Many faces of modal logic
(classical logics)

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1 Zero-order classical logic

1.1 Language

Symbols:

- Propositional variables/atoms: $p, q, r, \ldots$ $\qquad$ $\mathrm{At} \stackrel{\mathrm{def}}{=} \{p_i : i \in \omega\}$
- Logical symbols: $\neg, \wedge$
- auxiliary symbols: $), ($

The set Form of formulas is given in the following way:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi$$

Abbreviations:

$$\begin{array}{rcl}
\bot & \stackrel{\mathrm{def}}{\Leftrightarrow} & p \wedge \neg p \\
\top & \stackrel{\mathrm{def}}{\Leftrightarrow} & q \wedge \neg q \\
\varphi - \psi & \stackrel{\mathrm{def}}{\Leftrightarrow} & \varphi \wedge \neg\psi \\
(\varphi \vee \psi) & \stackrel{\mathrm{def}}{\Leftrightarrow} & \neg(\neg\varphi \wedge \neg\psi) \\
(\varphi \to \psi) & \stackrel{\mathrm{def}}{\Leftrightarrow} & \neg(\varphi \wedge \neg\psi) \qquad \Leftrightarrow \qquad \neg\varphi \vee \psi \\
(\varphi \leftrightarrow \psi) & \stackrel{\mathrm{def}}{\Leftrightarrow} & (\varphi \to \psi) \wedge (\psi \to \varphi) \\
\varphi(\psi/p) & \stackrel{\mathrm{def}}{\Leftrightarrow} & \text{substitute all the (`free') occurrences of } p \text{ with } \psi \\
\varphi \to \psi \to \chi & \stackrel{\mathrm{def}}{\Leftrightarrow} & \varphi \to (\psi \to \chi)
\end{array}$$

and we follow the notation convention according to which the outermost parentheses can be omitted.

1.2 Models

A zero-order model, or just simply, a propositional variable-assignment, or valuation is a $V : \mathrm{At} \to \{0, 1\}$ function.

The truth or validity of formulas are given in the following way:

$$\begin{array}{rcl}
V \models p & \iff & V(p) = 1 \\
V \models \neg\varphi & \iff & V \models \varphi \text{ is false} \\
V \models \varphi \wedge \psi & \iff & V \models \varphi \text{and} V \models \psi
\end{array}$$

The V function can also be extended to Form as well by the following way:

$$\begin{array}{rcl} V(\neg\varphi) & = & 1 - V(\varphi) \\ V(\varphi \wedge \psi) & = & \min(V(\varphi), V(\psi)) \end{array}$$

And the following equivalence can be proved by induction

$$V(\varphi) = 1 \iff V \models \varphi$$

In this situation we say that V satisfies φ. The formula φ is said to be satisfiable if there is such V, and φ is said to be *valid*, $\models \varphi$, iff

$$(\forall V : \mathrm{At} \to \{0,1\}) \qquad V \models \varphi$$

The formula set Γ is said to be satisfied by V, $V \models \Gamma$ if V satisfies all of its formulas, Γ is said to be valid if $\models \Gamma$, iff

$$(\forall V : \mathrm{At} \to \{0,1\}) \qquad V \models \Gamma$$

We say that Γ semantically implies φ, $\Gamma \models \varphi$, if

$$(\forall V : \mathrm{At} \to \{0,1\}) \quad (V \models \Gamma \text{ implies } V \models \varphi)$$

1.3 Axioms

$$\begin{array}{ll} \text{(PC1)} & \varphi \to \psi \to \varphi \\ \text{(PC2)} & (\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to \varphi \to \chi \\ \text{(PC3)} & \varphi \to \psi \to \varphi \end{array}$$

$$\text{(MP)} \quad \frac{\varphi, \varphi \to \psi}{\psi}$$

Not that all the presented symbols refer to formula *schemes*, i.e., $\varphi \to \psi \to \varphi$ claims the theoremhood of the elements of

$$\{\varphi \to \psi \to \varphi : \varphi, \psi \text{ are 0-order classical formulas}\}$$

We say that φ is derivable from a Γ set of formulas, $\Gamma \vdash \varphi$ if there is a finite list of formulas for which the following statement is true: For all element φ' it is true that

- φ' is an instance of (PC1)–(PC3)
- $\varphi' \in \Gamma$
- There is an earlier element ψ' of that list for which $\psi' \to \varphi'$ also occurs earlier than φ'.

