddly indexed variables occur only in the the pointing statements $a_{2i+1}{:}\, x_{2i+1}$ and in the double quantification $\exists a_{2i+1}\exists x_{2i+1}$. Therefore, since the subject of assignment-modification does not occur in atomic formulas other than $a_{2i+1}{:}\, x_{2i+1}$, the lemma is trivially true for these atomic formulas. For $a_{2i+1}{:}\, x_{2i+1}$, i.e., for $\mathrm{HSD}_\xi(e_i)$, we have to show that

$$
\begin{array}{lll}
& \mathfrak{M}, f(e_i), w \models a_{2i+1}{:}\, x_{2i+1} & \\
\iff & \mathfrak{M}, \mathrm{tr}_{\mathfrak{M}}(\eta[e_i \mapsto \{w\}]), w \models \mathrm{HSD}_\xi(e_i) & \text{def.of } \mathrm{tr}_{\mathfrak{M}}(\eta) \\
\iff & \mathfrak{M}, \eta[e_i \mapsto \{w\}], w \models e_i. & (4.6)
\end{array}
$$

  but the last is true by definition. Now the induction goes again straightforward for those formula construction that do not involve the oddly indexed clock and mathematical variables. For the double quantification, i.e., for the translation of ${\downarrow}e_i\varphi$ the proof is the following:

$$\begin{array}{lll}
 & \mathfrak{M}, f(\alpha,u), w \models \mathrm{HSD}_\xi(\downarrow e_i\varphi) & \text{assumption} \\
\iff & \mathfrak{M}, f(\alpha,u), w \models \exists a_{2i+1}\exists x_{2i+1}\big(a_{2i+1}\colon x_{2i+1} \wedge \mathrm{HSD}_\xi(\varphi)\big) & \text{def.of HSD}_\xi \\
\iff & (\exists\alpha\in\mathbb{C})(\exists u\in U)\ \alpha(w)=u & \\
 & \mathfrak{M}, f(\alpha,u), w \models \mathrm{HSD}_\xi(\varphi) & \text{def.of} \models \\
\iff & \mathfrak{M}, f(e_i), w \models \mathrm{HSD}_\xi(\varphi) & \text{ind.hip.} \\
\iff & c_1^{\eta[e_i\mapsto\{w\}]}(e_i)(w) = c_2^{\eta[e_i\mapsto\{w\}]}(e_i) \text{ and} & \\
 & \mathfrak{M}, f(e_i), w \models \mathrm{HSD}_\xi(\varphi) & (4.7) \\
\iff & c_1^{\eta[e_i\mapsto\{w\}]}(e_i)(w) = c_2^{\eta[e_i\mapsto\{w\}]}(e_i) \text{ and} & \\
 & \mathfrak{M}, f(e_i), w \models a_{2i+1}\colon x_{2i+1} \wedge \mathrm{HSD}_\xi(\varphi) & \text{def.of} \models \\
\iff & \mathfrak{M}, f(e_i), w \models \exists a_{2i+1}\exists x_{2i+1}\big(a_{2i+1}\colon x_{2i+1} \wedge \mathrm{HSD}_\xi(\varphi)\big) & \text{def.of} \models \\
\iff & \mathfrak{M}, f(e_i), w \models \mathrm{HSD}_\xi(\downarrow e_i\varphi) & \text{def.of HSD}_\xi
\end{array}$$

Thus we proved Lemma. 63. ■

Thus we proved Thm. 60. ■

Corollary 64. *Theorem 60 holds for the propositional variable-free version of* BCL*.*

Corollary 65. *Any classical theory extending the standard image of* CL_0*,* (UP) *and* $\forall e\forall e'\, e \,\overset{\uparrow}{\vee}\, e'$ *is expressible in* CL_0 *in the following sense:*
For any $\Gamma \subseteq \mathcal{L}_P^{CFOL}$*,* $\varphi \in \mathcal{L}_P^{CFOL}$*,*

$$\text{if } (UP) \cup \{\forall e \mathrm{ST}_e(\psi) : \psi \in \mathsf{CL}_0\} \cup \{\forall e\forall e'\, e \,\overset{\uparrow}{\vee}\, e'\} \cup \Gamma \vdash \varphi,$$

$$\text{then } \mathsf{CL}_0 \cup \{\mathrm{HSD}_\xi \circ \mathrm{HT}(\gamma) \ :\ \gamma \in \Gamma\} \vdash \mathrm{HSD}_\xi \circ \mathrm{HT}(\varphi).$$

Remark 66. Corollary 65 basically says that instead of building axiom systems in extensions of CL_0, we can do that in its corresponding classical language as well, since no matter what axiom or set of axioms do we take there, we can translate it back into the modal language of BCL.

Chapter 5

First-order Axiomatizations

5.1 Classical language and models

5.1.1 Logical abbreviations

Notation 31 (Vector-notation, projections)**.** If $\vec{x} = \langle x_1, \ldots, x_n \rangle$, then we denote the ith member of $\vec{x}$ by $\vec{x}_i$ or $(\vec{x})_i$.

If f is a function with a codomain of some set of n-tuples, then for any $1 \leq k \leq n$,

$$f_k(\vec{x}) \stackrel{\text{def}}{=} (f(\vec{x}))_k$$

We will use the following abbreviations as well: If $i \leq j \leq n$, then for any n-tuple $\vec{x}$,

$$f_{i-j}(\vec{x}) \stackrel{\text{def}}{=} \langle v_i(\vec{x}), v_{i+1}(\vec{x}), \ldots, v_j(\vec{x}) \rangle$$

$$f_{i_1,i_2,\ldots,i_n}(\vec{x}) \stackrel{\text{def}}{=} \langle v_{i_1}(\vec{x}), v_{i_2}(\vec{x}), \ldots, v_{i_n}(\vec{x}) \rangle$$

We also use the vector-notation in syntax; if P is an n-ary predicate then

$$P(\langle x_1, \ldots, x_n \rangle) \stackrel{\text{def}}{=} P(x_1, \ldots, x_n)$$

Notation 32 (Bounded quantifications)**.** We use the $\in$ symbol and binary relations to bound quantification:

$$\begin{array}{rcl}
(\forall v_1, v_2, \ldots, v_n \in \varphi)\psi & \stackrel{\text{def}}{\Leftrightarrow} & \forall v_1, \ldots, v_n((\varphi(v_1) \wedge \varphi(v_2) \wedge \cdots \wedge \varphi(v_n)) \rightarrow \psi) \\
(\forall \langle v_1, v_2, \ldots, v_n \rangle \in \varphi)\psi & \stackrel{\text{def}}{\Leftrightarrow} & \forall v_1, \ldots, v_n(\varphi(v_1, v_2, \ldots, v_n) \rightarrow \psi) \\
(\forall v_2 \varphi v_1)\psi & \stackrel{\text{def}}{\Leftrightarrow} & \forall v_2(\varphi(v_1, v_2) \rightarrow \psi)
\end{array}$$

In Chapter 5.2, we will frequently define functions in the object language, but most of the time these functions will be partial. The following notational conventions will make the life easier there.

Notation 33 (Functions, partial functions)**.** Let v an arbitrary variable, and $\vec{v}$ is an n-tuple of arbitrary variables. A formula $F(\vec{v}, v')$ is a *function* in the system Γ, iff

$$\Gamma \vdash \exists y(F(\vec{v}, v_1) \wedge \forall z(F(\vec{v}, v_2) \rightarrow v_1 = v_2)),$$

We call $\mathrm{F}(\vec{w}, \vec{a}, \vec{x}, y)$ a *partial function* in Γ, if

$$\Gamma \vdash \forall y, z(\mathrm{F}(\vec{w}, \vec{a}, \vec{x}, y) \wedge \mathrm{F}(\vec{w}, \vec{a}, \vec{x}, z) \rightarrow y = z).$$

We refer to the only v' which satisfy $\varphi(\vec{v}, v')$ with the lower case, one-argumentless $f(\vec{v})$. Formally:

$$\varphi(f(\vec{v})) \overset{\text{def}}{\Leftrightarrow} \exists y(F(\vec{v}, v') \wedge \varphi(v'))$$

So if $F(\vec{w}, \vec{a}, \vec{x}, y)$ is only a partial function, then the truth of $\varphi(\mathrm{f}(\vec{w}, \vec{a}, \vec{x}))$ implies that $f(\vec{w}, \vec{a}, \vec{x})$ *is defined*, and has the property φ. Roughly speaking, using this notation, we will never have to excuse ourselves using partial functions.

5.1.2 Classical language of clocks

We are going to use the following classical first-order 3-sorted language:

- Symbols:
  - Pointer variables: $a, b, c, \ldots$ $\quad ClVar \overset{\text{def}}{=} \{a_i : i \in \omega\}$
  - Mathematical variables: $x, y, z, \ldots$ $\quad MVar \overset{\text{def}}{=} \{x_i : i \in \omega\}$
  - Event variables: $e, e', e'', \ldots$ $\quad NVar \overset{\text{def}}{=} \{e_i : i \in \omega\}$
  - Mathematical function and relation symbols: $+$, $\cdot$, $\leq$
  - Event predicate: $\prec$
  - Clock predicate: In (Optional)
  - Intersort predicate: P
  - Logical symbols: $\neg, \wedge, =, \exists$
- Terms:

$$\tau ::= \; x \;\mid\; \tau_1 + \tau_2 \;\mid\; \tau_1 \cdot \tau_2$$

- Formulas:

$$\varphi ::= \; a = b \mid \tau = \tau' \mid \tau \leq \tau' \mid e = e' \mid e \prec e' \mid \mathrm{In}(a) \mid \mathrm{P}(e, a, \tau) \mid \\ \neg\varphi \mid \varphi \wedge \psi \mid \exists x\varphi \mid \exists a\varphi \mid \exists e\varphi$$

On rare occasions we will denote event variables with symbols different from e, e', $e_1, \ldots$ In these cases, the context always clarifies that the used symbols refer to event variables.

Remark 67. In the light of Theorem **??**, we could explicitly define inertial observers as the geodetic observers. We do not choose this way, by the following reasons: A tétel még készülőben

- We are able to construct our axiomatizations without using the very special geodetic property, and using the more geometrical 'line-like' properties of inertials/geodetics.
- We think that the equivalence of inertiality and geodeticity in Minkowski spacetimes should be on the "theoremhood" rather than the "assumption" side of an axiomatic approach to relativity theories.

Abbreviations

$$
\begin{array}{rcl@{\qquad}rcl}
a(e)=\tau & \stackrel{\text{def}}{\Leftrightarrow} & \mathrm{P}(a,e,\tau) & e \ll e' & \stackrel{\text{def}}{\Leftrightarrow} & e \prec e' \land \exists a(e\mathcal{E}a \land e'\mathcal{E}a) \\
e\mathcal{E}a & \stackrel{\text{def}}{\Leftrightarrow} & \exists x\, \mathrm{P}(a,e,x) & e \underline{\ll} e' & \stackrel{\text{def}}{\Leftrightarrow} & e \ll e' \lor e = e' \\
\mathrm{wline}_a & \stackrel{\text{def}}{=} & \{e : \exists x\, \mathrm{P}(a,e,x)\} & e \nearrow e' & \stackrel{\text{def}}{\Leftrightarrow} & e \prec e' \land \neg\exists a(e\mathcal{E}a \land e'\mathcal{E}a) \\
\mathrm{D}_e & \stackrel{\text{def}}{=} & \{a : \exists x\, \mathrm{P}(a,e,x)\} & e \underset{=}{\nearrow} e' & \stackrel{\text{def}}{\Leftrightarrow} & e \nearrow e' \lor e = e' \\
a \approx b & \stackrel{\text{def}}{\Leftrightarrow} & \forall e(e\mathcal{E}a \leftrightarrow e\mathcal{E}b) & \overrightarrow{e_1e_2e_3} & \stackrel{\text{def}}{\Leftrightarrow} & e_1 \nearrow e_2 \land e_2 \nearrow e_3 \land e_1 \nearrow e_3
\end{array}
$$

D_e is the *domain* of event e, the relation $a \approx b$ is referred as the *cohabitation* of clocks a and b, and $\overrightarrow{e_1e_2e_3}$ is the *directed lightlike betweenness* predicate.

5.1.3 Intended classical clock models

$$\mathfrak{M}^c = \left(\mathbb{R}^4, \mathbb{C}, \mathbb{R}, \prec^{\mathfrak{M}^c}, \mathrm{In}^{\mathfrak{M}^c} +, \cdot, \leq, \mathrm{P}^{\mathfrak{M}^c}\right)$$

where

- $\mathbb{C}$ is the set of those $\alpha : \mathbb{R}^4 \to \mathbb{R} \cup \{\Theta\}$, for which α^{-1}-s are timelike curves that follows the measure system of $\mathbb{R}^4$, i.e.,
  - α^{-1} is differentiable function w.r.t. Euclidean metric:

$$(\forall x \in U)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in U)$$
$$|x-y| \leq \delta \Rightarrow \frac{|\alpha^{-1}(x) - \alpha^{-1}(y)|}{|x-y|} \leq \varepsilon,$$

  - $(\alpha^{-1})'$ is continuous:

$$(\forall x \in U)(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in U)$$
$$|x-y| < \delta \Rightarrow \left|(\alpha^{-1})'(x) - (\alpha^{-1})'(y)\right| < \varepsilon$$

  - $(\alpha^{-1})'$ is timelike: $\mu \circ \left(\alpha^{-1}\right)'(x) > 0$ for all $x \in \mathbb{R}$.
  - Measure system of $\mathbb{R}^4$: $\mu(\alpha^{-1}(x), \alpha^{-1}(x+y)) = y$.
- $\vec{x} \prec^{\mathfrak{M}^c} \vec{y} \stackrel{\text{def}}{\Leftrightarrow} \mu(\vec{x}, \vec{y}) \geq 0$ and $x_1 < y_1$,
- $\mathrm{In}^{\mathfrak{M}^c} \stackrel{\text{def}}{=} \left\{ \alpha \in \mathbb{C} : \begin{array}{r} (\exists x, y \in U)(\forall z \in U)(\exists \lambda \in U) \\ \alpha^{-1}(z) = \alpha^{-1}(x) + \lambda \cdot (\alpha^{-1}(x) - \alpha^{-1}(y)) \end{array} \right\}$
- $\mathrm{P}^{\mathfrak{M}^c} = \{\langle \vec{x}, \alpha, y \rangle \in \mathbb{R}^4 \times \mathbb{C}_I \times \mathbb{R} \,:\, \alpha(\vec{x}) = y\}$,

The non-accelerating intended model $\mathfrak{M}^c_{\mathrm{I}}$ is the largest submodel of $\mathfrak{M}^c$ in which the domain of clocks is $\mathrm{In}^{\mathfrak{M}^c}$.

5.1.4 Goals

- Construct coordinate systems for inertial clocks.
- Construct coordinate systems for accelerating clocks.
- Find axiomatic base SClTh for these coordinate construction procedures.

- Extend SClTh into a complete axiomatization of $\mathrm{Th}(\mathfrak{M}^c_I)$.
- Extend SClTh into a complete axiomatization of $\mathrm{Th}(\mathfrak{M}^c)$ or show that it cannot be axiomatized.
- Compare $\mathrm{Th}(\mathfrak{M}^c_I)$ to SpecRel in terms of definitional equivalences.
- Compare $\mathrm{Th}(\mathfrak{M}^c)$ to AccRel in terms of definitional equivalences.

5.2 Coordinatization

In this section we work in $\mathrm{Th}(\mathfrak{M}^c)$.

5.2.1 How to build a coordinate system?

During this section keep in mind that we use mostly partial functions, so recall the remarks on Notation 33.

To define coordinatization we have to create the notions of *space and time relative to observers*. To define notions related to time is not a hard job anymore since we can use the structure of $\mathbb{R}$. To construct observer-relative space and the Coordinatization predicate, we follow ideas similar to the paper of Andréka and Németi [2014]. This idea can be summarized in the following steps:

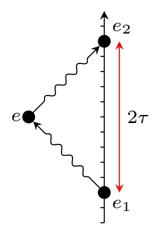
1. **Space:** We define the (spatial) points of clocks. The space of a clock will be the set of its *inertial synchronized co-movers* (or shortly, iscm-s).[1]
2. **Geometry:** We define the *betweenness* and *equidistance* relations, the two primitive relation of Tarski and Givant [1999]. This makes us able to talk about the geometrical structure of the space of any clock.
3. **Coordinate Systems:** We define orthogonality to identify coordinate systems as a 4-tuple of iscm-s, representing the origin and the three direction of the three axes.
4. **Coordinatization:** We use the distances from the axes and a sign-function to build coordinates for every events.
5. **Check:** We check that this coordinatization predicate is good indeed. In Theorem 91 we prove that it is a bijection between W and U^4 for any coordinate system, and in Section 5.3.5 we show that we can use it to interpret the worldview predicate W of SpecRel. We will check this in an axiomatic environment.

