Temporal Logic

Definability and Completeness

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Definability

Models

Give a countermodel
a) for every formula what we labelled 'strange', such that
b) for some formula what we labelled 'fine'.
(i.e., give a model in which the formula in question is not true (i.e., false in some world of it))

A **model** $\mathfrak{M}$ is a pair $\langle \mathfrak{F}, V \rangle$ where

- $\mathfrak{F}$ is a frame $\mathfrak{F} = \langle W, R \rangle$,
- V is an evaluation $V : \text{At} \to \mathcal{P}(W)$.

We define the **satisfaction** or **local truth** relation in the following way:

$$\begin{array}{lcl} \mathfrak{M}, w \models p & \overset{\text{def}}{\Leftrightarrow} & w \in V(p) \\ \mathfrak{M}, w \models \neg\varphi & \overset{\text{def}}{\Leftrightarrow} & \text{it is not true that } \mathfrak{M}, w \models \varphi \\ \mathfrak{M}, w \models \varphi \wedge \psi & \overset{\text{def}}{\Leftrightarrow} & \mathfrak{M}, w \models \varphi \text{ and } \mathfrak{M}, w \models \psi \\ \mathfrak{M}, w \models \mathbf{F}\varphi & \overset{\text{def}}{\Leftrightarrow} & \exists v \big(w < v \wedge \mathfrak{M}, v \models \varphi \big) \\ \mathfrak{M}, w \models \mathbf{P}\varphi & \overset{\text{def}}{\Leftrightarrow} & \exists v \big(v < w \wedge \mathfrak{M}, v \models \varphi \big) \end{array}$$

We define the **global truth** or just simply the **truth** relation based on the local truth:

$$\mathfrak{M} \models \varphi \iff \forall w\, \mathfrak{M}, w \models \varphi$$

And the most important: we say that φ is valid of $\mathfrak{F}$ iff it is true *no matter what are the meanings of its atomic particles*:

$$\mathfrak{F} \models \varphi \iff \forall V\, \mathfrak{F}, V \models \varphi$$

Why is the latter so important? Because only the structure matters here. So by investigating validities, we will able to investigate the structure of time, while we keep the local perspective of the modal language.

A-B Correspondences (modal definability)

Difficulty	Name	TL formula	FOL formula	Name
Easy	**T**	$\Box\varphi \to \varphi$	$\forall w\ wRw$	reflexive
Easy	**4**	$\Box\varphi \to \Box\Box\varphi$	$\forall wvu.\ wRvRu \to wRu$	transitive
Normal	**Den**	$\Box\Box\varphi \to \Box\varphi$	$\forall wv.\ wRu \to (\exists v)wRvRu$	dense
Easy	**B**	$\varphi \to \Box\Diamond\varphi$	$\forall wv.\ wRv \to vRw$	symmetric
Normal	**E**	$\Diamond\varphi \to \Box\Diamond\varphi$	$\forall wv.\ u\text{Я}wRv \to vRu$	euclidean
Normal	**G**	$\Diamond\Box\varphi \to \Box\Diamond\varphi$	$\forall wvu.\ v\text{Я}wRu \to (\exists u')(vRu'\text{Я}u)$	convergent
Normal	**.3**	$\Diamond\varphi \wedge \Diamond\psi \to (\Diamond(\varphi \wedge \Diamond\psi)\vee \Diamond(\varphi \wedge \psi)\vee \Diamond(\Diamond\varphi \wedge \psi))$	$\forall wvu.\ v\text{Я}wRu \to (vRu \vee v\text{Я}u \vee u = v)$	no branching to the right
Hard	**.3**	$\Box(\underline{\Box}\varphi \to \psi)\vee \Box(\underline{\Box}\psi \to \varphi)$	$\forall wvu.\ v\text{Я}wRu \to (vRu \vee v\text{Я}u \vee u = v)$	no branching to the right
Easy	**D**	$\Box\varphi \to \Diamond\varphi$	$\forall w\exists v\ wRv$	serial
Easy	**D$^+$**	$\Box(\Box\varphi \to \varphi)$	$\forall wv.\ wRv \to vRv$	secondary reflexive
Beautiful	**GL**	$\Box(\Box\varphi \to \varphi) \to \Box\varphi$	$\forall wvu(wRvRu \to wRu) \wedge \neg\exists P(\forall w \in P)(\exists v\text{Я}w)\, P(v)$	Noetherian SPO
Beautiful	**Grz**	$\Box(\Box(\varphi \to \Box\varphi) \to \to \varphi) \to \varphi$	$\forall w\ wRw\wedge \forall wvu\ (wRvRu \to wRu) \wedge \neg\exists P(\forall w \in P)(\exists v\text{Я}w)(w \neq v \wedge P(v))$	reflexive Noetherian partial ordering
Easy	**V**	$\Box\varphi$	$\forall wv\ \neg wRv$	empty
Easy	**Tr**	$\varphi \to \Box\varphi$	$\forall wv.\ wRv \to w = v$	diagonal
Normal	**1.1**	$\Diamond\varphi \to \Box\varphi$	$\forall wvu.\ v\text{Я}wRu \to v = u$	partial function
Normal	**ijkl**	$\Diamond^i\Box^j\varphi \to \Box^k\Diamond^l\varphi$	$\forall wvu.\ v\text{Я}^i wR^k u \to (\exists u')(vR^j u'\text{Я}^l u)$	*ijkl*-convergent

A-B Correspondences (modal definability)

Difficulty	Name	TL formula	FOL formula	Name
		?	$\forall w\ \neg wRw$	irreflexive
		?	$\forall wvu.\ wRvRu \to \neg wRu$	intransitive
		?	$\forall wv.\ wRv \to \neg vRw$	antisymmetric
		?	$\neg\forall wvu.\ v\reflectbox{R}wRu \to (vRu \vee v\reflectbox{R}u \vee u = v)$	there are branches

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We will show that none of them are definable.

FRAME / MODEL OPERATIONS

1. Disjoint union

   Glueing frames together – that act is modally invisible btw.

2. Submodel generation

   Erasing things from the frame in a modally invisible way

3. Zig-zag mapping (“*p*-morphism”, “bounded morphism”)

   Super handy modally invisible transformation

4. Ultrafilter extension

   Putting all contingency into the frame – beautiful advanced stuff, but we won’t discuss it.

GOLDBLATT-THOMASON THEOREM: A first-order definable class K of frames is modally definable iff its validity is is closed under zig-zag mapping, subframe generation, disjoint unions and reflects ultrafilter extensions.

Homomorphisms

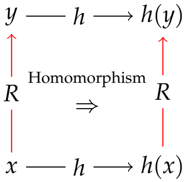
Homomorphism h from $\mathfrak{M}$ to $\mathfrak{M}'$ are $h : W \to W'$ such that they

1. preserve the valuation:

$$\mathfrak{M}, w \models p \quad \Longrightarrow \quad \mathfrak{M}', h(w) \models p$$

2. preserve the relation:

$$xRy \Longrightarrow h(x)R'h(y)$$

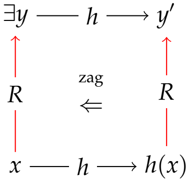


ZIG-ZAG-MORPHISMS

Zig-zag-morphisms h from $\mathfrak{M}$ to $\mathfrak{M}'$ are homomorphisms $h : W \to W'$ satisfying the same atoms and making zags:

$$\mathfrak{M}, w \models p \quad \Longleftarrow \quad \mathfrak{M}', h(w) \models p$$

$$h(x)R'y' \Rightarrow (\exists y \mathrel{\reflectbox{R}} x) h(y) = y'$$



Zig-zags = The upward arrows (R) force each other out.

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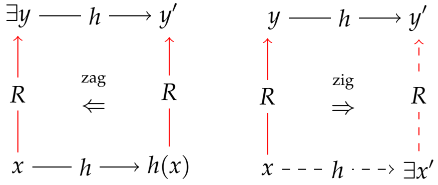

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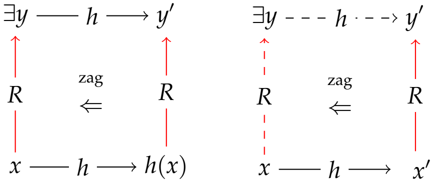

Zig-zags = The upward arrows (R) force each other out.

Invariance of truth

Truth is preserved under zig-zag-mapping.

$$\mathfrak{M}, w \models \varphi \iff \mathfrak{M}', h(w) \models \varphi$$

So φ is true in a world of a model iff it is true in the zig-zag image of it.

PROOF: By structural induction:

1. atoms p: $\mathfrak{M}, w \models p \iff \mathfrak{M}', h(w) \models p$ – but that's how we defined it!
2. negations $\neg\varphi$:

$$\begin{array}{ccc} \mathfrak{M}, w \models \neg\varphi & \iff & \mathfrak{M}, h(w) \models \neg\varphi \\ \Updownarrow & \text{ind.hip.} & \Updownarrow \\ \mathfrak{M}', w \not\models \varphi & \overset{\downarrow}{\iff} & \mathfrak{M}, h(w) \not\models \varphi \end{array}$$

3. conjunctions $\varphi \wedge \psi$: pretty much the same
4. $\Diamond\varphi$:

$$\begin{array}{ccc} \mathfrak{M}, w \models \Diamond\varphi & \iff & \mathfrak{M}', h(w) \models \Diamond\varphi \\ \Updownarrow & \text{zig+ind.hip.} & \Updownarrow \\ (\exists v \mathfrak{R} w)\ \mathfrak{M}, v \models \varphi & \overset{\downarrow}{\Rightarrow} & (h(v) \mathfrak{R} h(w)) \mathfrak{M}, h(v) \models \varphi \end{array}$$

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4 $\Diamond\varphi$:

$$\begin{array}{ccc} \mathfrak{M}, w \models \Diamond\varphi & \iff & \mathfrak{M}', h(w) \models \Diamond\varphi \\ \Updownarrow & \text{zag+ind.hip.} & \Updownarrow \\ (\exists v \mathfrak{R} w)\ \mathfrak{M}, v \models \varphi & \overset{\downarrow}{\Leftarrow} & (\exists u \mathfrak{R} h(w)) \mathfrak{M}, h(u) \models \varphi \end{array}$$

Invariance of validity

Frame-validity is invariant under **surjective** zig-zag-mapping:

$$\mathfrak{F} \models \varphi \iff h(\mathfrak{F}) \models \varphi$$

So φ is valid on all zig-zag image iff it is valid on the original.