Tautologies are those formulas for which $\vdash \varphi \overset{\text{def}}{\Leftrightarrow} \varnothing \vdash \varphi$ holds.

Remark 1 (syntactical compactness property)**.** If $\Gamma \vdash \varphi$, then there is a finite subset $\Gamma_0 \subseteq \Gamma$ for which $\Gamma_0 \vdash \varphi$.

Theorem 1 (Deduction-theorem). *By (PC1) and (PC2) the following rule is admissible:*

$$\frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \to \psi}$$

Theorem 2 (Substitutivity).

$$\frac{\vdash \varphi}{\vdash \varphi(\psi/p)} \text{ for arbitrary } p \in \mathrm{At} \text{ and } \psi \in \mathrm{Form}$$

Homework 1. Is it true that

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi(\psi/p)} \text{ for arbitrary } p \in \mathrm{At} \text{ and } \psi \in \mathrm{Form?}$$

Remark 2. Note that this is not the only standard in the literature. Some authors say that (PC1)–(PC3) are not formula schemes, only formulas containing the first three propositional variables p, q and r, and the rule of substitutivity is not a theorem but an unquestionable derivation rule of the system. Obviously, the set of derivable formulas of these two approaches are the same. There is, however, a notable semantical difference between the two approach.

Definition 1 (consistent formula sets). A set of formulas Γ is *inconsistent* iff $\Gamma \vdash \bot$, otherwise it is consistent.

Homework 2. Is it true (in both approach) that every satisfiable set of formulas are consistent?

1.4 Completeness

Theorem 3. *The axiom system described above is* sound *w.r.t. the semantics, i.e., if* $\Gamma \vdash \varphi$ *then* $\Gamma \models \varphi$*.*

Theorem 4. *The axiom system described above is* strongly complete *w.r.t. the semantics, i.e., if* $\Gamma \models \varphi$ *then* $\Gamma \vdash \varphi$*.*

Proof. Idea: We prove the contraposition "$\Gamma \nvdash \varphi$ implies $\Gamma \not\models \varphi$, i.e., there is a V for which $V(\varphi) \neq 1$."

Definition 2 (maximally consistent formula sets). Γ is maximally consistent iff it is consistent but any proper extension of it is inconsistent, i.e., $\Gamma \nvdash \bot$ but for all $\Gamma' \supset \Gamma$, $\Gamma' \vdash \bot$.

Remark 3. For every V, the set $\{\varphi : V(\varphi) = 1\}$ is maximally consistent.
The characteristic function of a maximally consistent set is a valuation.

Our job is to create valuation V that dissatisfies φ. All we can use is the fact that $\Gamma \nvdash \varphi$. Fortunately, this is implies $\Gamma \cup \{\neg\varphi\} \nvdash \bot$, for if $\Gamma \cup \{\neg\varphi\} \vdash \bot$ then by ded.thm. $\Gamma \vdash \neg\varphi \to \bot$, i.e., $\Gamma \vdash \neg\neg\varphi$ which implies $\Gamma \vdash \varphi$. (to prove try PC3.)

Homework* 1. Prove that.

So we have that $\Gamma \cup \{\neg\varphi\}$ is consistent.

Lemma 5 (Lindenbaum-lemma)**.** *Every consistent set can be extended to a maximally consistent set.*

Proof. Let Σ be a consistent set, i.e., $\Sigma \nvdash \bot$. Let ℓ be a countable (because we consider only countable languages) infinite enumeration that lists all the formulas. Consider now the following nested list of formula sets: Let $\Sigma_0 := \Sigma$, and let

$$\Sigma_{n+1} := \begin{cases} \Sigma \cup \{\ell_n\} & \text{if } \Sigma \cup \ell_n \nvdash \bot \\ \Sigma \cup \{\neg \ell_n\} & \text{otherwise} \end{cases}$$

Now let $\Sigma^+ := \bigcup_{i \in \omega} \Sigma_i$.

Homework* 2. Prove that Σ^+ is a maximally consistent set. Use synt. compactness property.