5.2.2 Space

Definition 34 (Distances)**.** The distance of an *inertial observer from an event* is defined via signalling, see Fig 5.1.

$$\delta^i(a,e)=\tau \overset{\mathrm{def}}{\Leftrightarrow} \mathrm{In}(a)\wedge\left(\exists e_1,e_2\in \mathrm{wline}_a\right)\left(e_1 \underset{=}{\rightsquigarrow} e \underset{=}{\rightsquigarrow} e_2 \wedge a(e_1)-a(e_2)=2\cdot\tau\right)$$

Figure 5.1: $\delta^i(a,e)=\tau$



[1] Here we note that the notion of space can be given more generally: simple inertial comovers are enough, but the more special synchronized subset simplifies the coordinatization process.

The *distance of inertials* can be defined in the following way:

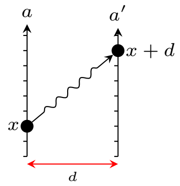
$$\delta^i(a, a') = \tau \overset{\text{def}}{\Leftrightarrow} (\forall w \in \mathrm{wline}_{a'})\delta^i(a, w) = \tau$$

Comovement is defined by having a distance:

$$a \overset{\text{i}}{\upuparrows} a' \overset{\text{def}}{\Leftrightarrow} \exists x \delta^i(a, a') = x$$

According to the theories of intended models, $\delta^i(a, e) = \tau$ is a total function and $\delta^i(a, a') = \tau$ is a partial, but not a total function. We will prove this in Proposition 74 later, when we will have the final axiom system to work with.

Figure 5.2: $a \overset{\text{syn}}{\upuparrows} a'$



Definition 35 (Inertial synchronized co-movers)**.** Clocks a and a' are *inertial synchronised co-movers* iff a' shows $x + \delta^i(a, a')$ whenever a' sees that a shows x. See Fig 5.2.

$$a \overset{\text{syn}}{\upuparrows} a' \overset{\text{def}}{\Leftrightarrow} (\forall w \in \mathrm{D}_a)(\forall w' \in \mathrm{D}'_a) \left(w \underset{=}{\rightsquigarrow} w' \rightarrow a'(w') = a(w) + \delta^i(a, a')\right)$$

(Note that comoving is ensured here by the pseudo-term $\delta^i(a, a')$!)

Now we are able to find representatives for points in spatial geometry for a clock a:

Definition 36 (Space)**.** Inertial synchronized comovers of a clock a will be called a *point* of a, and the set of all points of a will be called the *space* of a:

$$a' \in \mathrm{Space}_a \overset{\text{def}}{\Leftrightarrow} a \overset{\text{syn}}{\upuparrows} a'$$

5.2.3 Geometry

Now we define the basic primitives of [Tarski and Givant 1999] (The axioms can be found here in Table 5.1 on p.73 as well):

Definition 37 (Betweenness, Equidistance, Collinearity)**.** We say that a_2 is *between* a_1 and a_3 iff the shortest route from a_1 to a_3 leads through a_2:

$$\mathrm{B}(a_1, a_2, a_3) \overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) + \delta^i(a_2, a_3) = \delta^i(a_1, a_3)$$

Equidistance stands for equal distances:

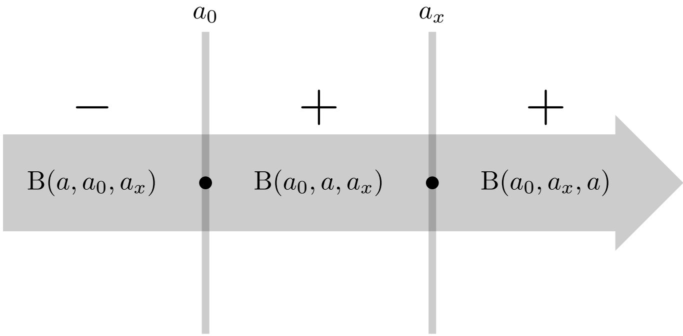
$$a_1 a_2 \equiv a_3 a_4 \overset{\text{def}}{\Leftrightarrow} \delta^i(a_1, a_2) = \delta^i(a_3, a_4)$$

Collinearity is the permutational closure of betweenness:

$$\mathrm{C}(a_1, a_2, a_3) \overset{\text{def}}{\Leftrightarrow} \mathrm{B}(a_1, a_2, a_3) \vee \mathrm{B}(a_3, a_1, a_2) \vee \mathrm{B}(a_2, a_3, a_1)$$

Remark 68. Recall that since δ^i is a partial function, all these relations implies the inertiality and the co-movement of all of its arguments.

Figure 5.4: Definition of the direction function

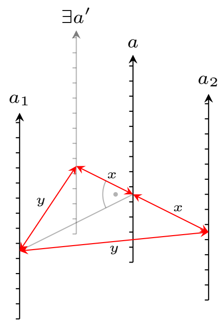


5.2.4 Coordinate systems

Definition 38 (Orthogonality)**.** Distinct lines determined by points a-a_1 and a-a_2 are *orthogonal* iff there is an a' such that a', a_1 and a_2 forms an isoscele triangle and a is in the middle of the segment a' and a_2, see Fig. 5.3:

$$\begin{aligned}\mathrm{Ort}(a, a_1, a_2) \overset{\text{def}}{\Leftrightarrow}\ & \delta^i(a, a_1) > 0 \land \delta^i(a_1, a_2) > 0 \land \delta^i(a, a_2) > 0 \\ & \land \exists a' \big(\mathrm{B}(a_2, a, a') \land \delta^i(a, a_2) = \delta^i(a, a') \land \delta^i(a_1, a_2) = \delta^i(a_1, a')\big)\end{aligned}$$

Figure 5.3: Right angle



Definition 39 (Distances from lines)**.** The distance of a clock a and a line given by the points (a_1, a_2) is τ iff the distance of a and its orthogonal projection on the line (a_1, a_2) is τ.

$$\delta^i(a, (a_1, a_2)) = \tau \overset{\text{def}}{\Leftrightarrow} \exists a'(\mathrm{Ort}(a', a, a_1) \land \mathrm{Ort}(a', a, a_2) \land \delta^i(a, a') = \tau)$$

Definition 40 (Coordinate systems)**.**

$$\mathrm{CoordSys}(a, a_x, a_y, a_z) \overset{\text{def}}{\Leftrightarrow} \mathrm{Ort}(a, a_x, a_y) \land \mathrm{Ort}(a, a_y, a_z) \land \mathrm{Ort}(a, a_x, a_z)$$

Definition 41 (Directed lines)**.** If a line is given by the points (a_0, a_x), then a point a of that line is in *negative direction* if a_0 is between a and a_x, is in *null-direction* if $a = a_0$, and is in *positive direction* otherwise, see Fig. 5.4:

$$\begin{aligned}\mathrm{sign}^-_{a_0,a_x}(a) = \tau \overset{\text{def}}{\Leftrightarrow}\ & (a \neq a_0 \land \mathrm{B}(a, a_0, a_x) \land \tau = -1) \lor (a = a_0 \land \tau = 0) \lor \\ & \big(a \neq a_0 \land (\mathrm{B}(a_0, a, a_x) \lor \mathrm{B}(a_0, a_x, a)) \land \tau = 1\big)\end{aligned}$$

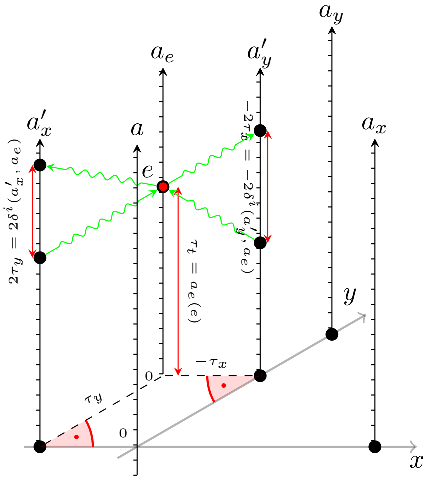
If a is not on the line given by (a_0, a_x), then we say that it is in the negative/null/positive direction iff its orthogonal projection on that line is in the negative/null/positive direction, respectively:

$$\mathrm{sign}_{a_0,a_x}(a) = \tau \overset{\text{def}}{\Leftrightarrow} \exists a'(\mathrm{Ort}(a', a, a_0) \land \mathrm{Ort}(a', a, a_x) \land \mathrm{sign}^-_{a_0,a_x}(a') = \tau)$$

5.2.5 Coordinatization

Definition 42 (Coordinatization)**.** See Fig. 5.5. The event e will be coordinatized on the spatiotemporal position $\langle \tau_t, \tau_x, \tau_y, \tau_z \rangle$ by the coordinate system

Figure 5.5: A 2D illustration of the coordinatization process



$\langle a, a_x, a_y, a_z \rangle$ iff there is a synchronized co-mover a_e of a that shows the time τ_t in e and $\tau_d = \mathrm{sign}_{a,a_d}(a_e) \cdot \delta^i(a_e, a, a_d)$ for $d \in \{x, y, z\}$.

$$
\begin{aligned}
\mathrm{Coord}_{a,a_x,a_y,a_z}(e) = (\tau_t, \tau_x, \tau_y, \tau_z) \;\overset{\text{def}}{\Leftrightarrow}&\\
(\exists a_e \in \mathrm{Space}_a)\Big(\mathrm{CoordSys}(a, a_x, a_y, a_z) \wedge \mathrm{P}(e, a_e, \tau_t) \wedge&\\
\mathrm{sign}_{a,a_x}(a_e) \cdot \delta^i(a_e, (a, a_x)) = \tau_x \wedge&\\
\mathrm{sign}_{a,a_y}(a_e) \cdot \delta^i(a_e, (a, a_y)) = \tau_y \wedge&\\
\mathrm{sign}_{a,a_z}(a_e) \cdot \delta^i(a_e, (a, a_z)) = \tau_z\Big)&
\end{aligned}
$$

5.3 Axiom system SClTh

AxReals The mathematical sort forms a real closed field, see [?]. mit hivatkozzunk?

$$
\begin{array}{llll}
 & (x+y)+z = x+(y+z) & & (x\cdot y)\cdot z = x\cdot(y\cdot z)\\
\exists 0 & x+0 = x & \exists 1 & x\cdot 1 = x\\
\exists(-x) & x+(-x) = 0 & x \neq 0 \to \exists x^{-1} & x\cdot x^{-1} = 1\\
 & x+y = y+x & & x\cdot y = y\cdot x
\end{array}
$$

$$x\cdot(x+y) = (x\cdot y)+(x\cdot z)$$

$$
\begin{array}{rr}
a \le b \wedge b \le a \to a = b & a \le b \to a+c \le b+c\\
a \le b \wedge b \le c \to a \le c & a \le b \wedge 0 \le c \to a\cdot c \le b\cdot c\\
\neg a \le b \to b \le a &
\end{array}
$$

$$\exists x(\forall y \in \varphi)x \le y \to \exists i(\forall y \in \varphi)(i \le y \wedge \forall i'((\forall y \in \varphi)(i' \le y \to i' \le i)) \tag{AxReals}$$

AxFull Every number occurs as a state of any clock in an event.

$$\forall a \forall x \exists e \quad \mathrm{P}(e, a, x) \tag{AxFull}$$

AxExt We do not distinguish between (1) indistinguishable clocks, (2) states of a particular clock in an event and (3) two events where a clock shows the same time.

$$
\begin{array}{lllcl}
(1) & \forall a, a' & \big(\forall e \forall x (\mathrm{P}(e,a,x) \leftrightarrow \mathrm{P}(e,a',x))\big) & \to & a = a' \\
(2) & \forall e \forall a \forall x, y & \big(\mathrm{P}(e,a,x) \land \mathrm{P}(e,a,y)\big) & \to & x = y \\
(3) & \forall e, e' \forall a \forall x & \big(\mathrm{P}(e,a,x) \land \mathrm{P}(e',a,x)\big) & \to & e = e'
\end{array}
\tag{AxExt}
$$

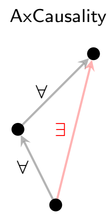
AxForward Clocks are ticking forward.

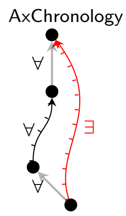
$$\forall a (\forall e, e' \in \mathrm{wline}_a) \quad \big(e \prec e' \leftrightarrow a(e) < a(e')\big) \tag{AxForward}$$

AxSynchron All clocks occupying the same worldline (i.e., cohabitants) use the same measure system, and for every clock, and delay, there is a cohabitant clock with that delay.

$$
\begin{array}{ll}
\forall a (\forall b \approx a) \exists x (\forall e \in \mathrm{wline}_a) & a(e) = b(e) + x \\
\forall a \forall x (\exists b \approx a)(\forall e \in \mathrm{wline}_a) & a(e) = b(e) + x
\end{array}
\tag{AxSynchron}
$$

AxCausality Causality is transitive.

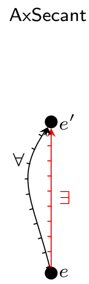


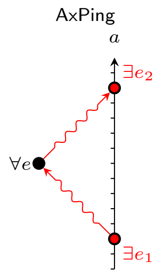


$$(e_1 \prec e_2 \land e_2 \prec e_3) \to e_1 \prec e_3 \tag{AxCausality}$$

AxChronology Interiors of two-way lightcones are filled with clocks crossing through the vertex.

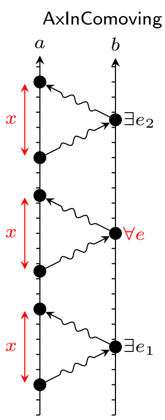
$$(e_1 \preceq e_2 \land e_2 \ll e_3 \land e_3 \preceq e_4) \to e_1 \ll e_4 \tag{AxChronology}$$





AxSecant Any two events that share a clock share an inertial clock as well.

$$e \ll e' \to (\exists a \in \mathrm{In})(e \mathcal{E} a \land e' \mathcal{E} a)) \tag{AxSecant}$$



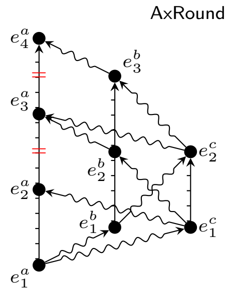
AxInComoving If an inertial clock measures an other inertial clock with the same distance twice, then they are comoving.

$$\big(a, b \in \mathrm{In} \land (\exists e_1, e_2 \in \mathrm{wline}_b)(e_1 \neq e_2 \land \delta^i(a, e_1) = \delta^i(a, e_2))\big) \to a \overset{\mathrm{i}}{\uparrow\uparrow} b \tag{AxInComoving}$$

AxPing Every inertial clock can send and receive a signal to any event.

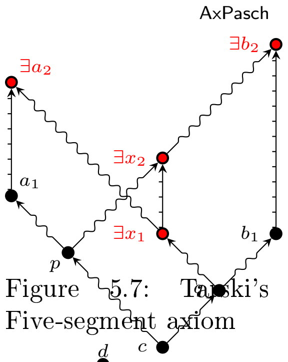
$$(\forall a \in \mathrm{In}) \forall e (\exists e_1, e_2 \in \mathrm{wline}_a) \qquad e_1 \underset{=}{\nearrow} e \underset{=}{\nearrow} e_2 \tag{AxPing}$$

AxRound Given comoving observers a, b and c, the travelling time of simultaneously sent signals on the route $\langle a, b, c, a\rangle$ and $\langle a, c, b, a\rangle$ are (the same, namely,) the average of the travelling time of the $\langle a, c, a\rangle$ and $\langle a, b, c, b, a\rangle$.



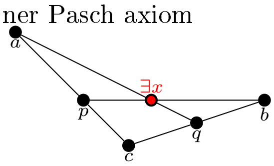
$$b \overset{\mathrm{i}}{\uparrow\uparrow} a \overset{\mathrm{i}}{\uparrow\uparrow} c \wedge \begin{pmatrix} e_1^a, e_2^a, e_3^a, e_3^{a\prime}, e_4^a \in \mathrm{wline}_a \\ e_1^b, e_2^b, e_3^b \in \mathrm{wline}_b \\ e_1^c, e_2^c \in \mathrm{wline}_c \end{pmatrix} \wedge \begin{pmatrix} e_1^a \nearrow e_1^b \nearrow e_2^c \nearrow e_3^a \\ e_1^a \nearrow e_1^c \nearrow e_2^b \nearrow e_3^{a\prime} \\ e_2^c \nearrow e_3^b \nearrow e_4^a \\ e_1^c \nearrow e_2^a \end{pmatrix} \rightarrow$$

$$\rightarrow \left(a(e_3^a) = a(e_3^{a\prime}) = \frac{a(e_2^a) + a(e_4^a)}{2} \right) \quad \text{(AxRound)}$$



AxPasch Pasch axiom for light signals, See Fig. 5.6.