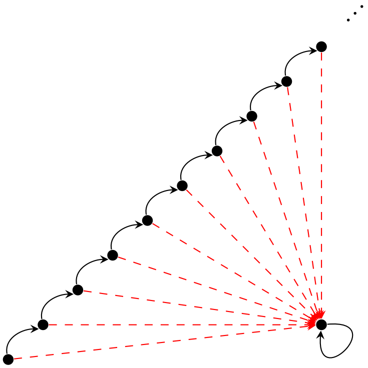
PROOF:

$$\begin{array}{ccc} \mathfrak{F} \not\models \varphi & \iff & h(\mathfrak{F}) \not\models \varphi \\ \Updownarrow & & \Updownarrow \\ \exists V \ \underbrace{\mathfrak{F}, V}_{\mathfrak{M}}, w \not\models \varphi & \overset{\text{inv.of.truth.}}{\underset{\downarrow}{\iff}} & \exists V' \ h(\mathfrak{F}), V', h(w) \not\models \varphi \end{array}$$

Here the V and V' are given by different reasons if we consider the different directions of the proof. One is given by the non-validity, and the other is determined by that; it is chosen to make a model zig-zag morphism from the frame zig-zag morphism h.

EXAMPLE

$$h : (\mathbb{N}, suc) \twoheadrightarrow (\{\bullet\}, \{(\bullet, \bullet)\})$$



this proves that irreflexivity is not definable! If there is a formula defining that property, then that should be valid on $(\mathbb{N}, <)$, and its validity should be preserved under h – so that formula would be valid on a reflexive frame, and that is wrong.

Example

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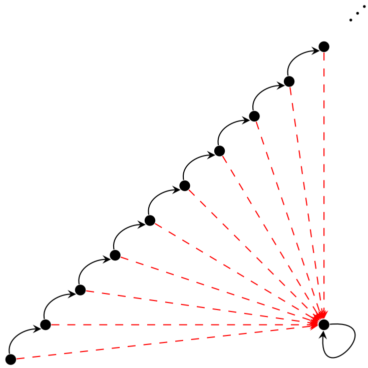

Prove that antisymmetry and intransitivity is not definable.

this proves that irreflexivity is not definable! If there is a formula defining that property, then that should be valid on $(\mathbb{N}, <)$, and its validity should be preserved under h – so that formula would be valid on a reflexive frame, and that is wrong.

Completeness

FREEDOM

$\mathfrak{M}$ **has more freedom** than $\mathfrak{M}'$ iff $\mathfrak{M}$ has less truth then $\mathfrak{M}'$, i.e.,

$$\mathrm{Th}(\mathfrak{M}) \subseteq \mathrm{Th}(\mathfrak{M}') \quad \text{where} \quad \mathrm{Th}(\mathfrak{M}) \stackrel{\text{def}}{=} \{\varphi : \mathfrak{M} \models \varphi\}$$

Are there models $\mathfrak{M}$ that are **absolutely free**, i.e.,

$$\forall \mathfrak{M}' \qquad \mathrm{Th}(\mathfrak{M}) \subseteq \mathrm{Th}(\mathfrak{M}')$$

Give two models $\mathfrak{M}, \mathfrak{M}'$, s.t. $\mathrm{Th}(\mathfrak{M}) \subseteq \mathrm{Th}(\mathfrak{M}')$

That would mean that such an absolutely free model is **free from any contingencies** that are definable in the modal language. If something can be falsified in some model, then it will be false in the absolutely free one as well. Therefore, such a model would act like a **universal countermodel**.

CHARACTERIZATION

The minimal temporal logic is $\mathbf{K} \stackrel{\text{def}}{=} \{\varphi : \forall \mathfrak{F}\ \mathfrak{F} \models \varphi\}$, i.e., the set of validities of frames in general. How can we create a logic from a set of formulas?

We will write $\models_{\mathbf{K}} \varphi$ instead of $\varphi \in \mathbf{K}$.

We will write $\psi_1, \ldots, \psi_n \models_{\mathbf{K}} \varphi$ instead of $\models_{\mathbf{K}} (\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$.

We will write $\Gamma \models_{\mathbf{K}} \varphi$ instead of $(\exists \psi_1, \ldots, \psi_n \in \Gamma)\ \psi_1, \ldots, \psi_n \models_{\mathbf{K}} \varphi$.

An absolutely free model $\mathfrak{M}$ then **characterizes K**:

$$\mathfrak{M} \models \varphi \iff \models_{\mathbf{K}} \varphi$$

So “valid on every frame” = “true on $\mathfrak{M}$”.

COMPLETENESS

It would be an interesting/important question whether **K**, the logic of all frames, is finitely axiomatizable or not, i.e.,

is there a finite list of axioms and rules with which one could derive all the validities of **K**? Or in other words,

is there a finite syntactical characterization of that logic too?

We will see that finite axiomatizability, strong completeness hinges on the fact that during the construction of the absolutely free model we use only finitely many axiom schemes and derivation rules!

A concrete free model

We will construct an absolutely free model $\mathfrak{M}_{\mathbf{K}}$ from **K**. In other words, we will construct a model whose

- every world will be a syntactical object: a special set of formulas.
- alternative relation will be a syntactical relation: some special subset of that formula set is a subset of another one.
- valuation will be given by the membership relation: for an atomic sentence,

to be **true** in a set of formulas

is the same as

to be **in** that set of formulas

We will invent the alternative relation and the notion of worlds in a way to make this property true for any kind of formulas, not only for the atoms, so to prove a statement like this

$$\varphi \in \Gamma \iff \mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$$

This will be the so-called **free** model of **K**, the property above will be called **Truth Lemma**.

A FREE MODEL OF K

Show that if $\Gamma = \{\varphi : w \models \varphi\}, \Gamma' = \{\varphi : w' \models \varphi\}$ $wRw' \iff \mathbf{G}^-(\Gamma) \subseteq \Gamma'$ is **not** tru

$$\mathfrak{M}_\mathbf{K} \stackrel{\text{def}}{=} (W_\mathbf{K}, R_\mathbf{K}, V_\mathbf{K})$$

where

- $W_\mathbf{K} \stackrel{\text{def}}{=} \{\Gamma \ : \ \Gamma \text{ is a maximally } \mathbf{K}\text{-consistent set}\}$, i.e.,
  - Every Γ is **K**-consistent: $\Gamma \not\models_\mathbf{K} \bot$
  - These Γ's is so huge that if you try to put one more formula in it, you would make it **K**-inconsistent:

$$(\forall \varphi \not\ni \Gamma) \quad \Gamma \cup \{\varphi\} \models_\mathbf{K} \bot$$

- $\Gamma R_\mathbf{K} \Gamma'$ iff Γ' contains φ whenever Γ contains $\mathbf{G}\varphi$, formally:

$$\Gamma R_\mathbf{K} \Gamma' \stackrel{\text{def}}{\Leftrightarrow} \mathbf{G}^-(\Gamma) \subseteq \Gamma' \qquad \text{where } \mathbf{G}^-(\Gamma) \stackrel{\text{def}}{=} \{\varphi \ : \ \mathbf{G}\varphi \in \Gamma\}$$

- $\Gamma \in V_\mathbf{K}(p) \stackrel{\text{def}}{\Leftrightarrow} p \in \Gamma$

FREE ALTERNATIVE RELATION

The followings are equivalent:

- $\Gamma R_{\mathbf{K}} \Gamma'$
- Γ' contains α whenever Γ contains $\mathbf{G}\alpha$, formally:

$$\mathbf{G}^{-}(\Gamma) \subseteq \Gamma' \qquad \text{where } \mathbf{G}^{-}(\Gamma) \stackrel{\text{def}}{=} \{\alpha : \mathbf{G}\alpha \in \Gamma\}$$

- Γ contains $\mathbf{F}\beta$ whenever Γ' contains β, formally:

$$\Gamma \supseteq \mathbf{F}^{+}(\Gamma') \qquad \text{where } \mathbf{F}^{+}(\Gamma') \stackrel{\text{def}}{=} \{\mathbf{F}\beta : \beta \in \Gamma'\}$$

- Γ contains γ whenever Γ' contains $\mathbf{H}\gamma$, formally:

$$\Gamma \supseteq \mathbf{H}^{-}(\Gamma') \qquad \text{where } \mathbf{H}^{-}(\Gamma') \stackrel{\text{def}}{=} \{\gamma : \mathbf{H}\gamma \in \Gamma'\}$$

- Γ' contains $\mathbf{P}\delta$ whenever Γ contains δ, formally:

$$\mathbf{P}^{+}(\Gamma) \subseteq \Gamma' \qquad \text{where } \mathbf{P}^{+}(\Gamma) \stackrel{\text{def}}{=} \{\mathbf{P}\delta : \delta \in \Gamma\}$$

FREE ALTERNATIVE RELATION

$$\frac{\mathbf{P}^+(\Gamma) \subseteq \Gamma' \Rightarrow \mathbf{G}^-(\Gamma) \subseteq \Gamma' \qquad \Gamma \supseteq \mathbf{F}^+(\Gamma') \Rightarrow \Gamma \supseteq \mathbf{H}^-(\Gamma')}{}$$

$\mathbf{G}\varphi \in \Gamma$	assumption	$\mathbf{H}\varphi \in \Gamma'$	assumption
$\mathbf{PG}\varphi \in \Gamma'$	by $\mathbf{P}^+(\Gamma) \subseteq \Gamma'$	$\mathbf{FH}\varphi \in \Gamma$	by $\Gamma \supseteq \mathbf{F}^+(\Gamma')$
$\varphi \in \Gamma'$	because $\models_{\mathbf{K}} \mathbf{PG}\varphi \to \varphi$	$\varphi \in \Gamma$	because $\models_{\mathbf{K}} \mathbf{FH}\varphi \to \varphi$

FREE ALTERNATIVE RELATION

$$\frac{\mathbf{P}^+(\Gamma) \subseteq \Gamma' \Rightarrow \mathbf{G}^-(\Gamma) \subseteq \Gamma' \qquad\qquad \Gamma \supseteq \mathbf{F}^+(\Gamma') \Rightarrow \Gamma \supseteq \mathbf{H}^-(\Gamma')}{}$$

$\mathbf{G}\varphi \in \Gamma$	assumption	$\mathbf{H}\varphi \in \Gamma'$	assumption
$\mathbf{PG}\varphi \in \Gamma'$	by $\mathbf{P}^+(\Gamma) \subseteq \Gamma'$	$\mathbf{FH}\varphi \in \Gamma$	by $\Gamma \supseteq \mathbf{F}^+(\Gamma')$
$\varphi \in \Gamma'$	because $\models_{\mathbf{K}} \mathbf{PG}\varphi \to \varphi$	$\varphi \in \Gamma$	because $\models_{\mathbf{K}} \mathbf{FH}\varphi \to \varphi$

Prove that
$\models_{\mathbf{K}} \mathbf{PG}\varphi \to \varphi$ and
$\models_{\mathbf{K}} \mathbf{FH}\varphi \to \varphi$

Prove the remaining directions!