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Let $\Gamma^+_{\neg\varphi}$ a maximally consistent set that extends $\Gamma \cup \{\neg\varphi\}$. The existence of this set is provided by Lindenbaum's lemma. Now the valuation that refutes φ is the characteristic function of that function

$$V_{\neg\varphi}(\psi) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } \psi \in \Gamma^+_{\neg\varphi} \\ 0 & \text{otherwise} \end{cases}$$

Homework 3. Show that $V_{\neg\varphi}(\varphi) = 0$.

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Remark 4. Note that the key to the completeness proof was the following: Since $\Gamma \cup \{\neg\varphi\}$ was consistent, there exists a set in which **"truth is membership"**, i.e.,

$$V_{\neg\varphi} \models \psi \iff \text{“}\Gamma^+_{\neg\varphi} \models \psi\text{”} \iff \psi \in \Gamma^+_{\neg\varphi}$$

We will follow this idea from now in every completeness proof we discuss.

1.5 Kripkean models for classical propositional logic

The Reader may also noticed during her/his logical education that logical rules and the boolean rules of set theoretical operations are similar. One explanation is that we use logical operations in the definition of $\cap, \cup, \setminus$, and an other explanation is that this logic is complete w.r.t. (full) set algebras.

A Kripkean model of classical propositional logic is $\mathfrak{M} = \langle W, V \rangle$ triples, where $W \neq \varnothing$, and $V : \mathrm{At} \to \wp W$ is a valuation that assigns subsets of W to the propositional variables.

We refer to the elements of W by $w, v, u, \ldots$ and we call them *worlds*. All these worlds are considered to be models of the previous interpretation in the sense that φ is true in w iff $w \in V(p)$.

The truth or validity of formulas *in a world* are given in the following way:

$$\begin{array}{rcl} \mathfrak{M}, w \models p & \iff & w \in V(p) \\ \mathfrak{M} \models \neg\varphi & \iff & \mathfrak{M} \models \varphi \text{ is false} \\ \mathfrak{M} \models \varphi \wedge \psi & \iff & \mathfrak{M} \models \varphi \text{and} \mathfrak{M} \models \psi \end{array}$$

The V function can also be extended to Form as well by the following way:

$$\begin{array}{rcl} V(\neg\varphi) & = & W \setminus V(\varphi) \\ V(\varphi \wedge \psi) & = & V(\varphi) \cap V(\psi) \end{array}$$

And the following equivalence can be proved by induction

$$V(\varphi) = X \iff (\forall w \in X)\mathfrak{M}, w \models \varphi$$

So $V(\varphi)$ is the set of worlds in which φ is true.

We say that φ satisfiable iff there is a $\mathfrak{M}$ and a w for which $\mathfrak{M}, w \models \varphi$. The formula φ is said to be true in $\mathfrak{M}$ iff $V(\varphi) = W$. The formula φ is said to be valid iff it is true in all models. A set of formulas is satisfiable iff there is a single $\mathfrak{M}$ and w for which every formula of it is true. Γ is true in $\mathfrak{M}$ iff every formula of it is true in $\mathfrak{M}$. A set of formulas is valid if every formulas of it is valid.

1.6 Kripke completeness by canonical model construction

We will do more that just give a countermodel: **we will give one single countermodel which dissatisfies all the non-valid formulas**. Because of that property of it, we will call this model a *canonical* model.

The idea is, again, the construction via the "truth is membership" principle. The canonical worlds (worlds of the canonical model) will be maximally consistent formulas sets. All of them.

The canonical model is $\mathfrak{C} = \langle \mathfrak{W}, \mathfrak{V} \rangle$, where $\mathfrak{W}$ is the set of all maximally consistent formula sets, and we define the valuation to fit to the "truth is membership" idea: $\mathfrak{V}(p) = \{\Gamma \in \mathfrak{W} : p \in \Gamma\}$.

This is a model indeed: W is not empty because there are consistent sets (soundness thm) and by the Lindenbaum lemma.

Theorem 6 ("Truth is Membership"). *For every $\Gamma \in \mathfrak{W}$ the following holds:*

$$\varphi \in \Gamma \iff \Gamma \in \mathfrak{V}(\varphi) \iff \mathfrak{M}_c, \Gamma \models \varphi$$

Homework 4. Prove that statement.