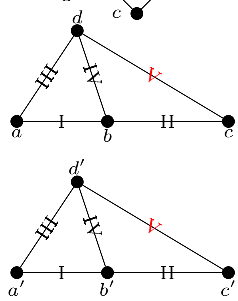
Figure 5.6: Tarski's Inner Pasch axiom

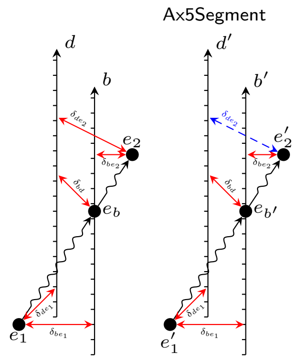


$$\left(a \overset{\mathrm{i}}{\uparrow\uparrow} b \wedge (\exists a_1 \in \mathrm{wline}_a)(\exists b_1 \in \mathrm{wline}_b)(\overrightarrow{cpa_1} \wedge \overrightarrow{cqb_1})\right) \rightarrow$$
$$\rightarrow (\exists x \overset{\mathrm{i}}{\uparrow\uparrow} a)(\exists x_1, x_2 \in \mathrm{wline}_x)(\exists a_2 \in \mathrm{wline}_a)(\exists b_2 \in \mathrm{wline}_b)(\overrightarrow{px_2b_2} \wedge \overrightarrow{qx_1a_2}) \quad \text{(AxPasch)}$$

Ax5Segment If there are two pairs of observers b,d and b',d' such that two light signals $e_1 \nearrow e_2$ and $e_1' \nearrow e_2'$ crosses the worldlines of b and b', respectively, then b and b' agree on the distance of e_2 and e_2, respectively, whenever they agree on the distance of e_1 and d, e_1' and d', respectively. Compare that axiom with Tarski's Five-segment axiom on Fig. 5.7

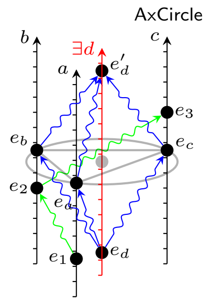
Figure 5.7: Tarski's Five-segment axiom



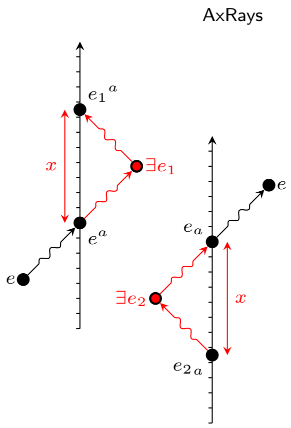


$$d \overset{\mathrm{i}}{\uparrow\uparrow} d' \wedge b \overset{\mathrm{i}}{\uparrow\uparrow} b' \wedge e_b \mathcal{E} b \wedge e_b' \mathcal{E} b' \wedge \overrightarrow{e_1 e_b e_2} \wedge \overrightarrow{e_1' e_b' e_2'} \wedge$$
$$\wedge\, \delta^i(b, e_1) = \delta^i(b', e_1') \wedge \delta^i(b, e_2) = \delta^i(b', e_2') \wedge$$
$$\wedge\, \delta^i(d, e_1) = \delta^i(d', e_1') \wedge \delta^i(b, d) = \delta^i(b', d')) \rightarrow$$
$$\rightarrow \delta^i(d, e_2) = \delta^i(d', e_2') \quad \text{(Ax5Segment)}$$

AxCircle For every three non-collinear inertial observer there is a fourth one that measures them with the same distance.



$$(\forall a, b, c \in \mathrm{In})\big((a \overset{\mathrm{i}}{\uparrow\uparrow} b \overset{\mathrm{i}}{\uparrow\uparrow} c \wedge$$
$$\wedge\, \exists e_1, e_2, e_3(e_1 \mathcal{E} a \wedge e_2 \mathcal{E} b \wedge e_3 \mathcal{E} c \wedge e_1 \nearrow e_2 \nearrow e_3 \wedge \neg e_1 \nearrow e_3)) \rightarrow$$
$$\rightarrow \exists d \exists e_a, e_b, e_c, e_d, e_d'(e_a \mathcal{E} a \wedge e_b \mathcal{E} b \wedge e_c \mathcal{E} c \wedge e_d \mathcal{E} d \wedge e_d' \mathcal{E} d \wedge$$
$$\wedge\, e_d \nearrow e_a \nearrow e_d' \wedge e_d \nearrow e_b \nearrow e_d' \wedge e_d \nearrow e_c \nearrow e_d')\big) \quad \text{(AxCircle)}$$



AxRays For every observer, for any positive x and every direction (given by a light signal) there are lightlike separated events in the past and the future whose distances are exactly x.

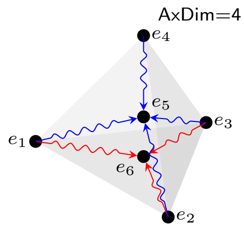
$$(\forall x > 0)\forall a \forall e \exists e_1 \exists e_2(\exists e^a, e_a \in \mathrm{wline}_a)$$
$$\overrightarrow{e_2 e_a e} \wedge \delta^i(a, e_2) = x \wedge \overrightarrow{e e^a e_1} \wedge \delta^i(a, e_1) = x \quad \text{(AxRays)}$$

AxDim$\geq n$ The dimension of the spacetime is at least n. The formula says that $n-1$ lightcones never intersect in only one event.

$$\forall e_1,\ldots,e_n \left(\bigwedge_{i\leq n-1} e_i \rightsquigarrow e_n \rightarrow \exists e_{n+1} \left(\bigwedge_{i\leq n-1} e_i \rightsquigarrow e_n \wedge e_n \neq e_{n+1} \right)\right) \tag{AxDim$\geq n$}$$

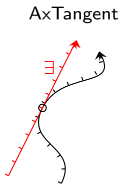
AxDim$\leq n$ The dimension of the spacetime is at most n. The formula says that there are n lightcones that intersect at most in one event.

$$\exists e_1,\ldots,e_{n+1} \left(\bigwedge_{i\leq n} e_i \rightsquigarrow e_{n+1} \wedge \forall e_{n+2} \left(\bigwedge_{i\leq n} e_i \rightsquigarrow e_{n+2} \rightarrow e_{n+1} = e_{n+2} \right)\right) \tag{AxDim$\leq n$}$$



AxDim=4 The dimension of the spacetime is exactly 4; 3 lightcones never intersect in only one event and there are 4 lightcones intersect in at most one event.

$$AxDim{\leq}4 \wedge AxDim{\geq}4 \tag{AxDim=4}$$



AxTangent For every event e of every clock a there is an inertial clock b that occurs in e and its velocity is the same as the local instantaneous velocity of a according to any inertial observer.

(AxTangent)

AxNoAcceleration Every clock is inertial.

$$\forall a \quad \mathrm{In}(a) \tag{AxNoAcceleration}$$

AxAcceleration For every coordinate system $\langle a, a_x, a_y, a_z\rangle$ and every definable timelike curve φ there is a clock having that wordline according to $\langle a, a_x, a_y, a_z\rangle$.

(AxAcceleration)

Definition 43 (Axiom systems)**.** We (re)define SClTh to be the following sets of axioms.

$$\mathrm{SClTh} \stackrel{\text{def}}{=} \left\{ \begin{array}{llll} \mathsf{AxFull} & \mathsf{AxCausality} & \mathsf{AxRays} & \mathsf{Ax5Segment} \\ \mathsf{AxExt} & \mathsf{AxChronology} & \mathsf{AxPing} & \mathsf{AxCircle} \\ \mathsf{AxForward} & \mathsf{AxSecant} & \mathsf{AxRound} & \mathsf{AxDim{=}4} \\ \mathsf{AxSynchron} & \mathsf{AxInComoving} & \mathsf{AxPasch} & \mathsf{AxTangent} \end{array} \right\}$$

$$\mathrm{SClTh}^{\mathrm{NoAcc}} \stackrel{\text{def}}{=} (\mathrm{SClTh} - \{\mathsf{AxSecant},\ \mathsf{AxTangent}\}) \cup \{\mathsf{AxNoAcceleration}\}$$

$$\mathrm{SClTh}^{\mathrm{Acc}} \stackrel{\text{def}}{=} \mathrm{SClTh} \cup \{\mathsf{AxAcceleration}\}$$

5.3.1 Theorems

Our plan is the following:

1. Kronheimer-Penrose axioms: We are working with causal spaces.
2. Signalling (radar-distance) is unique.
3. AxLocExp:For every observer, there is a point (local iscm) in every event. That is equiderivable with AxInComoving.
4. There is a clock in every event (Immediate)
5. Straight signals arrive sooner.
6. $\overset{\text{syn}}{\uparrow\uparrow}$ is an equivalence relation and δ^i is a metric on $\overset{\text{syn}}{\uparrow\uparrow}$-related clocks.
7. There are no two iscms/points in an event.
8. 'Equivalence' of $\overrightarrow{e_a e_b e_c}$ and $\mathrm{B}(a, b, c)$.
9. Tarski's axioms.
10. Coordinatization is a bijection between W and Q^4.
11. Radar-based spatial distance and elapsed time defines the same quantities as coordinate based definition. (Simplifying the coordinate-system based SpecRel axioms)
12. Proving 'Simple-SpecRel'.

Proposition 69. *$\langle W, \preceq, \ll, \underset{=}{\nearrow} \rangle$ is a causal space (see [Kronheimer and Penrose 1967]), i.e., the following statements are all true:*

Assumptions:
AxForward
AxCausality
AxChronology

$$
\begin{aligned}
&e \preceq e\\
&(e_1 \preceq e_2 \wedge e_2 \preceq e_3) \to e_1 \preceq e_3\\
&(e_1 \preceq e_2 \wedge e_2 \preceq e_1) \to e_1 = e_2\\
&\neg e \ll e\\
&e_1 \ll e_2 \to e_1 \preceq e_2\\
&(e_1 \preceq e_2 \wedge e_2 \ll e_3) \to e_1 \ll e_3\\
&(e_1 \ll e_2 \wedge e_2 \preceq e_3) \to e_1 \ll e_3\\
&e_1 \underset{=}{\nearrow} e_2 \leftrightarrow (e_1 \preceq e_2 \wedge \neg e_1 \ll e_2)
\end{aligned}
$$

And the following statements are also hold:

$$\neg e \prec e \tag{5.1}$$

$$(e_1 \prec e_2 \wedge e_2 \ll e_3) \to e_1 \ll e_3 \tag{5.2}$$

$$(e_1 \ll e_2 \wedge e_2 \prec e_3) \to e_1 \ll e_3 \tag{5.3}$$

Proof. All the defining properties of the causal spaces are straightforward consequences of AxCausality, (5.1), (5.2) and (5.3) or true simply by the definitions of $\preceq$, $\ll$ and $\underset{=}{\nearrow}$.

- (5.1) comes from AxForward; $e \prec e$ would lead to $a(e) < a(e)$.
- (5.2): is AxChronology where $e_1 \neq e_2$ and $e_3 = e_4$.

- (5.3): is AxChronology where $e_1 = e_2$ and $e_3 \neq e_4$.

$\blacksquare$

Assumptions:
AxChronology

Proposition 70. *Signalling is unique:*

$$\forall e \forall a (\forall e_a, e'_a \in \mathrm{wline}_a)(e_a \nearrow e \wedge e'_a \nearrow e) \rightarrow e_a = e'_a$$
$$\forall e \forall a (\forall e^a, e^{a\prime} \in \mathrm{wline}_a)(e \nearrow e^a \wedge e \nearrow e^{a\prime}) \rightarrow e^a = e^{a\prime}$$

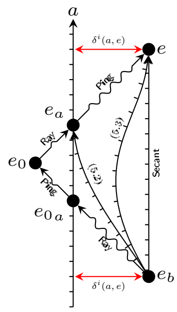
Proof. Suppose that $e_a \neq e'_a$ but $e_a \nearrow e \wedge e'_a \nearrow e$ and $e_a, e'_a \in \mathrm{wline}_a$. Then by definition $e_a \ll e'_a$ or $e_a \gg e'_a$. By AxChronology, $e_a \ll e$ or $e'_a \ll e$ which contradicts to the assumption. The proof is similar for the symmetrical formula as well. $\blacksquare$

The following theorem is equiderivable with AxInComoving above SClTh.

Proposition 71 (AxLocExp)**.** *For every inertial observer, there is a synchronized inertial observer (i.e., a point) in any event.*

Assumptions:
Proposition 69
AxPing
AxRays
AxSecant

$$(\forall a \in \mathrm{In})\forall e \exists b \qquad e\mathcal{E}b \wedge a \overset{\mathrm{syn}}{\uparrow\uparrow} b \tag{AxLocExp}$$



Proof. Let $a \in \mathrm{In}$ and e be arbitrary. If $e \in \mathrm{wline}_a$ then we are ready. Suppose now that $e \notin \mathrm{wline}_a$. By AxPing, there are $e_a, e^a \in \mathrm{wline}_a$ s.t. $e_a \nearrow e \nearrow e^a$. Let $x \stackrel{\mathrm{def}}{=} a(e^a) - a(e_a)$. Note that $\delta^i(a, e) = x$ is true. By AxCausality and AxForward and by the assumption that $e \notin \mathrm{wline}_a$, this x is strictly positive. By AxRays, there is an e_0 s.t. e_0 is 1 distance away from a and $\overrightarrow{e_0 e_a e}$. By AxPing, there is an $e_{0a} \in \mathrm{wline}_a$ s.t. $e_{0a} \nearrow e_0$. By AxRays again, there is an event e_b s.t. $\overrightarrow{e_b e_{0a} e_0}$ and $\delta^i(a, e_b) = x$. Since $e_b \nearrow e_{0a} \ll e_a \nearrow e$, by AxChronology we have $e_b \ll e$. By AxSecant, there is an inertial clock b through e_b and e. Now since both a and b are inertial and $\delta^i(a, e_b) = x$ and $\delta^i(a, e) = x$, by AxInComoving, $a \overset{\mathrm{i}}{\uparrow\uparrow} b$, and by AxSynchron again, there is an a-synchronized b' cohabitant of b here as well; that is the clock having delay x. $\blacksquare$

Proposition 72. *There is a clock in every event.*

Assumptions:
AxFull
AxSecant
Proposition 71

$$\forall e \exists c \quad e\mathcal{E}c$$

Proof. Let e be an arbitrary event. There is a clock a in some event e_0 by AxFull (and by the tautology $\exists a\, a = a$). By AxSecant, there is an inertial clock at e_0 as well. By Proposition 71, there is an inertial comover of a at e. $\blacksquare$

Assumptions:
AxFull
AxSecant
Proposition 71
AxExt

Corollary 73. *The pointing relation* P *is a surjective function* $\mathrm{P} : \mathbb{C} \times U \rightarrow W$.