Truth Lemma

$$\varphi \in \Gamma \iff \mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$$

$p \in \Gamma$	$\iff$	$\Gamma \in V_{\mathbf{K}}(p)$	by def. of $V_{\mathbf{K}}$
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$	by def. of $\models$
$\neg\varphi \in \Gamma$	$\iff$	$\varphi \notin \Gamma$	Γ is consistent
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \not\models \varphi$	ind.hip.
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \neg\varphi$	by def. of $\models$
$\varphi \wedge \psi \in \Gamma$	$\iff$	$\varphi \in \Gamma$ and $\psi \in \Gamma$	Γ is maximally cons.
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$ and $\mathfrak{M}_{\mathbf{K}}, \Gamma \models \psi$	ind.hip.
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi \wedge \psi$	by def. of $\models$

Truth Lemma

$$\varphi \in \Gamma \iff \mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$$

$$
\begin{array}{rcll}
\mathbf{G}\varphi \in \Gamma & \iff & \varphi \in \mathbf{G}^{-}(\Gamma) & \text{def.of } \mathbf{G}^{-} \\
\text{we should prove the other direction!} & \Longrightarrow & (\forall \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) & \\
 & & \qquad \text{implies } \varphi \in \Gamma'\big] & \\
 & \iff & (\forall \Gamma' \reflectbox{R}_{\mathbf{K}} \Gamma)\ \varphi \in \Gamma' & \text{def.of } R_{\mathbf{K}} \\
 & \iff & (\forall \Gamma' \reflectbox{R}_{\mathbf{K}} \Gamma)\ \mathfrak{M}_{\mathbf{K}}, \Gamma' \models \varphi & \text{ind.hip.} \\
 & \iff & \mathfrak{M}_{\mathbf{K}}, \Gamma \models \mathbf{G}\varphi & \text{by def. of } \models
\end{array}
$$

$$
\begin{array}{rcll}
\mathbf{H}\varphi \in \Gamma & \iff & \varphi \in \mathbf{H}^{-}(\Gamma) & \text{def.of } \mathbf{H}^{-} \\
\text{we should prove the other direction!} & \Longrightarrow & (\forall \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{H}^{-}(\Gamma) & \\
 & & \qquad \text{implies } \varphi \in \Gamma'\big] & \\
 & \iff & (\forall \Gamma' \in W_{\mathbf{K}})\big[\mathbf{G}^{-}(\Gamma') \subseteq \Gamma & \\
 & & \qquad \text{implies } \varphi \in \Gamma'\big] & \text{the equivalences} \\
 & \iff & (\forall \Gamma' R_{\mathbf{K}} \Gamma)\ \varphi \in \Gamma' & \text{def.of } R_{\mathbf{K}} \\
 & \iff & (\forall \Gamma' R_{\mathbf{K}} \Gamma)\ \mathfrak{M}_{\mathbf{K}}, \Gamma' \models \varphi & \text{ind.hip.} \\
 & \iff & \mathfrak{M}_{\mathbf{K}}, \Gamma \models \mathbf{H}\varphi & \text{by def. of } \models
\end{array}
$$

Existence Lemma

$$\mathbf{G}\varphi \in \Gamma \iff \varphi \in \mathbf{G}^{-}(\Gamma) \Longleftarrow (\forall \Gamma' \in W_{\mathbf{K}})\left[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ implies } \varphi \in \Gamma'\right]$$

Take the contraposition instead!

$$\mathbf{G}\varphi \notin \Gamma \iff \varphi \notin \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\left[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \varphi \notin \Gamma'\right]$$

Since we have maximally consistent sets,

$$\neg\mathbf{G}\varphi \in \Gamma \iff \neg\varphi \in \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\left[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \neg\varphi \in \Gamma'\right]$$

Let $\varphi := \neg\psi$ ("if it is true for any φ, it is true for any negation $\neg\psi$"):

$$\neg\mathbf{G}\neg\psi \in \Gamma \iff \neg\neg\psi \in \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\left[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \neg\neg\psi \in \Gamma'\right]$$

i.e., our meditational object will be

$$\mathbf{F}\psi \in \Gamma \iff \psi \in \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\left[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \psi \in \Gamma'\right]$$

So we have to prove that the presence of a $\mathbf{F}\psi$ in a m.c.s. enforce the existence of an other, **related** m.c.s. set, in which φ is contained.

Existence Lemma

$$\mathbf{F}\psi \in \Gamma \iff \psi \in \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \psi \in \Gamma'\big]$$

$\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$ is $\mathbf{K}$-consistent. For if

$\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$	$\models_{\mathbf{K}}$	$\bot$	indirect assumption
$\mathbf{G}^{-}(\Gamma)$	$\models_{\mathbf{K}}$	$\neg\varphi$	Deduction theorem
$\exists \chi_1, \cdots, \chi_n$	$\models_{\mathbf{K}}$	$\neg\varphi$	def.of $\mathbf{G}^{-}(\Gamma) \models_{\mathbf{K}}$
	$\models_{\mathbf{K}}$	$(\chi_1 \wedge \cdots \wedge \chi_n) \rightarrow \neg\varphi$	def.of $(\chi_1 \wedge \cdots \wedge \chi_n) \models_{\mathbf{K}}$
	$\models_{\mathbf{K}}$	$\mathbf{G}(\chi_1 \wedge \cdots \wedge \chi_n) \rightarrow \mathbf{G}\neg\varphi$	See below!
	$\models_{\mathbf{K}}$	$(\mathbf{G}\chi_1 \wedge \cdots \wedge \mathbf{G}\chi_n) \rightarrow \mathbf{G}\neg\varphi$	See below!
$\mathbf{G}\chi_1, \cdots, \mathbf{G}\chi_n$	$\models_{\mathbf{K}}$	$\mathbf{G}\neg\varphi$	def.of $(\mathbf{G}\chi_1 \wedge \cdots \wedge \mathbf{G}\chi_n) \models_{\mathbf{K}}$
Γ	$\models_{\mathbf{K}}$	$\mathbf{G}\neg\varphi$	$\chi \in \mathbf{G}^{-}(\Gamma) \Leftrightarrow \mathbf{G}\chi \in \Gamma$
Γ	$\models_{\mathbf{K}}$	$\neg\mathbf{F}\varphi$	Duality
$\Gamma \cup \{\mathbf{F}\varphi\}$	$\models_{\mathbf{K}}$	$\bot$	Deduction theorem
Γ	$\models_{\mathbf{K}}$	$\bot$	we assumed that $\mathbf{F}\varphi \in \Gamma$

Remember that we relied on basically the following two logical rule:

$$\models_{\mathbf{K}} (\mathbf{G}\varphi \wedge \mathbf{G}\psi) \rightarrow \mathbf{G}(\varphi \wedge \psi) \qquad \frac{\models_{\mathbf{K}} \varphi \rightarrow \psi}{\models_{\mathbf{K}} \mathbf{G}\varphi \rightarrow \mathbf{G}\psi}$$

Existence Lemma

$$\mathbf{P}\psi \in \Gamma \iff \psi \in \mathbf{H}^-(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{H}^-(\Gamma) \text{ and } \psi \in \Gamma'\big]$$

$\mathbf{H}^-(\Gamma) \cup \{\varphi\}$ is **K**-consistent. For if

$\mathbf{H}^-(\Gamma) \cup \{\varphi\}$	$\models_{\mathbf{K}}$	$\bot$	indirect assumption
$\mathbf{H}^-(\Gamma)$	$\models_{\mathbf{K}}$	$\neg\varphi$	Deduction theorem
$\exists \chi_1, \cdots, \chi_n$	$\models_{\mathbf{K}}$	$\neg\varphi$	def.of $\mathbf{H}^-(\Gamma) \models_{\mathbf{K}}$
	$\models_{\mathbf{K}}$	$(\chi_1 \wedge \cdots \wedge \chi_n) \rightarrow \neg\varphi$	def.of $(\chi_1 \wedge \cdots \wedge \chi_n) \models_{\mathbf{K}}$
	$\models_{\mathbf{K}}$	$\mathbf{H}(\chi_1 \wedge \cdots \wedge \chi_n) \rightarrow \mathbf{H}\neg\varphi$	See below!
	$\models_{\mathbf{K}}$	$(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \rightarrow \mathbf{H}\neg\varphi$	See below!
$\mathbf{H}\chi_1, \cdots, \mathbf{H}\chi_n$	$\models_{\mathbf{K}}$	$\mathbf{H}\neg\varphi$	def.of $(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \models_{\mathbf{K}}$
Γ	$\models_{\mathbf{K}}$	$\mathbf{H}\neg\varphi$	$\chi \in \mathbf{H}^-(\Gamma) \Leftrightarrow \mathbf{H}\chi \in \Gamma$
Γ	$\models_{\mathbf{K}}$	$\neg\mathbf{P}\varphi$	Duality
$\Gamma \cup \{\mathbf{P}\varphi\}$	$\models_{\mathbf{K}}$	$\bot$	Deduction theorem
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Remember that we relied on basically the following two logical rule:

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$\exists \chi_1, \cdots, \chi_n$	$\models_{\mathbf{K}}$	$\neg\varphi$	def.of $\mathbf{H}^-(\Gamma) \models_{\mathbf{K}}$
	$\models_{\mathbf{K}}$	$(\chi_1 \wedge \cdots \wedge \chi_n) \to \neg\varphi$	def.of $(\chi_1 \wedge \cdots \wedge \chi_n) \models_{\mathbf{K}}$
	$\models_{\mathbf{K}}$	$\mathbf{H}(\chi_1 \wedge \cdots \wedge \chi_n) \to \mathbf{H}\neg\varphi$	See below!
	$\models_{\mathbf{K}}$	$(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \to \mathbf{H}\neg\varphi$	See below!
$\mathbf{H}\chi_1, \cdots, \mathbf{H}\chi_n$	$\models_{\mathbf{K}}$	$\mathbf{H}\neg\varphi$	def.of $(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \models_{\mathbf{K}}$
Γ	$\models_{\mathbf{K}}$	$\mathbf{H}\neg\varphi$	$\chi \in \mathbf{H}^-(\Gamma) \Leftrightarrow \mathbf{H}\chi \in \Gamma$
Γ	$\models_{\mathbf{K}}$	$\neg\mathbf{P}\varphi$	Duality
$\Gamma \cup \{\mathbf{P}\varphi\}$	$\models_{\mathbf{K}}$	$\bot$	Deduction theorem
Γ	$\models_{\mathbf{K}}$	$\bot$	we assumed that $\mathbf{P}\varphi \in \Gamma$

Remember that we relied on basically the following two logical rule:

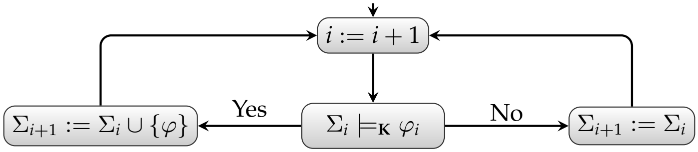
$$\models_{\mathbf{K}} (\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi) \qquad \frac{\models_{\mathbf{K}} \varphi \to \psi}{\models_{\mathbf{K}} \mathbf{H}\varphi \to \mathbf{H}\psi}$$

Prove that these are valid indeed!