Now the completeness proof goes like this:

Since $\Gamma \cup \{\neg\varphi\}$ is consistent, it can be extended to a maximally consistent set by the Lindenbaum lemma. This means that some possible world $\Gamma^+_{\neg\varphi} \in \mathfrak{W}$ for which $\Gamma \cup \{\neg\varphi\} \subseteq \Gamma^+_{\neg\varphi}$. By the "truth is membership" theorem we have the desired $\mathfrak{M}_c, \Gamma^+_{\neg\varphi} \models \Gamma$ but $\mathfrak{M}_c, \Gamma^+_{\neg\varphi} \not\models \varphi$ result. (This counts as a counter model, since $\mathfrak{M}_c$ is a model indeed.)

1.7 Admissible Kripke semantics

Remember the step when we said that V "is a valuation that assigns subsets of W to the propositional variables. Why did we do that? Because $\wp W$ has the desired property we need, that is, it is logically well-behaving: It is closed under conjunction (intersection) and negation (complementation). But also note that it is also an overkill. The property "the set of subsets that are closed under intersection and complementation" does not characterizes uniquely the powersets. A trivial example is $\{\varnothing, W\}$ for any W.

Homework 5. Consider the set ω of natural numbers. Let us call a set cofinite iff its complementer is finite. Show that the set $A = \{X \subseteq \omega : \; X \text{ is finite or cofinite}\}$ is closed under intersection and complementation over ω.

Generalized Kripke semantics is when we builds that idea into the notion of models. Right now, in classical logic the difference between these models are only philosophical, but later we will see that the similar ideas in modal and second-order logic are crucial in axiomatizability.

A generalized/admissible Kripkean model of classical propositional logic is $\mathfrak{M} = \langle W, V, A \rangle$ triples, where $W \neq \varnothing$, and $V : \mathrm{At} \to A$ is a valuation that assigns "logically well behaving" subsets of W to the propositional variables. The collection of logically well-behaving sets are A, where logical well-behavior abbreviates the following two stipulations:

$$\begin{array}{rcl} X \in A & \Longrightarrow & W \setminus X \\ X, Y \in A & \Longrightarrow & X \cap Y \end{array}$$

Two trivial example for A are $A = \{\varnothing, W\}$ and $A = \wp W$. If $A = \wp W$, then we call the Kripkean model *full*. (So Kripkean models are the same as those admissible models that are full – therefore, we have more models in this way, that is why we call them generalized models. The word "admissible" refers to the fact that a valuation can not render arbitrary sets to the atoms, only the admissible ($\in A$) ones.)

All the other notions (truth, validity, ...) are the same.

The completeness proof is also the same, though generalized canonical model is itself different; we have to specify the third component, the logically well-behaved sets. Again, we follow the "truth is membership" principle. The canonical model is $\mathfrak{C} = \langle \mathfrak{W}, \mathfrak{V}, \mathfrak{A} \rangle$, where $\mathfrak{W}$ and $\mathfrak{V}(p)$ are the same, and $\mathfrak{A} = \{\mathfrak{V}(\varphi) : \varphi \in \mathrm{Form}\} = \{X \subseteq \mathfrak{W} : \exists \varphi \quad X = \{\Gamma \in \mathfrak{W} : \varphi \in \Gamma\}\}$.

This is a generalized model indeed: The set A of admissible sets are admissible indeed because the restrictions correspond to the criteria of maximal consistency.

2 One-sorted first-order classical logic

2.1 Language

- Symbols:
  - Variables: $x_1, x_2, x_3, \ldots$ $\qquad$ $\mathrm{Var} \stackrel{\text{def}}{=} \{x_i : i \in \omega\}$
  - Constants: $\mathsf{a}_1, \mathsf{a}_2, \ldots$ $\qquad$ $\mathrm{Const} \stackrel{\text{def}}{=} \{\mathsf{a}_i : i \in \omega\}$
  - Function symbols: $f_1, f_2, \ldots$ $\qquad$ $\mathrm{Func} \stackrel{\text{def}}{=} \{\mathsf{f}_i : i \in \omega\}$
  - Mathematical predicate symbol: $P_1, P_2, \ldots$ $\qquad$ $\mathrm{Pred} \stackrel{\text{def}}{=} \{\mathsf{P}_i : i \in \omega\}$
  - Logical symbols: $\neg, \wedge, =, \exists$
- Terms:

$$\tau ::= x \mid \mathsf{a} \mid f(\tau, \tau')$$

- Formulas:

$$\varphi ::= \ \tau = \tau' \ \mid \ P(\tau_1, \tau_2, \dots, \tau_n)$$
$$\neg\varphi \ \mid \ \varphi \wedge \psi \ \mid \ \exists x \varphi \ \mid$$