Proof. It is a function by Proposition AxExt, and is surjective by 72. $\blacksquare$

Assumptions:
AxPing

Proposition 74. $\delta^i(a, e) = \tau$ *is a total function.*

Proof. This is true by AxPing: Since every observer can ping an event, it is always defined, and functionality comes from Proposition 70. $\blacksquare$

Proposition 75. $\delta^i(a, a') = \tau$ *is a partial function.*

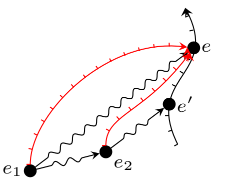
Proof. It is a partial function *by definition.* $\blacksquare$

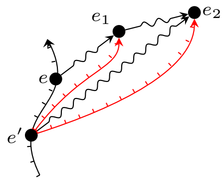
Proposition 76. *Straight signals arrive sooner:*

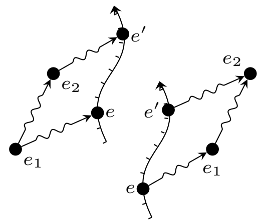
$$\forall a \forall e_1, e_2, e, e' (e\mathcal{E}a \land e'\mathcal{E}a \land e_1 \underset{=}{\nearrow} e \land e_1 \underset{=}{\nearrow} e_2 \underset{=}{\nearrow} e') \to a(e) \le a(e') \tag{5.4}$$
$$\forall a \forall e_1, e_2, e, e' (e\mathcal{E}a \land e'\mathcal{E}a \land e' \underset{=}{\nearrow} e_2 \land e \underset{=}{\nearrow} e_1 \underset{=}{\nearrow} e_2) \to a(e) \le a(e') \tag{5.5}$$

Assumptions:
AxPing
AxCausality
AxChronology
AxForward

Indirect proof







Proof.
- For (5.4) suppose indirectly that $a(e) > a(e')$. Then by AxForward, $e \succ e'$, and since they share the clock a, $e' \ll e$. If $e_2 = e_1$ or $e_2 = e'$ then by AxPing, $e = e'$ or $e = e'$ which contradicts to $e \succ e'$. So we have the chain
$$e_1 \nearrow e_2 \nearrow e' \ll e$$
This implies $e_2 \preceq e'$, and by AxCausality, $e_1 \preceq e'$. From (5.2) we have that (1) $e_2 \ll e$ and then (2) $e_1 \ll e$ which contradicts to $e_1 \nearrow e$.
- For (5.5) suppose indirectly that $a(e') < a(e)$. Then by AxForward, $e' \prec e$, and since they share the clock a, $e' \ll e$. If $e_1 = e$ or $e_1 = e_2$ then by AxPing, $e = e'$ or $e = e'$ which contradicts to $e' \prec e$. So we have the chain
$$e' \ll e \nearrow e_1 \nearrow e_2$$
This implies $e \prec e_1$, and by AxCausality, $e' \succ e_1$. From (5.3) we have that (1) $e' \ll e_1$, and then (2) $e' \ll e_2$, which contradicts to $e' \nearrow e_2$.

■

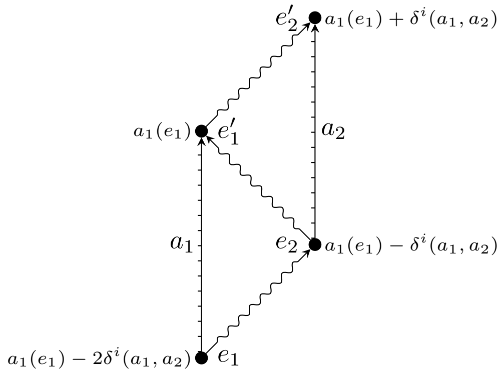
Proposition 77. $\overset{\text{syn}}{\upuparrows}$ *is an equivalence relation and* δ^i *is a(n U-relative) metric on* $\overset{\text{syn}}{\upuparrows}$ *related clocks, i.e.,*

Assumptions:
AxPing
AxExt
AxRound
AxForward
AxCausality
AxChronology
Proposition 71

$$\tag{5.6}$$
$$a \overset{\text{syn}}{\upuparrows} a \tag{5.7}$$
$$a_1 \overset{\text{syn}}{\upuparrows} a_2 \Rightarrow a_2 \overset{\text{syn}}{\upuparrows} a_1 \tag{5.8}$$
$$a_1 \overset{\text{syn}}{\upuparrows} a_2 \land a_2 \overset{\text{syn}}{\upuparrows} a_3 \Rightarrow a_1 \overset{\text{syn}}{\upuparrows} a_3 \tag{5.9}$$
$$\delta^i(a, a) = 0 \tag{5.10}$$
$$\delta^i(a_1, a_2) = 0 \Rightarrow a_1 = a_2 \tag{5.11}$$
$$\delta^i(a_1, a_2) = \delta^i(a_2, a_1) \tag{5.12}$$
$$\delta^i(a_1, a_2) + \delta^i(a_2, a_3) \ge \delta^i(a_1, a_3) \tag{5.13}$$

Proof.
- **Self-distance**, proof of (5.10): By $e \underset{=}{\nearrow} e \underset{=}{\nearrow} e$ we have $\delta^i(a, e) = a(e) - a(e) = 0$. The truth of $\delta^i(a, a) = 0$ is trivially implied by that fact.
- **Reflexivity of** $\overset{\text{syn}}{\upuparrows}$, proof of (5.7): By (5.10) we have $a(e') = a(e) + 0$ whenever $e \underset{=}{\nearrow} e'$, so $\overset{\text{syn}}{\upuparrows}$ is reflexive.
- **Symmetry of** $\overset{\text{syn}}{\upuparrows}$ **and** δ^i, proofs of (5.8) and (5.12), see Fig. 5.8. Suppose that $a_1 \overset{\text{syn}}{\upuparrows} a_2$, i.e.,
$$a_2(e_2) = a_1(e_1) + \delta^i(a_1, a_2) \text{ whenever } e_1 \underset{=}{\nearrow} e_2. \tag{5.14}$$
Take an arbitrary event $e_1'\mathcal{E}a_1$ s.t $e_2\mathcal{E}a_2$ and $e_2 \underset{=}{\nearrow} e_1'$. We have to show that $a_1(e_1') = a_2(e_2) + \delta(a_2, a_1)$. By AxPing, there is an $e_2'\mathcal{E}a_2$ such that

Figure 5.8: Proof of (5.8) and (5.12).



$e_2 \underset{=}{\rightsquigarrow} e'_1 \underset{=}{\rightsquigarrow} e'_2$. By (5.14), we have that $a_2(e'_2) = a_1(e'_1) + \delta^i(a_1, a_2)$. Also from AxPing we know that there is $e_1 \mathcal{E} a_1$ s.t. $e_1 \underset{=}{\rightsquigarrow} e_2 \rightsquigarrow e'_1$. Here by definition of δ^i, $a_1(e_1) = a_1(e'_1) - 2\delta^i(a_1, a_2)$, therefore, by (5.14) again we have that $a_2(e_2) = a_1(e'_1) - \delta^i(a_1, a_2)$. Therefore we showed

$$a_2(e_2) + \delta^i(a_1, a_2) = a_1(e'_1).$$

Note that here $\delta^i(a_1, a_2) = \delta^i(a_2, a_1)$ since $\delta^i(a_2, a_1) = a_2(e'_2) - a_2(e_2) = \delta^i(a_1, a_2)$, so we are ready with *both* (5.8) and (5.12).

- **Identity of indiscernibles**, proof of 5.11. Take arbitrary iscm's a_1 and a_2 for which $\delta^i(a_1, a_2) = 0$, i.e.,

  $$(\forall e \in \mathrm{wline}_{a_2})(\exists w_1, w_2 \in \mathrm{wline}_{a_1}) w_1 \underset{=}{\rightsquigarrow} e \underset{=}{\rightsquigarrow} w_2 \wedge a_1(w_2) - a_1(w_2) = 0$$

  but that means that $a_1(w_1) = a_1(w_2)$, and by AxExt, $w_1 = w_2$. It cannot be the case that $w_1 \rightsquigarrow e$ and $e \rightsquigarrow w_2 = w_1$, because by AxCausality we would have $w_1 \prec w_1$ which contradicts to the irreflexivity of $\prec$ (Prop. 69). It cannot be the case either that $w_1 = e \rightsquigarrow w_2$ or $w_2 = e \rightsquigarrow w_1$, since we know that w_1 and w_2 share the clock a. So the only possiblity is that $w_1 = e = w_2$. Since this is true for all $e \in \mathrm{wline}_{a_2}$, we have that $\mathrm{wline}_{a_2} \subseteq \mathrm{wline}_{a_1}$. Using (5.12) we have that $\mathrm{wline}_{a_2} = \mathrm{wline}_{a_1}$. Now since a_1 and a_2 are iscms, they show the same numbers in the same events, therefore $a_1 = a_2$ by AxExt.

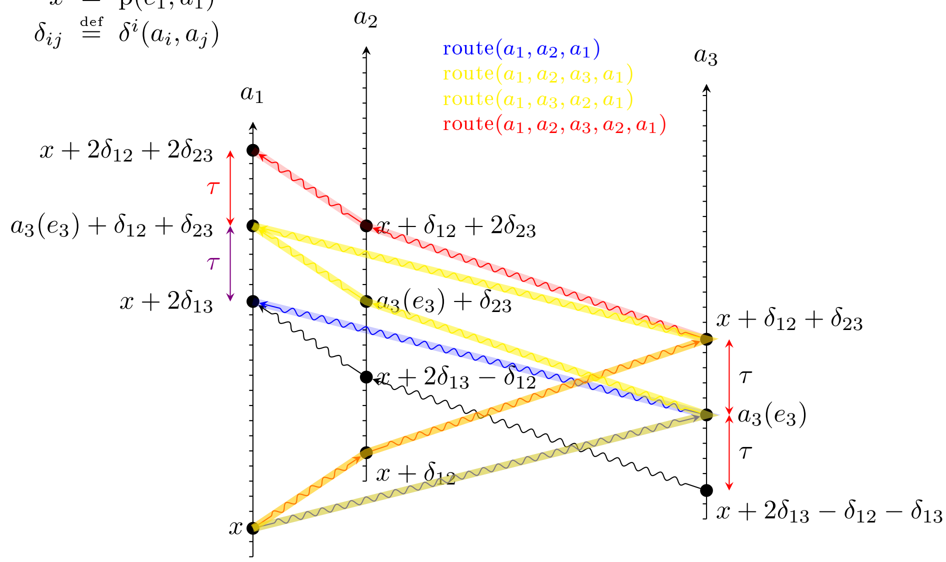
- **Transitivity of $\overset{\mathrm{syn}}{\uparrow\uparrow}$**, proof of (5.9): We start to circuit signals between a_1, a_2 and a_3 and track the time tags, see Fig. 5.9 Following the abbreviation of Fig. 5.9, we have to show that $a_3(e_3) = x + d_{13}$. To show that, it is enough to show that $a_3(e_3)$ is the average of $x + d_{12} + d_{23}$ and $x + 2d_{13} - d_{12} - d_{23}$, i.e., to show that a_3 measures the same elapsed time between them. Since we can project these distances along a_2 to a_1 by our assumption that $a_1 \overset{\mathrm{syn}}{\uparrow\uparrow} a_2 \overset{\mathrm{syn}}{\uparrow\uparrow} a_3$, it is enough to show that $a_3(e_3) + d_{12} + d_{23}$ is the average of $x + 2d_{12} + 2d_{23}$ and $x + 2d_{13}$. But this is true by AxRound.

- **Triangle inequality**, proof of (5.13): By AxPing, we can take $e_1 \in \mathrm{wline}_{a_1}$, $e_2 \in \mathrm{wline}_{a_2}$, $e_3, e_3^* \in \mathrm{wline}_{a_3}$ s.t. $e_1 \underset{=}{\rightsquigarrow} e_2 \underset{=}{\rightsquigarrow} e_3$ and $e_1 \rightsquigarrow e_3^*$. Since

Figure 5.9: Transitivity of $\overset{\text{syn}}{\uparrow\uparrow}$.

Abbreviations:

$$x \overset{\text{def}}{=} \mathrm{p}(e_1, a_1)$$
$$\delta_{ij} \overset{\text{def}}{=} \delta^i(a_i, a_j)$$



all clocks are iscm's of each other by (5.7)-(5.8)-(5.9), we have that

$$a_3(e_3) = a_1(e_1) + \delta^i(a_1, a_2) + \delta^i(a_2, a_3)$$
$$a_3(e_3^*) = a_1(e_1) + \delta^i(a_1, a_3)$$

Proposition 76 says that $a(e_3^*) \leq a(e_3)$, so

$$a_1(e_1) + \delta^i(a_1, a_3) \leq a_1(e_1) + \delta^i(a_1, a_2) + \delta^i(a_2, a_3)$$

which can be simplified to (5.13).

■

Proposition 78. *For any three distinct inertial comovers a, b and c, the clock b is between a and c iff a can send a light signal to c through b.*

Assumptions:
AxExt
??
AxChronology
Proposition 76
Proposition 70

$$\forall a_0 (\forall a, b, c \in \mathrm{Space}_{a_0})$$
$$a \neq b \neq c \wedge \mathrm{B}(a, b, c) \leftrightarrow \exists e_a, e_b, e_c (e_a \mathcal{E} a \wedge e_b \mathcal{E} b \wedge e_c \mathcal{E} c \wedge \overrightarrow{e_a e_b e_c})$$

Proof. $\Leftarrow$: Since we have iscm observers, and of course by AxExt, we have

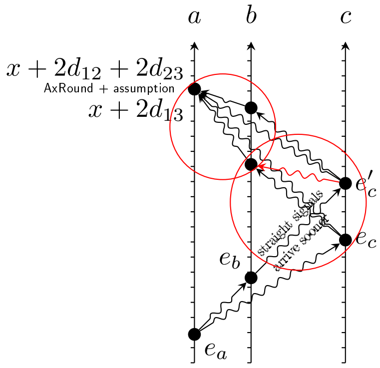
$$\begin{aligned} c(e_c) &= a(e_a) + \delta^i(a,b) + \delta^i(a,c) && \text{by } e_a \nearrow e_b \nearrow e_c \\ &= a(e_a) + \delta^i(a,c) && \text{by } e_a \nearrow e_c \end{aligned}$$

therefore $\delta^i(a,b) + \delta^i(b,c) = \delta^i(a,c)$.

The $\Rightarrow$ comes from the idea of the unique signalling Thm. 70; the assumption that there is no $\overrightarrow{e_a e_b e_c}$ while $\delta^i(a,b) + \delta^i(b,c) = \delta^i(a,c)$ leads to forbidden triangles as it is depicted on Fig. 5.10

■

Figure 5.10: ‘Equivalence’ of lightlike betweenness and triangle equality



Assumptions:
(5.10)
(5.11)
(5.9)

Proposition 79. *No clock has two different inertial synchronized comovers at the same event.*

$$(\forall a \in \mathrm{In})\forall e(\forall a_1, a_2 \in \mathrm{D}_e) \quad a_1 \overset{\mathrm{syn}}{\uparrow\uparrow} a \overset{\mathrm{syn}}{\uparrow\uparrow} a_2 \Rightarrow a_1 = a_2 \tag{5.15}$$

Proof. Let $e \in \mathrm{wline}_{a_1} \cap \mathrm{wline}_{a_2}$ be arbitrary but fixed. Let a_1 and a_2 be inertial comovers of a occurring at e. By (5.9), $a_1 \overset{\mathrm{syn}}{\uparrow\uparrow} a_2$. By the proof of (5.10) we know that $\delta^i(a_1, e) = \delta^i(a_2, e) = 0$. Since $a_1 \overset{\mathrm{syn}}{\uparrow\uparrow} a_2$ implies comovement, i.e., constant distance, $\delta^i(a_1, a_2) = 0$. By (5.11), $a_1 = a_2$. ∎

5.3.2 Geometry

To treat the sets Space_a as n dimensional Euclidean spaces we have two prove that they satisfy the (first-order) axioms of Euclidean geometry. We will use the axiom system of Tarski and Givant [1999]. Let $\forall\mathsf{EG}^n$ denote the set of the universal closures of the axioms of the n dimensional elementary geometry of Tarski and Givant [1999, p. 190.], i.e., the axioms 1 – 7, 8^n, 9^n and 10_2[2], see Table 5.1.

Let ξ be variable mapping that maps every variable of the language of $\forall\mathsf{EG}$ to a clock variable other than a_0, and let T_ξ be the following translation of the language of $\forall\mathsf{EG}^n$ to the language of SClTh:

$$\begin{array}{lcl}
\mathrm{T}_\xi(a = b) & \overset{\mathrm{def}}{\equiv} & \xi(a) = \xi(b) \\
\mathrm{T}_\xi(B(abc)) & \overset{\mathrm{def}}{\equiv} & \delta^i(\xi(a), \xi(b)) + \delta^i(\xi(b), \xi(c)) = \delta^i(\xi(a), \xi(c)) \\
\mathrm{T}_\xi(ab \equiv cd) & \overset{\mathrm{def}}{\equiv} & \delta^i(\xi(a), \xi(b)) = \delta^i(\xi(c), \xi(d)) \\
\mathrm{T}_\xi(\neg\varphi) & \overset{\mathrm{def}}{\equiv} & \neg\mathrm{T}_\xi(\varphi) \\
\mathrm{T}_\xi(\varphi \wedge \psi) & \overset{\mathrm{def}}{\equiv} & \mathrm{T}_\xi(\varphi) \wedge \mathrm{T}(\psi) \\
\mathrm{T}_\xi(\forall a\varphi) & \overset{\mathrm{def}}{\equiv} & (\forall \xi(a) \in \mathrm{Space}_{a_0})\mathrm{T}_\xi(\varphi)
\end{array}$$

Now under the *Tarski axioms for n-dimensional space of inertial clocks* we understand the following set of statements

[2]Here we used some results of **?** and **?**: we used axioms 7, 6 and 10_2 instead of 7_1, 15 and 10_2, respectively.