LINDENBAUM'S LEMMA

Since $\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$ is **K**-consistent, it can be extended into a maximally consistent set Γ'. Just list all the formulas and start the following procedure: take the first formula: Is it consistent with $\Sigma_0 \stackrel{\text{def}}{=} \mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$? If it is, then extend Σ_0 with that formula, if not, then don't. Repeat this into the infinity. Your m.c.s. will be $\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$ the one will contain every formula with which you would extend.

$$\Sigma_0 := \Gamma \cup \{\mathbf{F}\varphi\}$$

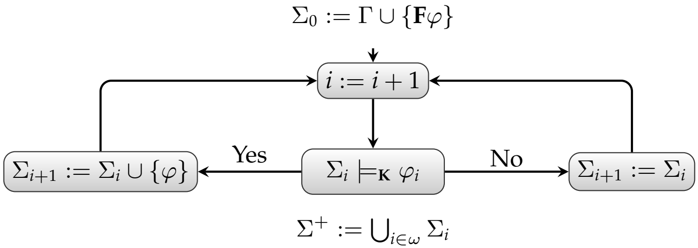


$$\Sigma^{+} := \bigcup_{i \in \omega} \Sigma_i$$

Similarly for **H**.

Lindenbaum's Lemma

Since $\mathbf{G}^-(\Gamma) \cup \{\varphi\}$ is **K**-consistent, it can be extended into a maximally consistent set Γ'. Just list all the formulas and start the following procedure: take the first formula: Is it consistent with $\Sigma_0 \overset{\text{def}}{=} \mathbf{G}^-(\Gamma) \cup \{\varphi\}$? If it is, then extend Σ_0 with that formula, if not, then don't. Repeat this into the infinity. Your m.c.s. will be $\mathbf{G}^-(\Gamma) \cup \{\varphi\}$ the one will contain every formula with which you would extend.



Similarly for **H**.

Prove that Σ^+ must be consistent!

Absolute freedom / Characterization / free model theorem

Only (Exactly) the valid formulas are true in $\mathfrak{M}_{\mathbf{K}}$.

$$\mathfrak{M}_{\mathbf{K}} \models \varphi \iff \models_{\mathbf{K}} \varphi$$

We show that

$$\mathfrak{M}_{\mathbf{K}} \not\models \varphi \iff \not\models_{\mathbf{K}} \varphi.$$

Since the construction called "free model" is a real **model** indeed, we have the $\Rightarrow$ direction.
If $\not\models_{\mathbf{K}} \varphi$, then $\{\neg\varphi\}$ is **K**-consistent. Therefore we can extend it to a maximally **K**-consistent $\Gamma^{\neg\varphi}$ set by Lindenbaum's lemma. But this set is a world in the free model $\mathfrak{M}_{\mathbf{K}}$. And since this world contains $\neg\varphi$, it is **true** in it by the Truth lemma:

$$\neg\varphi \in \Gamma^{\neg\varphi} \Longrightarrow \mathfrak{M}_{\mathbf{K}}, \Gamma^{\neg\varphi} \models \neg\varphi$$

And we are ready, since we found a world of $\mathfrak{M}_{\mathbf{K}}$ where $\neg\varphi$ is true, i.e., φ is not true neither in that world nor in the whole model.

Completeness

Consider the following tautology:

Every non-**K**-valid formula is falsifiable on some model

A completeness theorem has a similar form:

Every non-**K**-theorem is falsifiable on some model

where the word "theorem" refers to some syntactic derivation system. Using the absolute freedom of $\mathfrak{M}_{\mathbf{K}}$, we can freely interchange the second part of that sentence even in that opaque environment:

Every non-**K**-theorem is falsifiable on the free model $\mathfrak{M}_{\mathbf{K}}$

So we would have a finite syntactic characterization/axiomatization of **K** if we can define a finite derivational system satisfying that sentence. Remember that we used only the validity of the following temporal formulas and rules:

$$\mathbf{PG}\varphi \to \varphi \qquad \mathbf{FH}\varphi \to \varphi$$

$$(\mathbf{G}\varphi \wedge \mathbf{G}\psi) \to \mathbf{G}(\varphi \wedge \psi) \qquad (\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi)$$

$$\frac{\varphi \to \psi}{\mathbf{G}\varphi \to \mathbf{G}\psi} \qquad \frac{\varphi \to \psi}{\mathbf{H}\varphi \to \mathbf{H}\psi}$$

DERIVATION FOR K

We define the derivation relation $\vdash_{\mathbf{K}}$ inductively: It is the smallest relation satisfying the followings:

- $\vdash_{\mathrm{K}} \varphi \to .\psi \to \varphi$ for all φ, ψ,
- $\vdash_{\mathrm{K}} \varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$ for all φ, ψ, χ,
- $\vdash_{\mathrm{K}} \varphi \to \psi \to .\neg\psi \to \neg\varphi$ for all φ, ψ,
- $\vdash_{\mathrm{K}} \mathbf{PG}\varphi \to \varphi$ for all φ,
- $\vdash_{\mathrm{K}} \mathbf{FH}\varphi \to \varphi$ for all φ,
- $\vdash_{\mathrm{K}} (\mathbf{G}\varphi \wedge \mathbf{G}\psi) \to \mathbf{G}(\varphi \wedge \psi)$ for all φ, ψ,
- $\vdash_{\mathrm{K}} (\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi)$ for all φ, ψ,
- If $\vdash_{\mathrm{K}} \varphi$ and $\vdash_{\mathrm{K}} \varphi \to \psi$ then $\vdash_{\mathrm{K}} \psi$,
- If $\vdash_{\mathrm{K}} \varphi \to \psi$, then $\vdash_{\mathrm{K}} \mathbf{G}\varphi \to \mathbf{G}\psi$,
- If $\vdash_{\mathrm{K}} \varphi \to \psi$, then $\vdash_{\mathrm{K}} \mathbf{H}\varphi \to \mathbf{H}\psi$.

$$\psi_1, \ldots, \psi_n \vdash_{\mathbf{K}} \varphi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad \vdash_{\mathbf{K}} (\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi$$

$$\Gamma \vdash_{\mathbf{K}} \varphi \quad \stackrel{\text{def}}{\Leftrightarrow} \quad (\exists \psi_1, \ldots, \psi_n \in \Gamma)\psi_1, \ldots, \psi_n \vdash_{\mathbf{K}} \varphi$$

DERIVATION FOR **K**

<u>THEOREM</u>: $\Gamma \vdash_{\mathbf{K}} \varphi$ iff there is a finite list of formulas (called **proof**) such that for every formula of that list it is true that

- it is an axiom of **K**
- it is an element of Γ
- it can be derived from previous list-members using a rule of **K**.

You should try to prove it – this theorem is one of those for which it is true that everybody see why is it true, but not everybody can prove it step by step.

Strong completeness

THEOREM:

$$\Gamma \vdash_{\mathbf{K}} \varphi \quad \Longleftrightarrow \quad \Gamma \models_{\mathbf{K}} \varphi$$

To prove $\Leftarrow$, we show

$$\Gamma \nvdash_{\mathbf{K}} \varphi \quad \Longrightarrow \quad \Gamma \not\models_{\mathbf{K}} \varphi$$

From the premise we have $\Gamma \cup \{\neg\varphi\} \nvdash_{\mathbf{K}} \bot$. Then by replacing the sign $\models_{\mathbf{K}}$ with $\vdash_{\mathbf{K}}$ in Lindenbaum's lemma we can conclude that there is a maximally consistent set $\Sigma^{\Gamma\cup\{\neg\varphi\}}$ that contains $\Gamma \cup \{\neg\varphi\}$. Now, again, replace $\models_{\mathbf{K}}$ with $\vdash_{\mathbf{K}}$ everywhere in the definition of the free model: The resulting construction will be called **canonical model**. Since we used exactly the axioms of **K** in these proofs, we have the corresponding version of the free model theorem (called canonical model theorem) and the truth lemma as well. Then $\mathfrak{M}_{\mathbf{K}}$ has a world – that would be $\Sigma^{\Gamma\cup\{\varphi\}}$ – which contains every element of $\Gamma \cup \{\neg\varphi\}$, therefore, by the truth lemma,

$$(\forall\psi \in \Gamma) \quad \mathfrak{M}_{\mathbf{K}}, \Sigma^{\Gamma\cup\{\varphi\}} \models \psi \qquad \text{but} \qquad \mathfrak{M}_{\mathbf{K}}, \Sigma^{\Gamma\cup\{\varphi\}} \not\models \varphi$$

So since the free model is a model, we have the desired counter model for $\Gamma \models_{\mathbf{K}} \varphi$.

Prove $\Rightarrow$.

Canonical model

The canonical model

We will construct an absolutely free model $\mathfrak{M}_{\mathbf{K}}$ from $\vdash_{\mathbf{K}}$. In other words, we will construct a model whose

- every world will be a syntactical object: a special set of formulas.
- alternative relation will be a syntactical relation: some special subset of that formula set is a subset of another one.
- valuation will be given by the membership relation: for an atomic sentence,

to be **true** in a set of formulas

is the same as

to be **in** that set of formulas

We will invent the alternative relation and the notion of worlds in a way to make this property true for any kind of formulas, not only for the atoms, so to prove a statement like this

$$\varphi \in \Gamma \iff \mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$$

This will be the so-called **canonical** model of **K**, the property above will be called **Truth Lemma**.