Definition 3 (free occurences, sentences, etc.)**.** A term is closed if no variable occurs in it. In any formula $\exists x \varphi$, φ is called the *scope* of that particular token of $\exists x$. An occurrence of x is free in φ iff that occurrence of x does not lie in a scope of a $\exists x$ in φ. A variable x is free in a formula φ iff it has a free occurrence in φ. An x is free for $\tau(x_1, x_2, \dots, x_n)$ iff no free occurrence of x is in the scope of a $\exists x_1$, $\exists x_2$, $\dots \exists x_n$. (Roughly speaking, it is "not problematic" to substitute τ for x in φ.)

A first-order formula φ is a *first-order sentence* if it has no free variables.

Homework* 3. Define the metalinguistic notion "x is free in φ" and "x is free for $\tau(x_1, \dots, x_n)$ in φ" **inductively**.

2.2 Models

A FOL model is a (maybe infinite) tuple

$$\mathfrak{M} = \left(U, \mathsf{a}_1^{\mathfrak{M}}, \dots, f_1^{\mathfrak{M}}, \dots P_1^{\mathfrak{M}}, \dots\right),$$

or sometimes just denoted as a pair

$$\mathfrak{M} = \left(U, (\cdot)^{\mathfrak{M}}\right)$$

where $U \neq 0$, $a_i^{\mathfrak{M}} \in U$, $f_i^{\mathfrak{M}} : U^n \to U$, $P_i^{\mathfrak{M}} \subseteq U^m$ for some n and m for all i.

Assignments are functions of form $\sigma : \mathrm{Var} \to U$. σ is an x-variant of σ', $\sigma \overset{x}{\sim} \sigma'$ iff $\sigma(y) = \sigma'(y)$ for all $y \neq x$.

Extending the meaning of constants and functions to the meaning of arbitrary terms:

$$\begin{array}{rcll} x_i^{\mathfrak{M},\sigma} & \overset{\text{def}}{=} & \sigma(x_i) & \text{for all } i \in \omega \\ (f_i(\tau_1, \dots, \tau_n))^{\mathfrak{M},\sigma} & \overset{\text{def}}{=} & f_i^{\mathfrak{M}}(\tau_1^{\mathfrak{M},\sigma}, \dots, \tau_n^{\mathfrak{M},\sigma}) & \text{for all } i \end{array}$$

Truth w.r.t. an assignment:

$$\begin{array}{lcl} \mathfrak{M}, \sigma \models \tau_1 = \tau_2 & \iff & \tau_1^{\mathfrak{M},\sigma} = \tau_2^{\mathfrak{M},\sigma} \\ \mathfrak{M}, \sigma \models P_i(\tau_1, \dots, \tau_n) & \iff & \left(\tau_1^{\mathfrak{M},\sigma}, \dots, \tau_n^{\mathfrak{M},\sigma}\right) \in P_i^{\mathfrak{M}} \\ \mathfrak{M}, \sigma \models \neg\varphi & \iff & \mathfrak{M}, \sigma \not\models \varphi \\ \mathfrak{M}, \sigma \models \varphi \wedge \psi & \iff & \mathfrak{M}, \sigma \models \varphi \text{ and } \mathfrak{M}, \sigma \models \psi \\ \mathfrak{M}, \sigma \models \exists x \varphi & \iff & \text{there is a } \sigma' \overset{y}{\sim} \sigma; \mathfrak{M}, \sigma' \models \varphi \end{array}$$

φ is true in $\mathfrak{M}$, $\mathfrak{M} \models \varphi$, iff $\mathfrak{M} \models \varphi$ for all σ.

2.3 Generalized models