Table 5.1: Tarski's 11 axioms of elementary geometry

1. $ab \equiv ba$ (Reflexivity for $\equiv$)

2. $(ab \equiv pq \wedge ab \equiv rs) \to pq \equiv rs$ (Transitivity for $\equiv$)

3. $ab \equiv cc \to a = b$ (Identity for $\equiv$)

4. $\exists x(B(qax) \wedge ax \equiv bc)$ (Segment Construction)

5. $(a \neq b \wedge B(abc) \wedge B(a'b'c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge$

$\wedge ad \equiv a'd' \wedge bd \equiv b'd') \to cd \equiv c'd'$ (Five-segment)

6. $B(aba) \to a = b$ (Identity for B)

7. $(B(apc) \wedge B(bqc)) \to \exists x(B(pxb) \wedge B(qxa))$ (Pasch)

8^n. $\exists a, b, c, p_1, \ldots p_{n-1}\Bigg(\bigwedge_{i<j<n} p_i \neq p_j \wedge \bigwedge_{1<i<n} (ap_1 \equiv ap_i \wedge bp_1 \equiv bp_i \wedge cp_1 \equiv cp_i) \wedge$

$\wedge \neg(B(abc) \vee B(bca) \vee B(cab))\Bigg)$ (Lower n-dimension)

9^n. $\Bigg(\bigwedge_{i<j<n} p_i \neq p_j \wedge \bigwedge_{1<i<n} (ap_1 \equiv ap_i \wedge bp_1 \equiv bp_i \wedge cp_1 \equiv cp_i)\Bigg) \to$

$\to (B(abc) \vee B(bca) \vee B(cab))$ (Upper n-dimension)

10_2. $B(abc) \vee B(bca) \vee B(cab) \vee \exists x(ax \equiv bx \wedge ax \equiv cx)$ (Circumscribed tr.)

11. $\exists a \forall x, y(\alpha \wedge \beta \to B(axy)) \to \exists b \forall x, y(\alpha \wedge \beta \to B(aby))$ (Continuity scheme)

where α and β are first-order formulas, the first of which does not contain any free occurrences of a, b and y and the second any free occurrences of a, b, x.

(AxGeom) $\{\forall a_0 \mathrm{T}_\xi(\varphi) : \varphi \in \forall \mathsf{EG}^n\}$

Now we prove (AxGeom) to show that $\mathrm{Space}_\alpha^{\mathfrak{M}}$ is an Euclidean space for all $\alpha \in \mathbb{C}$.

Corollary 80 (Axiom 1.). *Reflexivity axiom for equidistance.*

Assumptions: (5.12)

$$\forall a_0 (\forall a, b \in \mathrm{Space}_{a_0}) \quad ab \equiv ba$$

Proof. That comes from the symmetry of δ^i, i.e., from (5.12). ■

Corollary 81 (Axiom 2.). *Transitivity of equidistance.*

Assumptions: Proposition 74

$$\forall a_0 (\forall a, b, c, d, e, f \in \mathrm{Space}_{a_0}) \; (ab \equiv cd \wedge ab \equiv fe) \rightarrow cd \equiv ef$$

Proof. By Proposition 74 this is just the consequence of the transitivity of $=$. ■

Corollary 82 (Axiom 3.). *Identity axiom for equidistance.*

Assumptions: Proposition 77

$$\forall a_0 (\forall abc \in \mathrm{Space}_{a_0}) \quad (ab \equiv cc \rightarrow a = b)$$

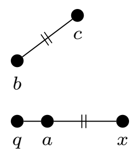
Proof. $\delta^i(a_3, a_3) = 0 = \delta^i(a_1, a_2)$ which implies $a_1 = a_2$ according the identity of indiscernibles provided by Proposition 77. ■

Corollary 83 (Axiom 4.). *Axiom of segment construction.*

Assumptions: AxPing, AxRays, Proposition 71

$$\forall a_0 (\forall a, b, c, q \in \mathrm{Space}_{a_0})(\exists x \in \mathrm{Space}_{a_0}) \; (bc \equiv ax \wedge \mathrm{B}(q, a, x))$$

Proof. Let $e_q \mathcal{E} q$ be arbitrary but fixed. By AxPing, there is an $e_a \mathcal{E} a$. s.t. $e_a \nearrow e_q$. By AxRays there is an e_x for which $\overrightarrow{e_d e_a e_q}$ and $\delta^i(a, e_x) = \delta^i(b, c)$. Proposition 71 then provides the desired iscm x in e_x. ■



Corollary 84 (Axiom 5.). *Five-segment axiom.*

Assumptions: AxPing, Proposition 78, Ax5Segment

$$\begin{aligned}\forall a_0 (\forall a, a', b, b', c, c', d, d' \in \mathrm{Space}_{a_0}) \\ (a \neq b \wedge \mathrm{B}(a, b, c) \wedge \mathrm{B}(a', b', c') \wedge ab \equiv a'b' \wedge bc \equiv b'c' \wedge \\ \wedge\, ad \equiv a'd' \wedge bd \equiv b'd') \rightarrow cd \equiv c'd'\end{aligned}$$

Proof. Suppose that we have the inertial comovers $a, b, c, d, a', b', c', d'$ with the properties described by the premise. By AxPing, b can ping a and c and b' can ping a' and c' s.t. the receiving event of the ping of a is the same as the sending event of the ping of c. By the equivalence of betweenness' provided by Proposition 78, all these events are on a lightline, therefore they satisfies the conditions of Ax5Segment, which provide the conclusion that $\delta^i(c, d) = \delta^i(c', d')$. ■

Corollary 85 (Axiom 6.). *Identity axiom for betweenness.*

Assumptions: Proposition 77

$$(\forall a, b \in \mathrm{Space}_{a_0}) \big(\mathrm{B}(a, b, a) \rightarrow a = b\big)$$

Proof. $\delta^i(a, b) + \delta^i(b, a) = \delta^i(a, a) = 0$ and the fact that $\delta^i(a, b) \geq 0$ implies that $\delta^i(a, b) = 0$. By the identity of indiscernibles provided by Proposition 77, $a = b$. ■

Corollary 86 (Axiom 7.). *Inner form of the Pasch axiom.*

Assumptions:
AxPing
Proposition 78
AxPasch

$$\forall a_0(\forall a,b,c,p,q \in \mathrm{Space}_{a_0})\ \big((\mathrm{B}(a,p,c) \wedge \mathrm{B}(b,q,c)) \to \\ \to (\exists d \in \mathrm{Space}_{a_0})(\mathrm{B}(p,d,b) \wedge \mathrm{B}(q,d,a))\big)$$

Proof. Suppose that a,b,c,p,q satisfies the premise. Then by AxPing a and b can ping an event of c in a way that these light signals will cross the wordline of p and q (the latter is provided by Proposition 78). Now AxPasch provides the existence of the desired clock x. ■

Corollary 87 (Axiom 8^n.). *Lower n-dimensional axiom:* UNDER CONSTRUCTION

Proof. UNDER CONSTRUCTION ■

Corollary 88 (Axiom 9^n.). *Upper n-dimensional axiom:* UNDER CONSTRUCTION

Proof. UNDER CONSTRUCTION ■

Corollary 89 (Axiom 10_2.). *Every triangle can be circumscribed:*

Assumptions:
Proposition 78, AxCircle

$$\mathrm{B}(a,b,c) \vee \mathrm{B}(b,c,a) \vee \mathrm{B}(c,a,b) \vee \exists x(ax \equiv bx \wedge ax \equiv cx)$$

Proof. It is enough to prove that $\exists x(ax \equiv bx \wedge ax \equiv cx)$ whenever $\mathrm{B}(a,b,c) \vee \mathrm{B}(b,c,a) \vee \mathrm{B}(c,a,b)$ is false. Suppose that this is false. Then by Proposition 78, they a, b and c can not be connected with a lightline. Therefore by AxCircle, there is an x s.t. this x has the same signalling distance from a, b and c, and that is what we needed. ■

Corollary 90 (Axiom 11.). *Tarski's axiom scheme of continuity (p.185.)*

Assumptions:
AxReals

$$\forall a_0(\exists a \in \mathrm{Space}_{a_0})(\forall c,d \in \mathrm{Space}_{a_0})(\varphi(c) \wedge \psi(d) \to \mathrm{B}(a,c,d)) \to \\ \to (\exists b \in \mathrm{Space}_{a_0})(\forall c,d \in \mathrm{Space}_{a_0})(\varphi(c) \wedge \psi(d) \to \mathrm{B}(c,b,d))$$

Proof. This comes from the continuity (or infimum-supremum) scheme of the real closed fields: The transition of that scheme to events is granted by AxRays, and the existence of the specific point through the event is granted by Proposition 71. ■

5.3.3 Coordinatization

Theorem 91 (Coordinatization). *For arbitrary coordinatesystem, the coordinatization function is a bijection between W and U^4. In other words, given an arbitrary but fixed coordinate system, the following statements are true:*

Totality *Every event is coordinatized with a 4-tuple.*

Surjectivity *Every 4-tuple is a coordinate of an event.*

Functionality *No event has two different coordinates.*

Injectivity *No 4-tuple is a coordinatization of 2 different events.*

Proof. Let a, a_x, a_y, a_z be an arbitrary but fixed coordinate system.

Totality Every event is coordinatized with a 4-tuple. Let e be an arbitrary event. By Proposition 71, we have a synchronized comover a_e of a in e. Then by definition, $a_e(e)$ will be the time coordinate. We can use Tarski's axioms to conclude that there are (unique) a'_x, a'_y and a'_z that are projections of the point a_e to the lines (a, a_x), (a, a_y) and (a, a_z), respectively. By (AxPing), these projections can ping a_e, i.e., they can measure the spatial distance between them and a_e (and e), and thus we will have the spatial coordinates of e as well.

Surjectivity Every 4-tuple is a coordinate of an event. Let (t, x, y, z) be an arbitrary 4-tuple. It follows from Tarski's axioms that there are planes there are inertial comovers a'_x, a'_y and a'_z of a on the axes (a, a_x), (a, a_y) and (a, a_z), respectively, such that $\delta^i(a, a_x) = x$, $\delta^i(a, a_y) = y$ and $\delta^i(a, a_t) = t$. For all $i \in \{x, y, z\}$ Let P_i denote the plane that contains a'_i and is orthogonal to the line (a, a_i). Then by Tarski's axioms, these planes has one (unique) intersection, a_e. By the definition of the Coord, any event of wline_{a_e} are coordinatized on the spatial coordinates (x, y, z). Now we know from (**??**) that there is an event e of wline_{a_e} such that $a(e) = t$.

Functionality No event has two different coordinates. In the proof of Totality, a_e is unique by Proposition 79. After that, as we noted above, Tarki's axioms provided the uniqueness of the projections as well, and this is enough for the uniqueness of the coordinates.

Injectivity No 4-tuple is a coordinate of 2 different events. From the proof of surjectivity we saw that a_e was unique. But for a given t, the e is also unique by (**??**). ■

5.3.4 Simplifying SpecRel

Note that a lot of physical quantities can be defined without referring to coordinate systems. Spatial distance, elapsed time and speed are nice examples of that. Here we are going to define these concepts and prove that they are indeed equivalent with the usual spacetime diagram-based definitions. These proofs will allow to identify the monstrous axioms of SpecRel with the lightweight propositions what we will call "Simple-SpecRel" in Section **??**.

Definition 44 (spatial distance)**.** We say that the spatial distance between events e and e' according to an inertial clock a is τ iff

$$\mathrm{sd}_a(e, e') = \tau \overset{\text{def}}{\Leftrightarrow} (\exists a' \in \mathrm{Space}_a)(a \in \mathrm{D}_e \land \delta^i(a, e') = \tau)$$

Proposition 92. $\mathrm{sd}_a(,)$ *is a total function for all* a*.*

Proof. There is such a' by Proposition 71, and this a' is unique by Proposition 79. ■

Proposition 93. *The spacetime diagram-based definition of spatial distance and our definition are the same.*

$$\begin{aligned}\mathrm{sd}_a(e, e') = \tau \iff & (\exists \langle a_x, a_y, a_z \rangle \in \mathrm{CoordSys(a)})\exists \vec{x}\vec{y} \\ & \mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \land \mathrm{Coord}_{a,a_x,a_y,a_z}(e') = \vec{y} \land \tau = |\vec{x}_{2-4} - \vec{y}_{2-4}|\end{aligned}$$

Proof. By Tarski's axioms of geometry, this is just Pythagoras's theorem:[3]

$$\delta^i(a_e, a_{e'})^2 = \delta^i(a_e, b)^2 + \delta^i(b, a_{e'})^2$$

where $b \in \mathrm{Space_a}$ is a clock with which

$$\mathrm{Ort}(a'_x, a, b) \wedge \mathrm{Ort}(a'_y, a, b) \wedge \mathrm{Ort}(a'_z, a, b)$$

where a'_x, a'_y, a'_z, are the projections of a_e to the axes of the coordinate system (See Fig. 5.5). ■

Definition 45 (elapsed time)**.** We say that the elapsed time between events e and e' according to an inertial clock a is τ iff

$$\mathrm{et}_a(e, e') = \tau \overset{\text{def}}{\Leftrightarrow} (\exists b, b' \in \mathrm{Space}_a)\ |b(e) - b'(e')| = \tau$$

Proposition 94. $\mathrm{et}_a(,)$ *is a total function for all* a*.*

Proof. That is true by the same reasons as Proposition 92. ■

Proposition 95. *The spacetime diagram-based definition of elapsed time and our definition are the same.*

$$\begin{aligned}\mathrm{et}_a(e, e') = \tau \iff & (\exists \langle a_x, a_y, a_z\rangle \in \mathrm{CoordSys(a)})\exists \vec{x}, \vec{y} \\ & \mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \wedge \mathrm{Coord}_{a,a_x,a_y,a_z}(e') = \vec{y} \wedge \tau = |\vec{x}_1 - \vec{y}_1|\end{aligned}$$

Proof. The clocks that measures the time in the events are the same in both definitions by Proposition 79, so practically, both formula refer to the same measurement. ■

Definition 46 (speed)**.** Speed is defined using the standard $v = \frac{\Delta s}{\Delta t}$ formula:

$$\mathrm{v}_a(e, e') \overset{\text{def}}{=} \frac{\mathrm{sd}_a(e, e')}{\mathrm{et}_a(e, e')}$$

5.3.5 Proving 'Simple-SpecRel'

The following theorems are important because of their resemblance to the axioms of SpecRelComp.

During the proofs we follow the notation of the definition of coordinatization predicate, e.g., we always refer to the inertial synchronized co-mover a that witness the event e by a_e.