A CANONICAL MODEL OF $\mathbf{K}$

$$\mathfrak{M}_{\mathbf{K}} \stackrel{\text{def}}{=} (W_{\mathbf{K}}, R_{\mathbf{K}}, V_{\mathbf{K}})$$

where

- $W_{\mathbf{K}} \stackrel{\text{def}}{=} \{\Gamma : \Gamma \text{ is a maximally } \mathbf{K}\text{-consistent set}\}$, i.e.,
  - Every Γ is **K**-consistent: $\Gamma \nvdash_{\mathbf{K}} \bot$
  - These Γ's is so huge that if you try to put one more formula in it, you would make it **K**-inconsistent:

$$(\forall \varphi \not\ni \Gamma) \quad \Gamma \cup \{\varphi\} \vdash_{\mathbf{K}} \bot$$

- $\Gamma R_{\mathbf{K}} \Gamma'$ iff Γ' contains φ whenever Γ contains $\mathbf{G}\varphi$, formally:

$$\Gamma R_{\mathbf{K}} \Gamma' \stackrel{\text{def}}{\Leftrightarrow} \mathbf{G}^{-}(\Gamma) \subseteq \Gamma' \qquad \text{where } \mathbf{G}^{-}(\Gamma) \stackrel{\text{def}}{=} \{\varphi : \mathbf{G}\varphi \in \Gamma\}$$

- $\Gamma \in V_{\mathbf{K}}(p) \stackrel{\text{def}}{\Leftrightarrow} p \in \Gamma$

CANONICAL ALTERNATIVE RELATION

The followings are equivalent:

- $\Gamma R_{\mathbf{K}} \Gamma'$
- Γ' contains α whenever Γ contains $\mathbf{G}\alpha$, formally:

$$\mathbf{G}^{-}(\Gamma) \subseteq \Gamma' \qquad \text{where } \mathbf{G}^{-}(\Gamma) \stackrel{\text{def}}{=} \{\alpha : \mathbf{G}\alpha \in \Gamma\}$$

- Γ contains $\mathbf{F}\beta$ whenever Γ' contains β, formally:

$$\Gamma \supseteq \mathbf{F}^{+}(\Gamma') \qquad \text{where } \mathbf{F}^{+}(\Gamma') \stackrel{\text{def}}{=} \{\mathbf{F}\beta : \beta \in \Gamma'\}$$

- Γ contains γ whenever Γ' contains $\mathbf{H}\gamma$, formally:

$$\Gamma \supseteq \mathbf{H}^{-}(\Gamma') \qquad \text{where } \mathbf{H}^{-}(\Gamma') \stackrel{\text{def}}{=} \{\gamma : \mathbf{H}\gamma \in \Gamma'\}$$

- Γ' contains $\mathbf{P}\delta$ whenever Γ contains δ, formally:

$$\mathbf{P}^{+}(\Gamma) \subseteq \Gamma' \qquad \text{where } \mathbf{P}^{+}(\Gamma) \stackrel{\text{def}}{=} \{\mathbf{P}\delta : \delta \in \Gamma\}$$

CANONICAL ALTERNATIVE RELATION

$$\frac{\mathbf{P}^+(\Gamma) \subseteq \Gamma' \Rightarrow \mathbf{G}^-(\Gamma) \subseteq \Gamma' \qquad\qquad \Gamma \supseteq \mathbf{F}^+(\Gamma') \Rightarrow \Gamma \supseteq \mathbf{H}^-(\Gamma')}{}$$

$\mathbf{G}\varphi \in \Gamma$	assumption	$\mathbf{H}\varphi \in \Gamma'$	assumption
$\mathbf{PG}\varphi \in \Gamma'$	by $\mathbf{P}^+(\Gamma) \subseteq \Gamma'$	$\mathbf{FH}\varphi \in \Gamma$	by $\Gamma \supseteq \mathbf{F}^+(\Gamma')$
$\varphi \in \Gamma'$	because $\vdash_{\mathbf{K}} \mathbf{PG}\varphi \to \varphi$	$\varphi \in \Gamma$	because $\vdash_{\mathbf{K}} \mathbf{FH}\varphi \to \varphi$

CANONICAL ALTERNATIVE RELATION

$\mathbf{P}^+(\Gamma) \subseteq \Gamma' \Rightarrow \mathbf{G}^-(\Gamma) \subseteq \Gamma'$		$\Gamma \supseteq \mathbf{F}^+(\Gamma') \Rightarrow \Gamma \supseteq \mathbf{H}^-(\Gamma')$	
$\mathbf{G}\varphi \in \Gamma$	assumption	$\mathbf{H}\varphi \in \Gamma'$	assumption
$\mathbf{PG}\varphi \in \Gamma'$	by $\mathbf{P}^+(\Gamma) \subseteq \Gamma'$	$\mathbf{FH}\varphi \in \Gamma$	by $\Gamma \supseteq \mathbf{F}^+(\Gamma')$
$\varphi \in \Gamma'$	because $\vdash_{\mathbf{K}} \mathbf{PG}\varphi \to \varphi$	$\varphi \in \Gamma$	because $\vdash_{\mathbf{K}} \mathbf{FH}\varphi \to \varphi$

Prove that $\vdash_{\mathbf{K}} \mathbf{PG}\varphi \to \varphi$ and $\vdash_{\mathbf{K}} \mathbf{FH}\varphi \to \varphi$

Prove the remaining directions!

TRUTH LEMMA

$$\varphi \in \Gamma \iff \mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$$

$p \in \Gamma$	$\iff$	$\Gamma \in V_{\mathbf{K}}(p)$	by def. of $V_{\mathbf{K}}$
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$	by def. of $\models$
$\neg\varphi \in \Gamma$	$\iff$	$\varphi \notin \Gamma$	Γ is consistent
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \not\models \varphi$	ind.hip.
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \neg\varphi$	by def. of $\models$
$\varphi \wedge \psi \in \Gamma$	$\iff$	$\varphi \in \Gamma$ and $\psi \in \Gamma$	Γ is maximally cons.
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$ and $\mathfrak{M}_{\mathbf{K}}, \Gamma \models \psi$	ind.hip.
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi \wedge \psi$	by def. of $\models$

TRUTH LEMMA

$$\varphi \in \Gamma \iff \mathfrak{M}_{\mathbf{K}}, \Gamma \models \varphi$$

$\mathbf{G}\varphi \in \Gamma$	$\iff$	$\varphi \in \mathbf{G}^{-}(\Gamma)$	def.of $\mathbf{G}^{-}$
we should prove the other direction!	$\Longrightarrow$	$(\forall \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma)$ implies $\varphi \in \Gamma'\big]$	
	$\iff$	$(\forall \Gamma' \reflectbox{R}_{\mathbf{K}} \Gamma)\ \varphi \in \Gamma'$	def.of $R_{\mathbf{K}}$
	$\iff$	$(\forall \Gamma' \reflectbox{R}_{\mathbf{K}} \Gamma)\ \mathfrak{M}_{\mathbf{K}}, \Gamma' \models \varphi$	ind.hip.
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \mathbf{G}\varphi$	by def. of $\models$
$\mathbf{H}\varphi \in \Gamma$	$\iff$	$\varphi \in \mathbf{H}^{-}(\Gamma)$	def.of $\mathbf{H}^{-}$
we should prove the other direction!	$\Longrightarrow$	$(\forall \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{H}^{-}(\Gamma)$ implies $\varphi \in \Gamma'\big]$	
	$\iff$	$(\forall \Gamma' \in W_{\mathbf{K}})\big[\mathbf{G}^{-}(\Gamma') \subseteq \Gamma$ implies $\varphi \in \Gamma'\big]$	the equivalences
	$\iff$	$(\forall \Gamma' R_{\mathbf{K}} \Gamma)\ \varphi \in \Gamma'$	def.of $R_{\mathbf{K}}$
	$\iff$	$(\forall \Gamma' R_{\mathbf{K}} \Gamma)\ \mathfrak{M}_{\mathbf{K}}, \Gamma' \models \varphi$	ind.hip.
	$\iff$	$\mathfrak{M}_{\mathbf{K}}, \Gamma \models \mathbf{H}\varphi$	by def. of $\models$

EXISTENCE LEMMA

$$\mathbf{G}\varphi \in \Gamma \iff \varphi \in \mathbf{G}^{-}(\Gamma) \Longleftarrow (\forall \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ implies } \varphi \in \Gamma'\big]$$

Take the contraposition instead!

$$\mathbf{G}\varphi \notin \Gamma \iff \varphi \notin \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \varphi \notin \Gamma'\big]$$

Since we have maximally consistent sets,

$$\neg\mathbf{G}\varphi \in \Gamma \iff \neg\varphi \in \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \neg\varphi \in \Gamma'\big]$$

Let $\varphi := \neg\psi$ ("if it is true for any φ, it is true for any negation $\neg\psi$"):

$$\neg\mathbf{G}\neg\psi \in \Gamma \iff \neg\neg\psi \in \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \neg\neg\psi \in \Gamma'\big]$$

i.e., our meditational object will be

$$\mathbf{F}\psi \in \Gamma \iff \psi \in \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \psi \in \Gamma'\big]$$

So we have to prove that the presence of a $\mathbf{F}\psi$ in a m.c.s. enforce the existence of an other, **related** m.c.s. set, in which φ is contained.

EXISTENCE LEMMA

$$\mathbf{F}\psi \in \Gamma \iff \psi \in \mathbf{G}^{-}(\Gamma) \Longrightarrow (\exists \Gamma' \in W_{\mathbf{K}})\big[\Gamma' \supseteq \mathbf{G}^{-}(\Gamma) \text{ and } \psi \in \Gamma'\big]$$

$\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$ is $\mathbf{K}$-consistent. For if

$\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$	$\vdash_{\mathbf{K}}$	$\bot$	indirect assumption
$\mathbf{G}^{-}(\Gamma)$	$\vdash_{\mathbf{K}}$	$\neg\varphi$	Deduction theorem
$\exists \chi_1, \cdots, \chi_n$	$\vdash_{\mathbf{K}}$	$\neg\varphi$	def.of $\mathbf{G}^{-}(\Gamma) \vdash_{\mathbf{K}}$
	$\vdash_{\mathbf{K}}$	$(\chi_1 \wedge \cdots \wedge \chi_n) \to \neg\varphi$	def.of $(\chi_1 \wedge \cdots \wedge \chi_n) \vdash_{\mathbf{K}}$
	$\vdash_{\mathbf{K}}$	$\mathbf{G}(\chi_1 \wedge \cdots \wedge \chi_n) \to \mathbf{G}\neg\varphi$	See below!
	$\vdash_{\mathbf{K}}$	$(\mathbf{G}\chi_1 \wedge \cdots \wedge \mathbf{G}\chi_n) \to \mathbf{G}\neg\varphi$	See below!
$\mathbf{G}\chi_1, \cdots, \mathbf{G}\chi_n$	$\vdash_{\mathbf{K}}$	$\mathbf{G}\neg\varphi$	def.of $(\mathbf{G}\chi_1 \wedge \cdots \wedge \mathbf{G}\chi_n) \vdash_{\mathbf{K}}$
Γ	$\vdash_{\mathbf{K}}$	$\mathbf{G}\neg\varphi$	$\chi \in \mathbf{G}^{-}(\Gamma) \Leftrightarrow \mathbf{G}\chi \in \Gamma$
Γ	$\vdash_{\mathbf{K}}$	$\neg\mathbf{F}\varphi$	Duality
$\Gamma \cup \{\mathbf{F}\varphi\}$	$\vdash_{\mathbf{K}}$	$\bot$	Deduction theorem
Γ	$\vdash_{\mathbf{K}}$	$\bot$	we assumed that $\mathbf{F}\varphi \in \Gamma$

Remember that we relied on basically the following two logical rule:

$$\vdash_{\mathbf{K}} (\mathbf{G}\varphi \wedge \mathbf{G}\psi) \to \mathbf{G}(\varphi \wedge \psi) \qquad \frac{\vdash_{\mathbf{K}} \varphi \to \psi}{\vdash_{\mathbf{K}} \mathbf{G}\varphi \to \mathbf{G}\psi}$$

Existence Lemma

$$\mathbf{P}\psi \in \Gamma \iff \psi \in \mathbf{H}^-(\Gamma) \Longrightarrow (\exists \Gamma' \in W_\mathbf{K})\big[\Gamma' \supseteq \mathbf{H}^-(\Gamma) \text{ and } \psi \in \Gamma'\big]$$

$\mathbf{H}^-(\Gamma) \cup \{\varphi\}$ is **K**-consistent. For if

$\mathbf{H}^-(\Gamma) \cup \{\varphi\}$	$\vdash_\mathbf{K}$	$\bot$	indirect assumption
$\mathbf{H}^-(\Gamma)$	$\vdash_\mathbf{K}$	$\neg\varphi$	Deduction theorem
$\exists \chi_1, \cdots, \chi_n$	$\vdash_\mathbf{K}$	$\neg\varphi$	def.of $\mathbf{H}^-(\Gamma) \vdash_\mathbf{K}$
	$\vdash_\mathbf{K}$	$(\chi_1 \wedge \cdots \wedge \chi_n) \to \neg\varphi$	def.of $(\chi_1 \wedge \cdots \wedge \chi_n) \vdash_\mathbf{K}$
	$\vdash_\mathbf{K}$	$\mathbf{H}(\chi_1 \wedge \cdots \wedge \chi_n) \to \mathbf{H}\neg\varphi$	See below!
	$\vdash_\mathbf{K}$	$(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \to \mathbf{H}\neg\varphi$	See below!
$\mathbf{H}\chi_1, \cdots, \mathbf{H}\chi_n$	$\vdash_\mathbf{K}$	$\mathbf{H}\neg\varphi$	def.of $(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \vdash_\mathbf{K}$
Γ	$\vdash_\mathbf{K}$	$\mathbf{H}\neg\varphi$	$\chi \in \mathbf{H}^-(\Gamma) \Leftrightarrow \mathbf{H}\chi \in \Gamma$
Γ	$\vdash_\mathbf{K}$	$\neg\mathbf{P}\varphi$	Duality
$\Gamma \cup \{\mathbf{P}\varphi\}$	$\vdash_\mathbf{K}$	$\bot$	Deduction theorem
Γ	$\vdash_\mathbf{K}$	$\bot$	we assumed that $\mathbf{P}\varphi \in \Gamma$

Remember that we relied on basically the following two logical rule:

$$\vdash_\mathbf{K} (\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi) \qquad \frac{\vdash_\mathbf{K} \varphi \to \psi}{\vdash_\mathbf{K} \mathbf{H}\varphi \to \mathbf{H}\psi}$$

Existence Lemma

$$\mathbf{P}\psi \in \Gamma \iff \psi \in \mathbf{H}^-(\Gamma) \Longrightarrow (\exists \Gamma' \in W_\mathbf{K})\left[\Gamma' \supseteq \mathbf{H}^-(\Gamma) \text{ and } \psi \in \Gamma'\right]$$

$\mathbf{H}^-(\Gamma) \cup \{\varphi\}$ is **K**-consistent. For if

$\mathbf{H}^-(\Gamma) \cup \{\varphi\}$	$\vdash_\mathbf{K}$	$\bot$	indirect assumption
$\mathbf{H}^-(\Gamma)$	$\vdash_\mathbf{K}$	$\neg\varphi$	Deduction theorem
$\exists \chi_1, \cdots, \chi_n$	$\vdash_\mathbf{K}$	$\neg\varphi$	def.of $\mathbf{H}^-(\Gamma) \vdash_\mathbf{K}$
	$\vdash_\mathbf{K}$	$(\chi_1 \wedge \cdots \wedge \chi_n) \to \neg\varphi$	def.of $(\chi_1 \wedge \cdots \wedge \chi_n) \vdash_\mathbf{K}$
	$\vdash_\mathbf{K}$	$\mathbf{H}(\chi_1 \wedge \cdots \wedge \chi_n) \to \mathbf{H}\neg\varphi$	See below!
	$\vdash_\mathbf{K}$	$(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \to \mathbf{H}\neg\varphi$	See below!
$\mathbf{H}\chi_1, \cdots, \mathbf{H}\chi_n$	$\vdash_\mathbf{K}$	$\mathbf{H}\neg\varphi$	def.of $(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n) \vdash_\mathbf{K}$
Γ	$\vdash_\mathbf{K}$	$\mathbf{H}\neg\varphi$	$\chi \in \mathbf{H}^-(\Gamma) \Leftrightarrow \mathbf{H}\chi \in \Gamma$
Γ	$\vdash_\mathbf{K}$	$\neg\mathbf{P}\varphi$	Duality
$\Gamma \cup \{\mathbf{P}\varphi\}$	$\vdash_\mathbf{K}$	$\bot$	Deduction theorem
Γ	$\vdash_\mathbf{K}$	$\bot$	we assumed that $\mathbf{P}\varphi \in \Gamma$

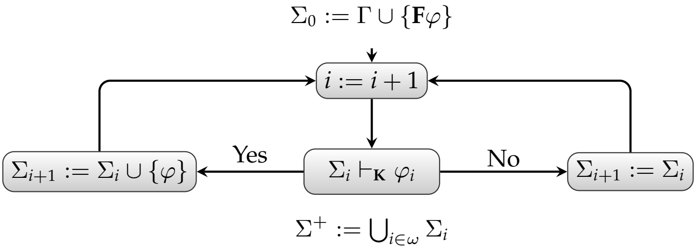
Remember that we relied on basically the following two logical rule:

$$\vdash_\mathbf{K} (\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi) \qquad \frac{\vdash_\mathbf{K} \varphi \to \psi}{\vdash_\mathbf{K} \mathbf{H}\varphi \to \mathbf{H}\psi}$$

Prove that these are valid indeed!

LINDENBAUM'S LEMMA

Since $\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$ is **K**-consistent, it can be extended into a maximally consistent set Γ'. Just list all the formulas and start the following procedure: take the first formula: Is it consistent with $\Sigma_0 \stackrel{\text{def}}{=} \mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$? If it is, then extend Σ_0 with that formula, if not, then don't. Repeat this into the infinity. Your m.c.s. will be $\mathbf{G}^{-}(\Gamma) \cup \{\varphi\}$ the one will contain every formula with which you would extend.



Similarly for **H**.

LINDENBAUM'S LEMMA

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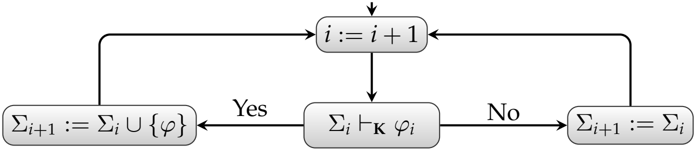
$$\Sigma_0 := \Gamma \cup \{\mathbf{F}\varphi\}$$



$$\Sigma^{+} := \bigcup_{i \in \omega} \Sigma_i$$

Similarly for $\mathbf{H}$.

Prove that Σ^{+} must be consistent!

ABSOLUTE FREEDOM / CHARACTERIZATION / CANONICAL MODEL THEOREM

Only (Exactly) the valid formulas are true in $\mathfrak{M}_{\mathbf{K}}$.

$$\mathfrak{M}_{\mathbf{K}} \models \varphi \iff \vdash_{\mathbf{K}} \varphi$$

We show that

$$\mathfrak{M}_{\mathbf{K}} \not\models \varphi \iff \nvdash_{\mathbf{K}} \varphi.$$

Since the construction called "canonical model" is a real **model** indeed, we have the $\Rightarrow$ direction.
If $\nvdash_{\mathbf{K}} \varphi$, then $\{\neg\varphi\}$ is **K**-consistent. Therefore we can extend it to a maximally **K**-consistent $\Gamma^{\neg\varphi}$ set by Lindenbaum's lemma. But this set is a world in the canonical model $\mathfrak{M}_{\mathbf{K}}$. And since this world contains $\neg\varphi$, it is **true** in it by the Truth lemma:

$$\neg\varphi \in \Gamma^{\neg\varphi} \Longrightarrow \mathfrak{M}_{\mathbf{K}}, \Gamma^{\neg\varphi} \models \neg\varphi$$

And we are ready, since we found a world of $\mathfrak{M}_{\mathbf{K}}$ where $\neg\varphi$ is true, i.e., φ is not true neither in that world nor in the whole model.

Logics

LOGICS

DEFINITION: A **normal temporal propositional logic** is a set of formulas that contains every **K**-valid formula and is closed under the rules of **K**.

THEOREM: **K** is the smallest normal temporal propositional logic.

DEFINITION: We denote the smallest normal temporal propositional logic that contains (the syntactically defined) **K** and φ with $\mathbf{K} + (\varphi)$

DEFINITION: A formula φ is **canonical** for a property P, iff besides that φ is valid on P-frames, 'by taking it as an axiom the canonical model of that new logic becomes P':

$$(\forall L \supseteq \mathbf{K} + (\varphi))\ \mathfrak{M}_L \text{ has the property } P$$

Axiomatizing Transitivity

Theorem: $\mathbf{G}\varphi \to \mathbf{GG}\varphi$ is canonical for $wRw'Rw'' \Rightarrow wRw''$

Proof: defining Let L be a n.t.p. logic that contains the scheme (4) and let Γ, Γ', Γ'' be arbitrary canonical worlds s.t. $\Gamma R_{\mathrm{L}} \Gamma' R_{\mathrm{L}} \Gamma''$. We have to prove that $\mathbf{G}^{-}(\Gamma) \subseteq \Gamma''$. Take a $\mathbf{G}\varphi \in \Gamma$. Then by (4), $\mathbf{GG}\varphi \in \Gamma$, therefore $\mathbf{G}\varphi \in \mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$ and $\varphi \in \mathbf{G}^{-}(\Gamma') \subseteq \Gamma''$.

Corollary: $\mathbf{K} + (4)$ axiomatizes the logic of transitive frames.

(Because $\mathfrak{M}_{\mathbf{K}+(4)}$ will count as a counter-model.)

Axiomatizing non-branching

<u>Theorem</u>: $\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \vee \mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi)$ is canonical for $(wRw_1 \text{ and } wRw_2) \Rightarrow (w_1Rw_2 \text{ or } w_1 = w_2 \text{ or } w_1 \reflectbox{R} w_2)$,

<u>Proof</u>: Let L be a n.t.p. logic containing the formula (H.3). Let Γ be arbitrary but fixed, and let Γ_1 and Γ_2 be arbitrary R_L-neighbours of Γ.

If $\Gamma_1 = \Gamma_2$, then we are ready. If $\Gamma_1 \neq \Gamma_2$, then suppose indirectly that they are not related by R_L at all. That would mean that there is a formula $\mathbf{H}\varphi \in \Gamma_1$ for which $\varphi \notin \Gamma_2$, and similarly, that there is a formula $\mathbf{H}\psi \in \Gamma_2$ for which $\psi \notin \Gamma_1$. So $\mathbf{H}\varphi, \neg\psi \in \Gamma_1$ and $\mathbf{H}\psi, \neg\varphi \in \Gamma_2$.