Proposition 96 (Simple-AxSelf)**.**

$$\forall a(\forall e \in \mathrm{wline}_a)(\forall \langle a_x, a_y, a_z\rangle \in \mathrm{CoordSys(a)})\exists t\ \mathrm{Coord}_{a,a_x,a_y,a_z}(e) = (t, 0, 0, 0)$$

(Note that t *here is exactly* $a(e)$*.)*

Proof. No matter how we choose a_x, a_y or a_z, the clock a_e can be chosen to be a itself, since $e \in \mathrm{wline}_a$. Since a is on all the axes (a, a_x), (a, a_y), (a, a_z), the distance of a from these lines are all 0, and $a(e)$ will be t. ■

[3] UNDER CONSTRUCTION

Proposition 97 (Simple-AxPh)**.**

$$(\forall a \in \mathrm{In})\forall e, e' \ (\mathrm{v}_a(e, e') = 1 \leftrightarrow e \rightsquigarrow e')$$

Proof. The $\leftarrow$ direction is trivial by the definition of $\mathrm{sd}(,)$ and $\mathrm{et}(,)$; they produce the same number for lightlike related events. For the other direction, if $\mathrm{v}_a(e, e') = 1$, then take an iscm a' into e'. This a' can ping e', so there will be an event e'' on $\mathrm{wline}_{a'}$ such that $e \rightsquigarrow e''$. By 70, $e' = e''$. ∎

Proposition 98 (Simple-AxEv)**.**

$$\forall e (\forall \langle a, a_x, a_y, a_z \rangle, \langle a', a'_x, a'_y, a'_z \rangle \in \mathrm{CoordSys})$$
$$\exists \vec{x}\ \mathrm{Coord}_{a,a_x,a_y,a_z}(e) = \vec{x} \rightarrow \exists \vec{y}\ \mathrm{Coord}_{a',a'_x,a'_y,a'_z}(e) = \vec{y}$$

Proof. That is true by the totality of coordinatization, i.e., by Proposition 91. ∎

Proposition 99 (Simple-AxSym)**.**

$$(\forall a, a' \in \mathrm{In})\forall e, e' \ (\mathrm{et}_a(e, e') = \mathrm{et}_{a'}(e, e') = 0 \rightarrow \mathrm{sd}_a(e, e') = \mathrm{sd}_{a'}(e, e'))$$

Proof. Here the local experimenters of a and a' coincide by Proposition 79. ∎

Proposition 100 (Simple-AxThExp)**.**

$$\forall a \forall e, e' \quad (\mathrm{v}_a(e, e') < 1 \rightarrow (\exists a' \in \mathrm{In}) e, e' \in \mathrm{wline}_{a'})$$

Proof. From AxPing, Propositions 71 and 70 and from the premise we have that there is an event $e'' \underset{=}{\nwarrow} e$ and a clock $a_{e'} \in \mathrm{D}_{e'} \cap \mathrm{D}_{e''}$. Now this event is in the chronological future of the causal future of e, so by AxChronology, it is in the chronological future of e as well. AxSecant then provides the existence of the desired inertial clock. ∎

5.4 Geodetic-Inertial equivalence

Definition 47 (Geodetic)**.** *Geodetic* clocks are the fastest clocks between any two events on their worldline.

$$\mathrm{Geo}(a) \overset{\mathrm{def}}{\Leftrightarrow} (\forall e, e' \in \mathrm{wline}_a)(\forall b \in \mathrm{D}_e \cap \mathrm{D}_{e'})|a(e) - a(e')| \geq |b(e) - b(e')|$$

UNDER CONSTRUCTION

5.5 Appendix: Definitional Equivalence of $\mathrm{SClTh}_{\mathrm{NoAcc}}$ and SpecRelComp

5.5.1 Language of SpecRelComp

Definition 48 (Language of SpecRelComp)**.**

- Body sort:
  - Body variables: $b_1, b_2, \dots \in BVar$
  - Body predicates: $\mathrm{Ob}, \mathrm{IOb}, \mathrm{Ph}$
- Mathematical sort:
  - Mathematical variables: $x, y, z, \dots \in MVar$
  - Mathematical functions: $+, \cdot$
  - Mathematical predicate: $\leq$
- Connection between sorts:
  - Intersort predicate: W
- Mathematical terms:

$$\tau ::= x \mid \mathsf{r} \mid \tau_1 + \tau_2 \mid \tau_1 \cdot \tau_2$$

- Formulas:

$$\begin{aligned}\varphi ::= b = b' \mid \tau_1 = \tau_2 \mid \tau_1 \leq \tau_2 \\ \mathrm{Ob}(b) \mid \mathrm{IOb}(b) \mid \mathrm{Ph}(b) \mid \mathrm{W}(b, b', \tau_t, \tau_x, \tau_y, \tau_z) \\ \neg\varphi \mid \varphi \wedge \psi \mid \exists b\varphi \mid \exists x\varphi\end{aligned}$$

5.5.2 Axioms of SpecRelComp

UNDER CONSTRUCTIONFor axioms and models of SpecRelComp of the axiom system $\mathsf{SpecRel} \cup \mathsf{Comp}$ see [Andréka et al. 2007]. (Note that the language of that paper contains one more sort for events. This, however, is definable, for more details on that see [Andréka et al. 2001], or, since we have to define it anyway to prove the definitional equivalence with SClTh, see the proof of Thm. 101 on p. 80.

5.5.3 Definitional equivalence with SpecRelComp

Plan

Theorem 101. SpecRelComp *and* SClTh *are definitionally equivalent, i.e., there are translations*

$$\begin{aligned}\mathrm{STC}_\xi &: \mathcal{L}_{\mathrm{SpecRelComp}} \to \mathcal{L}_{\mathrm{SClTh}} \\ \mathrm{CTS}_\zeta &: \mathcal{L}_{\mathrm{SClTh}} \to \mathcal{L}_{\mathrm{SpecRelComp}}\end{aligned}$$

and model-transformations

$$\begin{aligned}\mathrm{stc} &: \mathrm{Mod}(\mathrm{SpecRelComp}) \to \mathrm{Mod}(\mathrm{SClTh}) \\ \mathrm{cts} &: \mathrm{Mod}(\mathrm{SClTh}) \to \mathrm{Mod}(\mathrm{SpecRelComp})\end{aligned}$$

and assignment transformations f_ρ and g_ϱ such that the followings hold for all $\mathfrak{M}_s \in \mathrm{Mod}(\mathrm{SpecRelComp})$ and $\mathfrak{M}_c \in \mathrm{Mod}(\mathrm{SClTh})$ and for any $\varphi_s \in \mathcal{L}_{\mathrm{SpecRelComp}}$, $\varphi_c \in \mathcal{L}_{\mathrm{SClTh}}$:

$$\mathrm{stc}(\mathfrak{M}_s) \models \varphi_c \quad [\eta] \iff \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi_c)[f_\rho(\eta)] \tag{5.16}$$

$$\mathfrak{M}_c \models \mathrm{STC}_\zeta(\varphi_s)[g_\varrho(\mu)] \iff \mathrm{cts}(\mathfrak{M}_c) \models \varphi_s \quad [\mu] \tag{5.17}$$

$$\mathrm{SClTh} \vdash \varphi_c \implies \mathrm{SpecRelComp} \vdash \mathrm{CTS}_\xi(\varphi_c) \tag{5.18}$$

$$\mathrm{SpecRelComp} \vdash \varphi_s \implies \mathrm{SClTh} \vdash \mathrm{STC}_\zeta(\varphi_s) \tag{5.19}$$

Proof. 1. **Definition of** stc. Let

$$\mathfrak{M}_s = \left(B, \mathrm{IOb}^{\mathfrak{M}_s}, \mathrm{Ph}^{\mathfrak{M}_s}, \mathfrak{Q}, \mathrm{W}^{\mathfrak{M}_s}\right)$$

be an arbitrary but fixed model of SpecRel. We will introduce the transformation stc : Mod(SpecRel) $\to$ Mod(CTh), i.e., we will construct the corresponding CTh model $\mathrm{stc}(\mathfrak{M})$ from the information that $\mathfrak{M}$ contains. Such a CTh model will be given as

$$\mathrm{stc}(\mathfrak{M}_s) \stackrel{\mathrm{def}}{=} \left(\mathrm{stc}_W(\mathfrak{M}_s), \mathrm{stc}_\prec(\mathfrak{M}_s), \mathrm{IOb}^{\mathfrak{M}_s}, \mathfrak{Q}, \mathrm{stc}_{\mathrm{P}}(\mathfrak{M}_s)\right)$$

where the three undefined entity are the domain of events, the causality relation and the meaning of the pointing relation, respectively.

(a) **The event domain** $\mathrm{stc}_W(\mathfrak{M}_s)$**.** The idea is that an event will be identified as the set of bodies occurring *there*. To express the word 'there' in SpecRelComp, we have to use the worldview predicate W with parameters. To *name* the elements of the universe of the defined sort, we will use sets defined with 5 parameters:

$$ev_{o,t,x,y,z} \stackrel{\mathrm{def}}{=} \{b \in B : (o, b, t, x, y, z) \in \mathrm{W}^{\mathfrak{M}_s}\}$$

But we know that the *same* event can occur in different observers' different coordinate points. So we factorize over that set with the following equivalence relation.

$$\langle o_1, t_1, x_1, y_1, z_1\rangle \stackrel{e}{\simeq} \langle o_2, t_2, x_2, y_2, z_2\rangle \stackrel{\mathrm{def}}{\Leftrightarrow} \mathrm{w}^{\mathfrak{M}_s}_{o_1 o_2}(t_1, x_1, y_1, z_1) = (t_2, x_2, y_2, z_2)$$

Where $\mathrm{w}^{\mathfrak{M}_s}$ is the meaning of the worldview transformation defined in SpecRelComp. Now we are ready to define the universe of $\mathrm{stc}(\mathfrak{M}_s)$:

$$\mathrm{stc}_W(\mathfrak{M}_s) \stackrel{\mathrm{def}}{=} \{\langle o, t, x, y, z\rangle/\stackrel{e}{\simeq} \,:\, o \in \mathrm{IOb}^{\mathfrak{M}_s} \wedge t, x, y, z \in Q\}$$

(b) **The causality relation** $\mathrm{stc}_\prec(\mathfrak{M}_s)$ We use the usual definition of Minkowski distance, which is easily definable in SpecRelComp

$$\mu(\vec{x}, \vec{y}) \stackrel{\mathrm{def}}{\Leftrightarrow} (\vec{x}_1 - \vec{y}_1)^2 - (\vec{x}_2 - \vec{y}_2)^2 - (\vec{x}_3 - \vec{y}_3)^2 - (\vec{x}_4 - \vec{y}_4)^2$$

$$\mathrm{stc}_\prec(\mathfrak{M}_s) \stackrel{\mathrm{def}}{=} \Big\{\langle (o, \vec{x})/\stackrel{e}{\simeq}, (o', \vec{x}')/\stackrel{e}{\simeq}\rangle \in \mathrm{stc}_W(\mathfrak{M}_s)^2 : \mu^{\mathfrak{M}_s}(\mathrm{w}^{\mathfrak{M}_s}_{oo'}(\vec{x}), \vec{x}') \geq 0 \text{ and } \left(\mathrm{w}^{\mathfrak{M}_s}_{oo'}(\vec{x})\right)_1 < \vec{x}'_1\Big\}$$

(c) **Meaning of pointing** $\mathrm{stc_P}(\mathfrak{M}_s)$ Pointing statements comes straight from the worldview-transformation:

$$\mathrm{stc_P}(\mathfrak{M}_s) \stackrel{\text{def}}{=} \Big\{\langle (o,t,x,y,z)/{\simeq}, o', t'\rangle \,:\, \mathrm{w}^{\mathfrak{M}_s}_{oo'}(t,x,y,z) = (t',0,0,0)\Big\}$$

2. **Definition of** ξ. Note that to quantify over $\mathrm{stc}_W(\mathfrak{M}_s)$, it is enough if we can quantify over the representants, i.e., over $\mathrm{IOb} \times Q^4$. But to do so, we'll need variables. For mathematical variables we map mathematical variables, for event variable e, we map a 5-tuples of ***different*** variables $(b, x_t, x_x, x_y, x_z) \in Var_b \times Var_m^4$, and for clock variable c we body variables $b \in Var_b$ in a way that no variable will be the representative of two different variables[4]:

$$\xi : \begin{array}{rcl} x_i & \mapsto & x_{5i} \\ a_i & \mapsto & b_{2i} \\ e_i & \mapsto & \langle b_{2i+1}, x_{5i+1}, x_{5i+2}, x_{5i+3}, x_{5i+4}\rangle \end{array}$$

For mathematical terms we define $\hat{\xi}$ to be the induced substitution based on ξ:

$$\begin{array}{rcl} \hat{\xi}(x) & \stackrel{\text{def}}{=} & \xi(x) \\ \hat{\xi}(\tau+\tau') & \stackrel{\text{def}}{=} & \hat{\xi}(\tau)+\hat{\xi}(\tau') \\ \hat{\xi}(\tau\cdot\tau') & \stackrel{\text{def}}{=} & \hat{\xi}(\tau)\cdot\hat{\xi}(\tau') \end{array}$$

3. **Definition of** CTS_ξ

$$\begin{array}{lcl}
\mathrm{CTS}_\xi(e=e') & \stackrel{\text{def}}{=} & \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) = \xi_{2-5}(e') \\
\mathrm{CTS}_\xi(e\prec e') & \stackrel{\text{def}}{=} & \mu\Big(\mathrm{w}_{\xi_1(e)\xi_1(e')}\big(\xi_{2-5}(e)\big), \xi_{2-5}(e')\Big) \geq 0 \,\wedge \\
 & & \qquad \wedge\, \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) \neq \xi_{2-5}(e') \\
\mathrm{CTS}_\xi(\tau=\tau') & \stackrel{\text{def}}{=} & \hat{\xi}(\tau)=\hat{\xi}(\tau') \\
\mathrm{CTS}_\xi(\tau\leq\tau') & \stackrel{\text{def}}{=} & \hat{\xi}(\tau)\leq\hat{\xi}(\tau') \\
\mathrm{CTS}_\xi(\mathrm{P}(e,a,\tau)) & \stackrel{\text{def}}{=} & \mathrm{w}_{\xi_1(e)\xi(a)}(\xi_{2-5}(e)) = (\hat{\xi}(\tau),0,0,0) \\
\mathrm{CTS}_\xi(\neg\varphi) & \stackrel{\text{def}}{=} & \neg\mathrm{CTS}_\xi(\varphi) \\
\mathrm{CTS}_\xi(\varphi\wedge\psi) & \stackrel{\text{def}}{=} & \mathrm{CTS}_\xi(\varphi)\wedge\mathrm{CTS}_\xi(\psi) \\
\mathrm{CTS}_\xi(\exists e\varphi) & \stackrel{\text{def}}{=} & \exists\xi_1(e)\exists\xi_2(e)\exists\xi_3(e)\exists\xi_4(e)\exists\xi_5(e)\mathrm{CTS}_\xi(\varphi) \\
\mathrm{CTS}_\xi(\exists a\varphi) & \stackrel{\text{def}}{=} & \exists\xi(a)\big(\mathrm{IOb}(v(a))\wedge\mathrm{CTS}_\xi(\varphi)\big) \\
\mathrm{CTS}_\xi(\exists x\varphi) & \stackrel{\text{def}}{=} & \exists\xi(x)\mathrm{CTS}_\xi(\varphi)
\end{array}$$

[4]The latter is an important constraint: suppose that $\xi_1(e_1) = \xi(a_1)$, i.e., the variable b_1 represents an inertial observer and a maybe different observer that coordinatizes the event e_1 in $\xi_{2,5}(e_1)$. It is easy to find a model of SpecRelComp with an assignment such that these two observers are different. Then their difference will not be expressible since the transformated assignment η will not be able differentiate between them, since we used the same variable b_1 to represent 'them'. This failure could be conjectured also syntactically, if we imagine the situation when we try to translate a formula $\exists e_1\exists a_1\varphi$, because that would result in a formula that starts with

$$\exists\xi_1(e_1)\exists\xi_2(e_1)\exists\xi_3(e_1)\exists\xi_4(e_1)\exists\xi_5(e_1)\exists\xi(a_1)(\dots\mathrm{CTS}_\xi(\varphi)\dots)$$

but here, by $\xi_1(e_1) = \xi(a_1)$, we have a *vacuous quantification*, which was not present in $\exists e_1\exists a_1\varphi$. The position in the proof when we will refer to that 'well-separated' property of ξ is when we are going to discuss the formulas $\exists e\varphi$.