In this case we would have that $\neg(\underline{\mathbf{H}}\psi \to \varphi) \in \Gamma_1$ and $\neg(\underline{\mathbf{H}}\varphi \to \psi) \in \Gamma_2$, therefore, since both of Γ_1 and Γ_2 are R_L-related to Γ we have that $\mathbf{P}\neg(\underline{\mathbf{H}}\psi \to \varphi) \in \Gamma$ and $\mathbf{P}\neg(\underline{\mathbf{H}}\varphi \to \psi) \in \Gamma$, i.e., even $\mathbf{P}\neg(\underline{\mathbf{H}}\psi \to \varphi) \wedge \mathbf{P}\neg(\underline{\mathbf{H}}\varphi \to \psi) \in \Gamma$, hence $\neg\mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi) \wedge \neg\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \in \Gamma$ which makes Γ inconsistent.

<u>Corollary</u>: $\mathbf{K} + (H.3)$ axiomatizes the logic of those frames where there are no branching in the past.

Short Summary

K

LOGIC OF FRAMES

(PC1) $\varphi \to .\psi \to \varphi$

(PC2) $\varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$

(PC3) $\varphi \to \psi \to .\neg\psi \to \neg\varphi$

(CP) $\mathbf{PG}\varphi \to \varphi$

(CF) $\mathbf{FH}\varphi \to \varphi$

(AP) $(\mathbf{G}\varphi \wedge \mathbf{G}\psi) \to \mathbf{G}(\varphi \wedge \psi)$

(AF) $(\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi)$

(MP) $$\frac{\begin{array}{c}\varphi \\ \varphi \to \psi\end{array}}{\psi}$$

(PLem) $$\frac{\varphi \to \psi}{\mathbf{H}\varphi \to \mathbf{H}\psi}$$

(FLem) $$\frac{\varphi \to \psi}{\mathbf{G}\varphi \to \mathbf{G}\psi}$$

A more traditional axiomatization: replace the (AF)-(AP)-(PLem)-(FLem) schemes with

(KP) $\mathbf{H}(\varphi \to \psi) \to (\mathbf{H}\varphi \to \mathbf{H}\psi)$

(KF) $\mathbf{G}(\varphi \to \psi) \to (\mathbf{G}\varphi \to \mathbf{G}\psi)$

(RNP) $$\frac{\varphi}{\mathbf{H}\varphi}$$

(RNF) $$\frac{\varphi}{\mathbf{G}\varphi}$$

And it is also usual to postulate the duals (contrapositions) of (CP) and (CF)

$\mathbf{K} + (4)$

LOGIC OF TRANSITIVE FRAMES

(PC1) $\varphi \to .\psi \to \varphi$

(PC2) $\varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$

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(MP) $$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

(PLem) $$\frac{\varphi \to \psi}{\mathbf{H}\varphi \to \mathbf{H}\psi}$$

(FLem) $$\frac{\varphi \to \psi}{\mathbf{G}\varphi \to \mathbf{G}\psi}$$

(4) $\mathbf{G}\varphi \to \mathbf{GG}\varphi$

$\mathbf{K}$ + (H.3)

LOGIC OF BACKWARD-NONBRANCHING FRAMES

(PC1) $\varphi \to .\psi \to \varphi$

(PC2) $\varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$

(PC3) $\varphi \to \psi \to .\neg\psi \to \neg\varphi$

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(MP) $$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

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(FLem) $$\frac{\varphi \to \psi}{\mathbf{G}\varphi \to \mathbf{G}\psi}$$

(H.3) $\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \vee \mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi)$

$\mathbf{K} + (\mathrm{G.3})$

Logic of forward-nonbranching frames

(PC1) $\varphi \to .\psi \to \varphi$

(PC2) $\varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$

(PC3) $\varphi \to \psi \to .\neg\psi \to \neg\varphi$

(CP) $\mathbf{PG}\varphi \to \varphi$

(CF) $\mathbf{FH}\varphi \to \varphi$

(AP) $(\mathbf{G}\varphi \wedge \mathbf{G}\psi) \to \mathbf{G}(\varphi \wedge \psi)$

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(G.3) $\mathbf{G}(\underline{\mathbf{G}}\varphi \to \psi) \vee \mathbf{G}(\underline{\mathbf{G}}\psi \to \varphi)$

$\mathbf{K} + (\mathrm{H.3}) + (\mathrm{G.3})$

LOGIC OF NONBRANCHING FRAMES

(PC1) $\varphi \to .\psi \to \varphi$

(PC2) $\varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$

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(FLem) $\dfrac{\varphi \to \psi}{\mathbf{G}\varphi \to \mathbf{G}\psi}$

(H.3) $\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \vee \mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi)$

(G.3) $\mathbf{G}(\underline{\mathbf{G}}\varphi \to \psi) \vee \mathbf{G}(\underline{\mathbf{G}}\psi \to \varphi)$

$\mathbf{K} + (4) + (\mathrm{H.3}) + (\mathrm{G.3})$

LOGIC OF TRANSITIVE NONBRANCHING FRAMES

(PC1) $\varphi \to .\psi \to \varphi$

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(4) $\mathbf{G}\varphi \to \mathbf{GG}\varphi$

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(G.3) $\mathbf{G}(\underline{\mathbf{G}}\varphi \to \psi) \vee \mathbf{G}(\underline{\mathbf{G}}\psi \to \varphi)$

Tricking the undefinable

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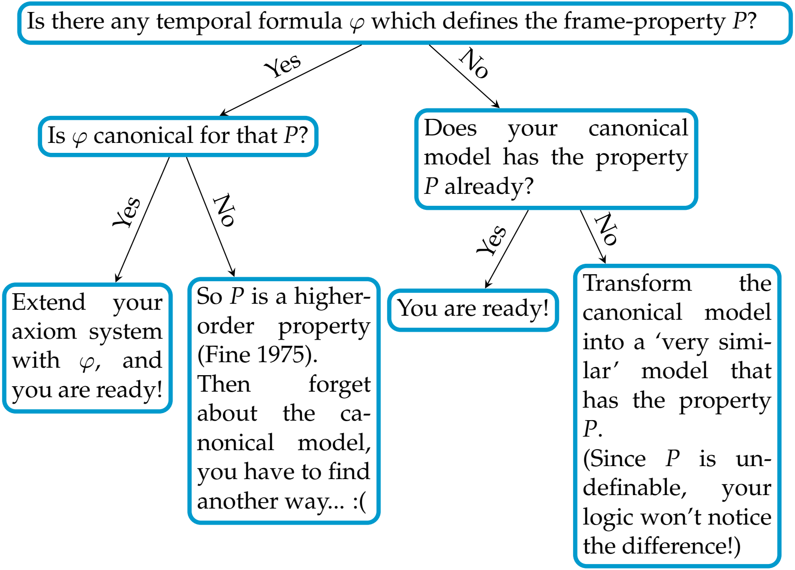
- Fortunately, **since there is no formula that would be sensitive to connectedness**, if we transform the canonical model in a way that only that parts of the model vanish that we don't use in the completeness theorem, and the remaining parts are connected, then the **logic** of that model would be the same.


Is there any temporal formula φ which defines the frame-property P?
Yes
No
Is φ canonical for that P?
Does your canonical model has the property P already?
Yes
No
Yes
No
Extend your axiom system with φ, and you are ready!
So P is a higher-order property (Fine 1975). Then forget about the canonical model, you have to find another way... :(
You are ready!
Transform the canonical model into a 'very similar' model that has the property P. (Since P is undefinable, your logic won't notice the difference!)

PLAYGROUND

Homeworks:

1. Prove that there are loops in the canonical model of **K**.

   HINT: Take the theory (set of true formulas), let us call it Γ, of a reflexive model in which there is only 1 world. Is Γ a maximally consistent set? Is there any world of the canonical model that contains Γ? Is it true that a canonical world that contains Γ sees itself in the canonical frame?

2. Prove that there are worlds that have no loops in the canonical model of **K**.

   HINT: Take the theory, Γ, of a non-reflexive model in which there is only 1 world.

3. Show that **K** is the logic of not-reflexive (this is not the same as irreflexivity!) frames.

4. Prove that the canonical model of **K** is not connected.

   HINT: Follow the previous hint!

5. Show that **K** is the logic of non-connected frames.

6. Prove that all these proofs work even if we take some or all the axioms from this list: (4), (H.3), (G.3).

Point-generated submodels

Point-generation

We can make a new model from an old one in a way that we forgot about the parts that are not accessible from a previously chosen point. That new model will be called the **point-generated submodel** of the old one.

DEFINITION: $\mathfrak{F}'$ is a **closed subframe** of $\mathfrak{F}$ ($\mathfrak{F}' \leq \mathfrak{F}$) iff

1. $W' \subseteq W$,
2. $R' = R \restriction W'$,
3. If $w \in W'$ and wRv, then $v \in W'$,
4. If $w \in W'$ and vRw, then $v \in W'$. (Because we are temporal!)

If this extends even to the valuation of models builded on these frames, then we have closed submodels. Formally: If we have two models $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ and $\mathfrak{M}' = \langle \mathfrak{F}', V' \rangle$ such that $\mathfrak{F}' \leq \mathfrak{F}$, and $V' = V \restriction W'$, then we say that $\mathfrak{M}'$ is a closed submodel of $\mathfrak{M}$, in symbols: $\mathfrak{M}' \leq \mathfrak{M}$.

The **closed subframe/submodel generated by the set of worlds** X in $\mathfrak{M}$ is the smallest closed submodel of $\mathfrak{M}$ containing X.

$$\langle X \rangle_{\mathfrak{M}} \stackrel{\text{def}}{=} \bigcap \{\mathfrak{M}' \leq \mathfrak{M} \;:\; X \subseteq W'\}$$

A model $\mathfrak{M}$ or a frame $\mathfrak{F}$ is point-generated iff there is a world w such that

$$\mathfrak{M} = \langle w \rangle_{\mathfrak{M}} \qquad \text{or} \qquad \mathfrak{F} = \langle w \rangle_{\mathfrak{F}}$$

Point-generation

<u>Proposition</u>: Every point-generated frames/models are connected.

<u>Proposition</u>: Every connected frames/models are point-generated.

Invariance

The temporal language is ‘blind’ for the submodel-generation.