4. **Definition of the assignment transformation** f_ρ Let ρ be an arbitrary choice function that chooses one representant from every equivalence class of $\mathrm{stc}_W(\mathfrak{M}_c)$, i.e., ρ satisfies the equation

$$\rho(\langle o,t,x,y,z\rangle/\overset{e}{\simeq}) \overset{e}{\simeq} \langle o,t,x,y,z\rangle \tag{5.20}$$

Now we define f_ρ to fit to ξ:

$$f_\rho(\eta): \begin{array}{rcl} b_{2i} & \mapsto & \eta(a_i) \\ b_{2i+1} & \mapsto & \rho_1 \circ \eta(e_i) \\ x_{5i} & \mapsto & \eta(x_i) \\ x_{5i+n} & \mapsto & \rho_{n+1} \circ \eta(e_i) \text{ for any } n \in \{1,2,3,4\} \end{array}$$

Now by the construction we have that

$$f_\rho(\eta) \circ \xi(e)/\overset{e}{\simeq} = \eta(e) \tag{5.21}$$

where we used the abbreviation

$$f_\rho(\eta)(\vec{v}) \overset{\text{def}}{=} \langle f_\rho(\eta)(\vec{v}_1), f_\rho(\eta)(\vec{v}_2), f_\rho(\eta)(\vec{v}_3), f_\rho(\eta)(\vec{v}_4), f_\rho(\eta)(\vec{v}_5)\rangle$$

By the construction of $f_\rho(\eta)$ we also have the equations

$$f_\rho(\eta) \circ \xi(a) = \eta(a) \tag{5.22}$$

and

$$f_\rho(\eta) \circ \xi(x) = \eta(x),$$

and if we take the natural extension $\hat{\eta}$ of the assignment function η for terms, i.e.,

$$\begin{array}{rcl} \hat{\eta}(x) & \overset{\text{def}}{=} & \eta(x) \\ \hat{\eta}(\tau+\tau') & \overset{\text{def}}{=} & \hat{\eta}(\tau) +^{\mathfrak{M}_s} \hat{\eta}(\tau') \\ \hat{\eta}(\tau\cdot\tau') & \overset{\text{def}}{=} & \hat{\eta}(\tau) \cdot^{\mathfrak{M}_s} \hat{\eta}(\tau') \end{array}$$

and we can generalize the above equation to

$$\widehat{f_\rho(\eta)} \circ \hat{\xi}(\tau) = \hat{\eta}(\tau) \tag{5.23}$$

5. **Proof of the equivalence** (5.16)

$$\mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi_c)[f_\rho(\eta)] \iff \mathrm{stc}(\mathfrak{M}_s) \models \varphi_c[\eta]$$

We prove this by induction on φ_c. Notice that the proof itself is of logical in nature; the proof goes through because the model-construction and the translations are defined to fit to each other. (So)

- $\varphi_c = e = e'$

$$\begin{array}{lll} & \mathfrak{M}_s \models \mathrm{CTS}_\xi(e=e')[f_\rho(\eta)] & \\ \iff & \mathfrak{M}_s \models \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) = (\xi_{2-5}(e'))[f_\rho(\eta)] & \text{def.of } \mathrm{CTS}_\xi \\ \iff & \mathrm{w}^{\mathfrak{M}_s}_{f_\rho(\eta)\circ\xi_1(e), f_\rho(\eta)\circ\xi_1(e')}(f_\rho(\eta)\circ\xi_{2-5}(e)) = f_\rho(\eta)\circ\xi_{2-5}(e') & \\ \iff & ev_{f_\rho(\eta)\circ\xi(e)} \overset{e}{\simeq} ev_{f_\rho(\eta)\circ\xi(e')} & \text{def.of } \overset{e}{\simeq} \\ \iff & \mathrm{stc}(\mathfrak{M}_s) \models e = e'[\eta] & (5.21) \end{array}$$

- $\varphi_c = e \prec e'$ is similar to $e = e'$.
- $\varphi_c = \tau = \tau'$

$$\begin{array}{rll} & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\tau = \tau')[f_\rho(\eta)] & \\ \iff & \mathfrak{M}_s \models \hat{\xi}(\tau) = \hat{\xi}(\tau')[f_\rho(\eta)] & \text{def.of CTS}_\xi \\ \iff & \mathfrak{M}_s \models \hat{\xi}(\tau) = \hat{\xi}(\tau')[f_\rho(\eta)] & \text{def.of } \hat{\xi} \\ \iff & \widehat{f_\rho(\eta)} \circ \hat{\xi}(\tau) = \widehat{f_\rho(\eta)} \circ \hat{\xi}(\tau') & \text{def.of} \models \\ \iff & \hat{\eta}(\tau) = \hat{\eta}(\tau') & (5.23) \\ \iff & \mathrm{stc}(\mathfrak{M}_s) \models \tau = \tau'[\eta] & \text{def.of} \models \end{array}$$

- $\varphi_c = \tau \leq \tau'$ is similar to $\tau = \tau'$
- $\varphi_c = \mathrm{P}(e, a, \tau)$

$$\begin{array}{rll} & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\mathrm{P}(e,a,\tau))[f_\rho(\eta)] & \\ \iff & \mathfrak{M}_s \models \mathrm{w}_{\xi_1(e)\xi(a)}(\xi_{2-5}(e)) = (\hat{\xi}(\tau), 0, 0, 0)[f_\rho(\eta)] & \text{def.of CTS}_\xi \\ \iff & \mathrm{w}^{\mathfrak{M}_s}_{f_\rho(\eta)\circ\xi_1(e), f_\rho(\eta)\circ\xi(a)}(f_\rho(\eta) \circ \xi_{2-5}(e)) = (\widehat{f_\rho(\eta)} \circ \hat{\xi}(\tau), 0, 0, 0) & \text{def.of} \models \\ \iff & \mathrm{w}^{\mathfrak{M}_s}_{f_\rho(\eta)\circ\xi_1(e), \eta(a)}(f_\rho(\eta) \circ \xi_{2-5}(e)) = (\hat{\eta}(\tau), 0, 0, 0) & (5.22), (5.23) \\ \iff & f_\rho(\eta) \circ \xi(e) \overset{e}{\simeq} \langle \eta(a), \hat{\eta}(\tau), 0, 0, 0\rangle & \text{def.of } \overset{e}{\simeq} \\ \iff & \eta(e) \overset{e}{\simeq} \langle \eta(a), \hat{\eta}(\tau), 0, 0, 0\rangle & (5.21) \\ \iff & \langle \eta(e), \eta(a), \hat{\eta}(\tau)\rangle \in \mathrm{stc}_{\mathrm{P}}(\mathfrak{M}_s) & \text{def.of } \mathrm{stc}_{\mathrm{P}}(\mathfrak{M}_s) \\ \iff & \mathrm{stc}(\mathfrak{M}_s) \models \mathrm{P}(e, a, \tau)[\eta] & \text{def.of} \models \end{array}$$

- $\varphi_c = \neg\varphi$ and $\varphi_c = \varphi \wedge \psi$ are straightforward.
- $\varphi_c = \exists e \varphi$ Here we will need a lemma:

  Lemma 102.

$$\mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)[\xi(e) \mapsto \langle b, t, x, y, z\rangle]] \iff$$
$$\iff \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e \mapsto \langle b, t, x, y, z\rangle / \overset{e}{\simeq}])]$$

  Proof. We use the following abbreviations:

$$g \overset{\text{def}}{=} f_\rho(\eta)[\xi(e) \mapsto \langle b, t, x, y, z\rangle] \quad \text{and} \quad g' \overset{\text{def}}{=} f_\rho(\eta[e \mapsto \langle b, t, x, y, z\rangle / \overset{e}{\simeq}])$$

  It is clear that

$$g(v) \;=\; g'(v) \text{ for every variable } v \text{ not occurring in } \xi(e) \qquad (5.24)$$

  Now we cannot be sure whether $g' \circ \xi(e) = \langle b, t, x, y, z\rangle$, but we know from the equations (5.20) and (5.21) that

$$g' \circ \xi(e) \overset{e}{\simeq} \langle b, t, x, y, z\rangle. \qquad (5.25)$$

  By the construction of ξ, in the formula $\mathrm{CTS}_\xi(\varphi)$ the sole purpose of any variable that occur in $\xi(e)$ is to represent the event e and are not used to represent bodies or numbers; the variables that refers to numbers as numbers and bodies as bodies, are shifted to positions x_{5i} and b_{2i} but no variable of $\xi(e)$ has even indexes by the construction of ξ. This observation will be enough to prove Lemma 102.

  We prove by induction on the construction of φ. The observation (5.24) make every case of this induction trivial in which e does not occur, so it is enough to check the cases $e = e'$, $e' = e$, $e \prec e'$, $e' \prec e$, $\mathrm{P}(e, a, \tau)$, $\exists e \varphi$ where e and e' are different variables.

– $e = e'$:

$$\begin{aligned}
& \mathfrak{M}_s \models \mathrm{CTS}_\xi(e = e')[g] && \text{assumption}\\
\iff\ & \mathfrak{M}_s \models \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) = \xi_{2-5}(e')[g] && \text{def.of } \mathrm{CTS}_\xi\\
\iff\ & \mathrm{w}^{\mathfrak{M}_s}_{g\circ\xi_1(e)g\circ\xi_1(e')}(g\circ\xi_{2-5}(e)) = g\circ\xi_{2-5}(e') && \text{def.of } \models\\
\iff\ & g\circ\xi(e) \overset{e}{\simeq} g\circ\xi(e') && \text{def.of } \overset{e}{\simeq}\\
\iff\ & \langle b,t,x,y,z\rangle \overset{e}{\simeq} g'\circ\xi(e') && \text{def.of } g,\ (5.24)\\
\iff\ & g'\circ\xi(e) \overset{e}{\simeq} g'\circ\xi(e') && (5.25)\\
\iff\ & \mathfrak{M}_s \models \mathrm{w}_{\xi_1(e)\xi_1(e')}(\xi_{2-5}(e)) = \xi_{2-5}(e')[g'] && \text{def.of } \models\\
\iff\ & \mathfrak{M}_s \models \mathrm{CTS}_\xi(e = e')[g'] && \text{def.of } \mathrm{CTS}_\xi
\end{aligned}$$

– $e' = e$, $e \prec e'$, $e' \prec e$ and $\mathrm{P}(e,a,\tau)$ are similar to $e = e'$.

– $\exists e\varphi$:

$$\begin{aligned}
& \mathfrak{M}_s \models \mathrm{CTS}_\xi(\exists e\varphi)[g] && \text{assumption}\\
\iff\ & \mathfrak{M}_s \models \exists\xi(e)\mathrm{CTS}_\xi(\varphi)[g] && \text{def.of } \mathrm{CTS}_\xi\\
\iff\ & (\exists\langle b',t',x',y',z'\rangle \in B\times Q^4)\\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[g[\xi(e)\mapsto\langle b',t',x',y',z'\rangle]] && \text{def.of } \models\\
\iff\ & (\exists\langle b',t',x',y',z'\rangle \in B\times Q^4) && \text{modification } [\xi(e)\mapsto\\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)[\xi(e)\mapsto\langle b',t',x',y',z'\rangle]] && \text{overwrote the one in}\\
\iff\ & (\exists\langle b',t',x',y',z'\rangle \in B\times Q^4)\\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e\mapsto\langle b',t',x',y',z'\rangle/\overset{e}{\simeq}])] && \text{(nested) ind.hip.}\\
\iff\ & (\exists\langle b',t',x',y',z'\rangle \in B\times Q^4)\\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e\mapsto\langle b,t,x,y,z\rangle/\overset{e}{\simeq}][e\mapsto\langle b',t',x',y',z'\rangle/\overset{e}{\simeq}])] && \text{substitution will be c}\\
\iff\ & (\exists\langle b',t',x',y',z'\rangle \in B\times Q^4)\\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e\mapsto\langle b,t,x,y,z\rangle/\overset{e}{\simeq}])[\xi(e)\mapsto\langle b',t',x',y',z'\rangle])] && \text{(nested)ind.hip.}\\
\iff\ & (\exists\langle b',t',x',y',z'\rangle \in B\times Q^4) && \text{def.of } g\\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[g'[\xi(e)\mapsto\langle b',t',x',y',z'\rangle]]\\
\iff\ & \mathfrak{M}_s \models \exists\xi(e)\mathrm{CTS}_\xi(\varphi)[g'] && \text{def.of } \models\\
\iff\ & \mathfrak{M}_s \models \mathrm{CTS}_\xi(\exists e\varphi)[g'] && \text{def.of } \mathrm{CTS}_\xi
\end{aligned}$$

This ends the proof of Lemma 102. ■

Now the main induction:

$$\begin{aligned}
& \mathfrak{M}_s \models \mathrm{CTS}_\xi(\exists e\varphi)[f_\rho(\eta)]\\
\iff\ & \mathfrak{M}_s \models \exists\xi(e)\mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)] && \text{def.of } \mathrm{CTS}_\xi\\
\iff\ & (\exists\langle b,t,x,y,z\rangle \in B\times Q^4)\\
& \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)[\xi(e)\mapsto\langle b,t,x,y,z\rangle]] && \text{def.of } \models\\
\iff\ & (\exists\langle b,t,x,y,z\rangle \in B\times Q^4)\\
& \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[e\mapsto\langle b,t,x,y,z\rangle/\overset{e}{\simeq}])] && \text{Lemma 102}\\
\iff\ & (\exists\langle b,t,x,y,z\rangle \in B\times Q^4)\\
& \qquad \mathrm{stc}(\mathfrak{M}_s) \models \varphi[\eta[e\mapsto\langle b,t,x,y,z\rangle/\overset{e}{\simeq}]] && \text{ind.hip.}\\
\iff\ & \mathrm{stc}(\mathfrak{M}_s) \models \exists e\varphi[\eta]
\end{aligned}$$

- $\varphi_c = \exists a\varphi$ Now the main induction:

$$
\begin{array}{lll}
& \mathfrak{M}_s \models \mathrm{CTS}_\xi(\exists a\varphi)[f_\rho(\eta)] & \\
\iff & \mathfrak{M}_s \models \exists\xi(a)(\mathrm{IOb}(\xi(a)) \land \mathrm{CTS}_\xi(\varphi))[f_\rho(\eta)] & \text{def.of } \mathrm{CTS}_\xi \\
\iff & (\exists b \in \mathrm{IOb}^{\mathfrak{M}_s}) & \\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta)[\xi(a) \mapsto b]] & \text{def.of } \models \\
\iff & (\exists b \in \mathrm{IOb}^{\mathfrak{M}_s}) & \\
& \qquad \mathfrak{M}_s \models \mathrm{CTS}_\xi(\varphi)[f_\rho(\eta[a \mapsto b])] & (5.22) \\
\iff & (\exists b \in \mathrm{IOb}^{\mathfrak{M}_s})\, \mathrm{stc}(\mathfrak{M}_s) \models \varphi[\eta[a \mapsto b]] & \text{ind.hip.} \\
\iff & \mathrm{stc}(\mathfrak{M}_s) \models \exists a\varphi[\eta] & \text{def.of } \models
\end{array}
$$

- $\varphi_c = \exists x\varphi$ is similar to $\exists a\varphi$.

6. **Definition of** cts. Let

$$\mathfrak{M}_c = \left(W, \prec^{\mathfrak{M}_c}, C, \mathfrak{Q}, \mathrm{P}^{\mathfrak{M}_c}\right)$$

be an arbitrary but fixed model of SClTh. We will introduce the transformation cts : Mod(SClTh) $\to$ Mod(SpecRelComp), i.e., we will construct the corresponding SpecRelComp model cts$(\mathfrak{M}_c)$ from the information that $\mathfrak{M}_c$ contains. Such a SpecRelComp model will be given as

$$\mathrm{cts}(\mathfrak{M}_c) \stackrel{\text{def}}{=} (\mathrm{cts}_B(\mathfrak{M}_c), \mathrm{stc}_{\mathrm{IOb}}(\mathfrak{M}_c), \mathrm{stc}_{\mathrm{Ph}}(\mathfrak{M}_c), \mathfrak{Q}, \mathrm{stc}_{\mathrm{W}}(\mathfrak{M}_c))$$

where the four undefined entity will be body domain and the meanings of predicates IOb, Ph and W, respectively.

   (a) **Body domain** $\mathrm{cts}_B(\mathfrak{M}_c)$. The first idea would be that a body will be identified with a set of events (the worldline). Even if we have a predicate variable sort for that purpose, we do not have the quantifiers for that sort, and thus we cannot translate the formulas of the form $\exists b\varphi$. We will sort out a lot of worldlines; we keep only those that are worldlines of observers or photons. In models of SpecRelComp, there are no other worldlines anyway. The worldlines of observers seems to be easy, the set

$$\{w \in W \,:\, (\exists q \in Q)(w, c, q) \in \mathrm{P}^{\mathfrak{M}_c}\}$$

   seems to be a fine candidate. But this won't be enough, since a SpecRelComp observer is very different from a clock. If we take a closer look on the axioms about the interaction of IOb and W, a SpecRelComp observer knows where is forward, where is right, where is up, while a clock does not know this alone; it needs (mutually orthogonal) partners c_x, c_y, c_z to represent these directions. So an observer is a coordinate system rather than a body drifting alone in the Minkowski spacetime. Thus we are going to identify an inertial observer with a 4-tuple of clocks c, c_x, c_y, c_z:

$$
\begin{aligned}
wl_{c,c_x,c_y,c_z} \stackrel{\text{def}}{=} \{e \in W \,:\, & (\exists q \in Q)(e, c, q) \in \mathrm{P}^{\mathfrak{M}_c} \text{ and} \\
& (c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c}\}
\end{aligned}
$$

where $\mathrm{CoordSys}^{\mathfrak{M}_c}$ is the meaning of the formula defined on p. 62.

But the worldlines of photons must be different, since no observer can travel as fast as the light, and there are no terms for photons in the language of SClTh. We will use the relation of light-like separation instead. Using that relation we can identify every photon with a pair of lightlike separated events $e_1 \nearrow^{\mathfrak{M}_c} e_2$. Let us define (in the object language) the lightline determined by (e_1, e_2):

$$e \in \mathrm{lline}(e_1, e_2) \overset{\mathrm{def}}{\Leftrightarrow} e_1 \nearrow e_2 \nearrow e \vee e_1 \nearrow e \nearrow e_2 \vee e \nearrow e_1 \nearrow e_2$$

Now take the meaning of that formula, i.e., let

$$\mathrm{lline}^{\mathfrak{M}_c}(w_1, w_2) \overset{\mathrm{def}}{=} \left\{ w \in W \ : \ \begin{array}{c} w_1 \nearrow^{\mathfrak{M}_c} w_2 \nearrow^{\mathfrak{M}_c} w \text{ or} \\ w_1 \nearrow^{\mathfrak{M}_c} w \nearrow^{\mathfrak{M}_c} w_2 \text{ or} \\ w \nearrow^{\mathfrak{M}_c} w_1 \nearrow^{\mathfrak{M}_c} w_2 \end{array} \right\}$$

Now we can merge the two concepts of wordlines (worldline of a photon as the lightline defined by two lightlike events, and the worldline of a observer defined via 5 clocks) in the following way: a body is determined by a 5-tuple $(c, c_x, c_y, c_z, c_t, e_1, e_2)$ in the following way: if $e_1 \nearrow^{\mathfrak{M}_c} e_2$ then it is the light-line determined by e_1, e_2, otherwise it is the worldline of c:

$$\begin{aligned} & wl_{c,c_x,c_y,c_z,e_1,e_2} \overset{\mathrm{def}}{=} \\ & \quad \left\{ e \in W \ : \ \left[e_1 \nearrow e_2 \text{ and } \left(\begin{array}{c} e_1 \nearrow^{\mathfrak{M}_c} e_2 \nearrow^{\mathfrak{M}_c} e \text{ or} \\ e_1 \nearrow^{\mathfrak{M}_c} e \nearrow^{\mathfrak{M}_c} e_2 \text{ or} \\ e \nearrow^{\mathfrak{M}_c} e_1 \nearrow^{\mathfrak{M}_c} e_2 \end{array} \right) \right] \right. \\ & \qquad \qquad \left. \text{or } \left[\begin{array}{c} \text{not } e_1 \nearrow e_2 \text{ and} \\ (\exists q \in Q)(e, c, q) \text{ and} \\ (c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c} \end{array} \right] \right\} \end{aligned}$$

According to that definition it is not true that every 6-tuple $c, c_x, c_y, c_z, e_1, e_2$ determines a body: this can happen when $e_1 \nearrow e_2$ is not true and the 4 clock do not constitute a coordinate system. So the set of suitable 6-tuples to name bodies will be

$$\begin{aligned} WL \overset{\mathrm{def}}{=} \{ & \langle c, c_x, c_y, c_z, e_1, e_2 \rangle \in C^4 \times W^2 \ : \\ & e_1 \nearrow^{\mathfrak{M}_c} e_2 \text{ or } (c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c} \} \end{aligned}$$

Also note that, according to the definitions, if both $e_1 \nearrow^{\mathfrak{M}_c} e_2$ and $(c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c}$, i.e., it is both capable of referring to a photon and an inertial observer, then we always refer with that tuple to the photon.

But a lot of 6-tuple can name the same worldline, so we have to find a suitable equivalence relation to factorize over the set of bodies to create the final domain of bodies.

$$(c, c_x, c_y, c_z, e_1, e_2) \overset{b}{\simeq} (c', c'_x, c'_y, c'_z, e'_1, e'_2) \overset{\text{def}}{\Leftrightarrow}$$
$$[e_1 \nearrow^{\mathfrak{M}_c} e_2 \text{ and } \mathrm{lline}^{\mathfrak{M}_c}(e_1, e_2) = \mathrm{lline}^{\mathfrak{M}_c}(e'_1, e'_2)]$$
$$\text{or} \begin{bmatrix} \text{not } e_1 \nearrow^{\mathfrak{M}_c} e_2 \text{ and} \\ wl_{c,c_x,c_y,c_z} = wl_{c',c'_x,c'_y,c'_z} \text{ and} \\ (c, c_x, c'_x), (c, c_y, c'_y), (c, c_z, c'_z) \in \mathrm{C}^{\mathfrak{M}_c} \end{bmatrix}$$

where C is the meaning of the collinearity relation, see Def. 37. This is an equivalence relation (definable in the language of clock logic). So the domain of the bodies will be

$$\mathrm{cts}_B(\mathfrak{M}_c) \overset{\text{def}}{=} WL/\overset{b}{\simeq}.$$

(b) **meaning of observer predicate** $\mathrm{cts}_{\mathrm{IOb}}(\mathfrak{M}_c)$

$$\mathrm{cts}_{\mathrm{IOb}}(\mathfrak{M}_c) \overset{\text{def}}{=} \{(c, c_x, c_y, c_z, e_1, e_2)/\overset{b}{\simeq} \in \mathrm{cts}_{\mathrm{B}} :$$
$$\text{not } e_1 \nearrow^{\mathfrak{M}} e_2 \text{ and } (c, c_x, c_y, c_z) \in \mathrm{CoordSys}^{\mathfrak{M}_c}\}$$

(c) **meaning of photon predicate** $\mathrm{cts}_{\mathrm{Ph}}$

$$\mathrm{cts}_{\mathrm{Ph}}(\mathfrak{M}_c) \overset{\text{def}}{=} \left\{(c, c_x, c_y, c_z, e_1, e_2)/\overset{b}{\simeq} \in \mathrm{cts}_{\mathrm{B}} : e_1 \nearrow^{\mathfrak{M}_c} e_2\right\}$$

(d) **meaning of worldview relation** $\mathrm{cts}_{\mathrm{W}}(\mathfrak{M}_c)$

$$\mathrm{cts}_{\mathrm{W}}(\mathfrak{M}_c) \overset{\text{def}}{=}$$
$$\{(c, c_x, c_y, c_z, e_1, e_2)/\overset{b}{\simeq}, (c', c'_x, c'_y, c'_z, e'_1, e'_2)/\overset{b}{\simeq}, \vec{x} \in \mathrm{cts}_{\mathrm{B}}(\mathfrak{M}_c)^2 \times Q^4 :$$
$$(\exists e \in wl_{c',c'_x,c'_y,c'_z,e'_1,e'_2})\mathrm{Coord}^{\mathfrak{M}_c}(c, c_x, c_y, c_z, e) = \vec{x}\}$$

where $\mathrm{Coord}^{\mathfrak{M}_c}$ is the meaning of the coordinatization relation defined on p. 63.

7. **Definition of ζ.** By the above construction of the domains, we choose ζ to be

$$\zeta : \begin{array}{ccc} x_i & \mapsto & x_i \\ b_i & \mapsto & \langle a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}, e_{2i}, e_{2i+1} \rangle \end{array}$$

8. **Definition of** STC_ζ.

$$
\begin{array}{lll}
\mathrm{STC}_\zeta(b=b') & \stackrel{\mathrm{def}}{=} & (\zeta_5(b) \rightsquigarrow \zeta_6(b') \wedge \mathrm{lline}(\zeta_{5-6}(b)) = \mathrm{lline}(\zeta_{5-6}(b'))) \vee \\
 & & \vee \big(\neg\zeta_5(b) \rightsquigarrow \zeta_6(b) \wedge \mathrm{wline}_{\zeta_1(b)} = \mathrm{wline}_{\zeta_1(b')} \wedge \\
 & & \wedge\, \mathrm{C}(\zeta_{1,2}(b), \zeta_2(b')) \wedge \mathrm{C}(\zeta_{1,3}(b), \zeta_3(b')) \wedge \mathrm{C}(\zeta_{1,4}(b), \zeta_4(b'))\big) \\
\mathrm{STC}_\zeta(\tau=\tau') & \stackrel{\mathrm{def}}{=} & \tau = \tau' \\
\mathrm{STC}_\zeta(\tau\leq\tau') & \stackrel{\mathrm{def}}{=} & \tau \leq \tau' \\
\mathrm{STC}_\zeta(\mathrm{IOb}(b)) & \stackrel{\mathrm{def}}{=} & \mathrm{CoordSys}(\zeta_{1-4}(b)) \wedge \neg\zeta_5(b) \rightsquigarrow \zeta_6(b) \\
\mathrm{STC}_\zeta(\mathrm{Ph}(b)) & \stackrel{\mathrm{def}}{=} & \zeta_5(b) \rightsquigarrow \zeta_6(b) \\
\mathrm{STC}_\zeta(\mathrm{W}(b,b',\vec{\tau})) & \stackrel{\mathrm{def}}{=} & (\exists e \in \mathrm{wline}_{\zeta(b')})\mathrm{Coord}_{\zeta_{1-4}(b)}(e) = \vec{\tau} \\
\mathrm{STC}_\zeta(\neg\varphi) & \stackrel{\mathrm{def}}{=} & \neg\mathrm{STC}_\zeta(\varphi) \\
\mathrm{STC}_\zeta(\varphi\wedge\psi) & \stackrel{\mathrm{def}}{=} & \mathrm{STC}_\zeta(\varphi) \wedge \mathrm{STC}_\zeta(\psi) \\
\mathrm{STC}_\zeta(\exists b\varphi) & \stackrel{\mathrm{def}}{=} & \exists\zeta(b)\Big(\big(\zeta_5(b) \rightsquigarrow \zeta_6(b) \vee \mathrm{CoordSys}(\zeta_{1-4}(b))\big) \wedge \mathrm{STC}_\zeta(\varphi)\Big) \\
\mathrm{STC}_\zeta(\exists x\varphi) & \stackrel{\mathrm{def}}{=} & \exists x \mathrm{STC}_\zeta(\varphi)
\end{array}
$$

9. Proof of the equivalence (5.17). Similar to step 5. (including a lemma like Lemma 102 in case of $\exists b$).

10. Proof of (5.19): Proving SpecRelComp in SClTh According to Propositions 95 and 93, every translation of every axiom of SpecRelComp is equivalent to its 'simple-'version described in Section **??**, so we are already done.

11. Proof of (5.18): Proving SClTh in SpecRelComp. This proof itself consists only of standard analytical geometrical calculations and basic facts about Minkowski geometry. Since in this report we focus on logical issues and signalling procedures in a logical environment, we omit this proof. UNDER CONSTRUCTION

■

Chapter 6

Possible continuations

Having these results, numerous new ways of future researches became possible.

6.1 Three new researches

General Relativity We plan to allow clock-variables to denote timelike *curves* instead of timelike lines, and approach kinematics of accelerating objects in Minkowski and locally Minkowski spacetimes, and compare the resulting systems to AccRel and GenRel (see [Madarász, Németi, and Székely 2006; Andréka, Madarász, Németi, and Székely 2012]). The novelty of our approach is that in this clock-based language, contrary to the other operational approaches like [Ax 1978; Szabó 2010; Andréka and Németi 2014], it is easy to differentiate between inertial and non-inertial agents: In the spirit of twin paradox, we can say that inertial agents are those who have the fastest clocks, i.e.,

$$\text{Inertial}(a) \overset{\text{def}}{\Leftrightarrow} \forall b,x,y,x',y' \big((\mathbf{P}(a{:}x \wedge b{:}y) \wedge a{:}x' \wedge b{:}y') \rightarrow |x - x'| \geq |y - y'| \big)$$

Branching spacetimes We would like to axiomatize Minkowski branching spacetimes using an Ockhamist approach. It can be shown that we can form nominals for histories (similarly to the non-relativistic [Blackburn and Goranko 2001]) and variables for causal chains from propositional variables, which makes the prior choice principle of Belnap [1992] possible to be formalized.

Operational definition of mass We plan to give an explicit definition of mass using counterfactual scenarios provided by branching spacetimes, and to do so we can use similar ideas to those that we used in [Molnár and Székely 2014]: we can define mass-standard objects and colliding experiments, and we can take postulates that grants the functional behaviour of a defined relation "b has the relativistic mass of r according to a".

6.2 Some more ideas

The following two researches are in a more remote future:

Epistemic logic in spacetime While the knowledge of agents concerning the *causal past* is easily representable by combining the causal past operator and the clock variables, the knowledge concerning information about the spacelike related events and the causal future can be harder than that. We can know facts about distant or future events by calculating with the physical laws and the accessible evidences about the causal past of that event. To construct a framework in which we can calculate with the epistemic possibilities, we can try the **S5**-logic of historical necessity, where a history (a maximally directed set) represents a possible epistemic state that fits to the information that is available to the agent in a given event.

Applications in Quantum mechanics Having propositional variables in the language, we can use this system to analyze quantum mechanical phenomena such as the EPR paradox, either in the way of Belnap [1992] or in another way in which the structure of spacetime also plays an important role.

6.3 Summary

Since the first-order temporal logic that was outlined in Section 1.5 is more expressive than the classical and modal systems, but still has a strong completeness theorem, it can serve as a common *supersystem* for the modal and classical logical systems in the foundation of physics. Once we can explore the definability theory of first-order modal theories, that could make the proper logical investigation possible between the modal and classical approaches.

Bibliography

H. Andréka and I. Németi. Comparing theories: the dynamics of changing vocabulary. A case-study in relativity theory. In A. Baltag and S. Smets, editors, *Johan van Benthem on logical and informational dynamics*, pages 143–172. Springer-Verlag, springer series outstanding contributions to logic vol 5 edition, 2014.

H. Andréka, J. X. Madarász, and I. Németi. Defining new universes in many-sorted logic. Technical report, Alfréd Rényi Institute of Mathematics, Hungar. Acad. Sci., Budapest, 2001.

H. Andréka, J. X. Madarász, and I. Németi. Logic of space-time and relativity theory. In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of spatial logics*, pages 607–711. Springer-Verlag, Dordrecht, 2007.

H. Andréka, J. X. Madarász, I. Németi, and G. Székely. Axiomatizing relativistic dynamics without conservation postulates. *Studia Logica*, 89(2):163–186, 2008.

H. Andréka, J. X. Madarász, I. Németi, and G. Székely. A logic road from special relativity to general relativity. *Synthese*, 186(3):633–649, 2012.

J. Ax. The elementary foundations of spacetime. *Found. Phys.*, 8(7-8):507–546, 1978.

N. Belnap. Branching space-time. *Synthese*, 92(3):385–434, 1992.

P. Blackburn and V. Goranko. Hybrid ockhamist temporal logic. In *Proc. of the Eighth International Symposium on Temporal Representation and Reasoning (TIME-01, Cividale del Friuli*, pages 14–16. IEEE Computer Society Press, 2001.

Michèle Friend. On the epistemological significance of the hungarian project. *Synthese*, pages 1–17, 2014. ISSN 0039-7857. doi: 10.1007/s11229-014-0608-x.

D. Gabbay, I. Hodkinson, and M. Reynolds. *Temporal Logic (Vol. 1): Mathematical Foundations and Computational Aspects.* Oxford University Press, Inc., New York, NY, USA, 1994. ISBN 0-19-853769-7.

R. Goldblatt. Diodorean modality in minkowski spacetime. *Studia Logica*, 39 (2-3):219–236, 1980.

R. Goldblatt. *Quantifiers, Propositions and Identity.* Cambridge University Press, 2011.

NaveenSundar Govindarajalulu, Selmer Bringsjord, and Joshua Taylor. Proof verification and proof discovery for relativity. *Synthese*, pages 1–18, 2014. ISSN 0039-7857. doi: 10.1007/s11229-014-0424-3. URL `http://dx.doi.org/10.1007/s11229-014-0424-3`.

E. H. Kronheimer and R. Penrose. On the structure of causal spaces. *Proc. Cambridge Philos. Soc.*, 63:481–501, 1967.

J. X. Madarász, I. Németi, and G. Székely. Twin paradox and the logical foundation of relativity theory. *Found. Phys.*, 36(5):681–714, 2006.

J. X. Madarász and G. Székely. Special Relativity over the Field of Rational Numbers. *International Journal of Theoretical Physics*, 52(5):1706–1718, 2013.

A. Molnár and G. Székely. Axiomatizing relativistic dynamics using formal thought experiments. *Synthese*, pages 1–40, 2014. doi: 10.1007/s11229-014-0545-8. (online first).

I. Shapirovsky and V. Shehtman. Modal logics of regions and minkowski spacetime. *Journal of Logic and Computation*, 15(4):559–574, 2005.

L. E. Szabó. Empirical Foundation of Space and Time. In M. Suárez, M. Dorato, and M. Rédei, editors, *EPSA07: Launch of the European Philosophy of Science Association*, pages 251–266. Springer, 2010.

G. Székely. The existence of superluminal particles is consistent with the kinematics of Einstein's special theory of relativity. *Reports on Mathematical Physics*, 72(2):133–152, 2013.

Alfred Tarski and Steven Givant. Tarski's system of geometry. *The Bulletin of Symbolic Logic*, 5(2):175–214, 1999.