<u>THEOREM</u>: Truth is invariant under submodel generation.

$$\mathfrak{M}' \leq \mathfrak{M} \qquad \Longrightarrow \qquad \mathfrak{M}, w \Vdash \varphi \iff M', w \Vdash \varphi$$

<u>THEOREM</u>:

$$\mathfrak{F}' \leq \mathfrak{F} \qquad \Longrightarrow \qquad \mathfrak{F} \Vdash \varphi \iff F' \Vdash \varphi$$

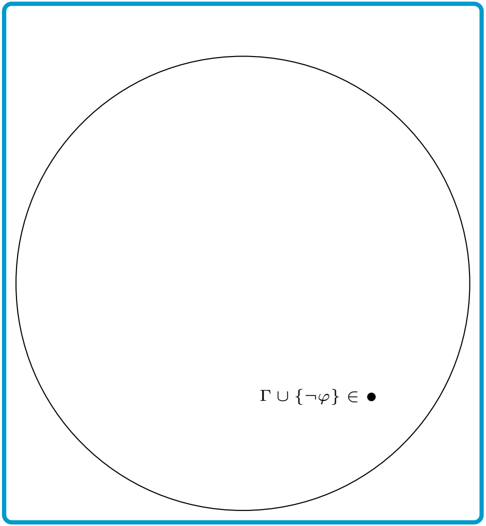
CONNECTED MODELS

THEOREM: **K** is the temporal logic of connected models (too):

For all **connected** $\mathfrak{M}$:

$$\mathbf{K} \vdash \varphi \iff \mathfrak{M} \models \varphi$$

Because for every non-theorem we have a **connected** counter-model! Not the canonical model (because it is not connected), but one of its sub-models!



Show that $\mathfrak{M}_\mathbf{K}$ is not connected!

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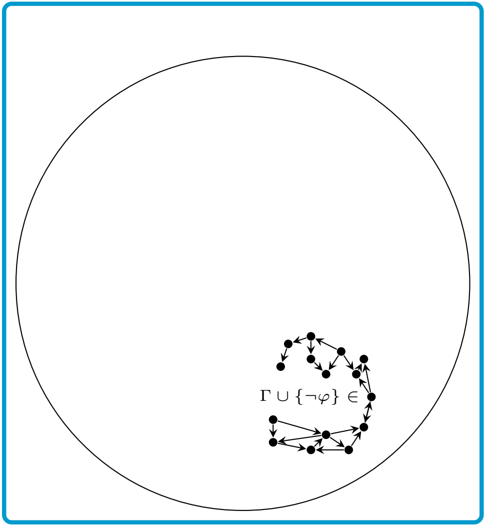

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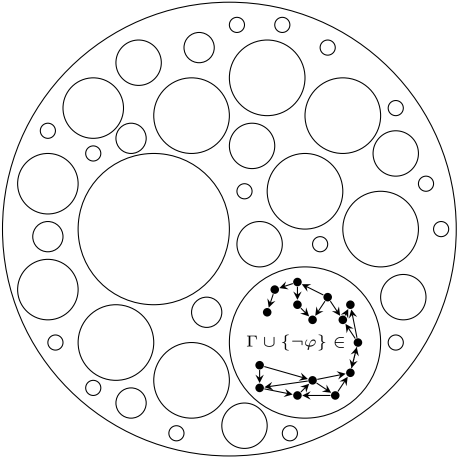
Point-generated submodels in $\mathfrak{M}_{\mathbf{K}}$



Show that $\mathfrak{M}_{\mathbf{K}}$ is not connected!

Unraveling

Logic of forests

THEOREM: $\mathbf{K} + (4)$ is the logic of forests (too).

IDEA:: We will transform the canonical model $\mathfrak{M}_{\mathbf{K}}$ into a forest forest($\mathfrak{M}_{\mathbf{K}}$) in a way that $\mathfrak{M}_{\mathbf{K}}$ will be a zig-zag image of forest($\mathfrak{M}_{\mathbf{K}}$). Therefore every formula will be satisfiable on a forest: on forest($\mathfrak{M}_{\mathbf{K}}$).

$$\text{forest}(\mathfrak{M}_{\mathbf{K}}) \stackrel{\text{def}}{=} (\vec{W}_{\mathbf{K}}, \vec{R}^t_{\mathbf{K}}, \vec{V}_{\mathbf{K}})$$

where

- $\vec{W}_{\mathbf{K}}$ is the set of all finite paths in $W_{\mathbf{K}}$:

$$\vec{W}_{\mathbf{K}} \stackrel{\text{def}}{=} \{\vec{w} : \vec{w} \text{ is an } n\text{-tuple s.t. } w_1 R w_2 R \dots R w_n\}$$

- $\vec{R}^t_{\mathbf{K}}$ is a transitive closure of $\vec{w}\vec{R}_{\mathbf{K}}\vec{v}$, where $\vec{w}\vec{R}_{\mathbf{K}}\vec{v}$ iff $\vec{v}$ is a continuation of $\vec{w}$.

$$\vec{w}\vec{R}_{\mathbf{K}}\vec{v} \stackrel{\text{def}}{\Leftrightarrow} (\exists u \in W_{\mathbf{K}})\ (\vec{w}, u) = \vec{v}$$

- $\vec{w} \in \vec{V}_{\mathbf{K}}$ iff in the end of the past $\vec{w}$, p is true ($\in V_{\mathbf{K}}(p)$).

$$(w_1, \dots, w_n) \in \vec{V}_{\mathbf{K}}(p) \stackrel{\text{def}}{\Leftrightarrow} w_n \in \vec{V}_{\mathbf{K}}(p)$$

Now the zigzag-morphism will be $h(w_1, \dots, w_n) = w_n$.

LOGIC OF FORESTS

THEOREM: $\mathbf{K} + (4)$ is the logic of forests (too).

IDEA:: We will transform the canonical model $\mathfrak{M}_\mathbf{K}$ into a forest forest$(\mathfrak{M}_\mathbf{K})$ in a way that $\mathfrak{M}_\mathbf{K}$ will be a zig-zag image of forest$(\mathfrak{M}_\mathbf{K})$. Therefore every formula will be satisfiable on a forest: on forest$(\mathfrak{M}_\mathbf{K})$.

Show that forest$(\mathfrak{M}_\mathbf{K})$ is irreflexive, intransitive, antisymmetric.

$$\text{forest}(\mathfrak{M}_\mathbf{K}) \stackrel{\text{def}}{=} (\vec{W}_\mathbf{K}, \vec{R}^t_\mathbf{K}, \vec{V}_\mathbf{K})$$

Show that h is a surjective zigzag-morphism.

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$$\vec{W}_\mathbf{K} \stackrel{\text{def}}{=} \{\vec{w} \;:\; \vec{w} \text{ is an } n\text{-tuple s.t. } w_1 R w_2 R \ldots R w_n\}$$

- $\vec{R}^t_\mathbf{K}$ is a transitive closure of $\vec{w}\vec{R}_\mathbf{K}\vec{v}$, where $\vec{w}\vec{R}_\mathbf{K}\vec{v}$ iff $\vec{v}$ is a continuation of $\vec{w}$.

$$\vec{w}\vec{R}_\mathbf{K}\vec{v} \stackrel{\text{def}}{\Leftrightarrow} (\exists u \in W_\mathbf{K})\ (\vec{w}, u) = \vec{v}$$

- $\vec{w} \in \vec{V}_\mathbf{K}$ iff in the end of the past $\vec{w}$, p is true ($\in V_\mathbf{K}(p)$).

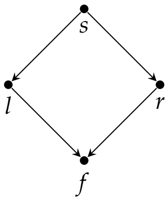
$$(w_1, \ldots, w_n) \in \vec{V}_\mathbf{K}(p) \stackrel{\text{def}}{\Leftrightarrow} w_n \in \vec{V}_\mathbf{K}(p)$$

Now the zigzag-morphism will be $h(w_1, \ldots, w_n) = w_n$.

Examples

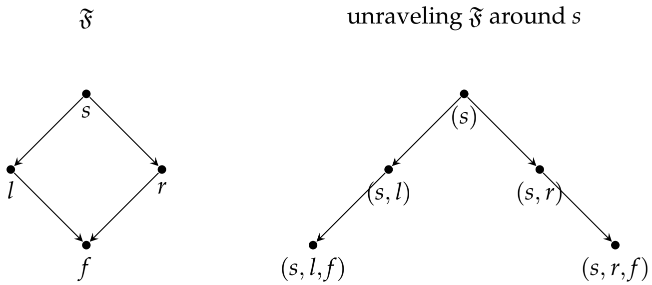
$\mathfrak{F}$

unraveling $\mathfrak{F}$ around s



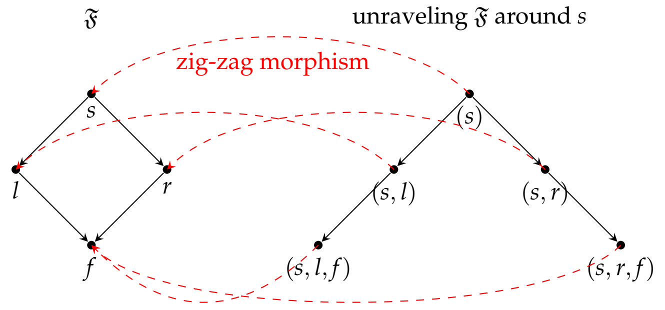
s	for	start
l	for	left
r	for	right
f	for	finish

Examples



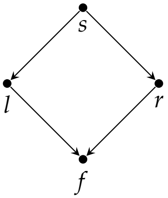
s	for	start
l	for	left
r	for	right
f	for	finish

Examples

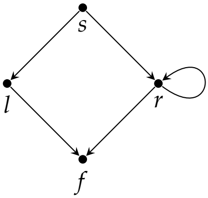


s	for	start
l	for	left
r	for	right
f	for	finish

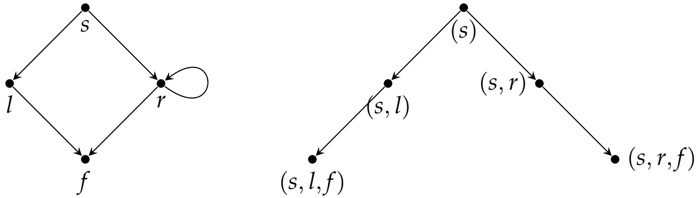
Examples



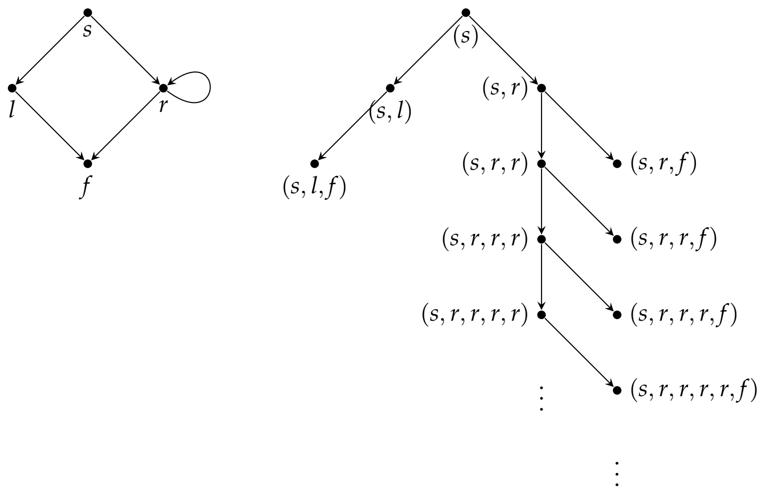
Examples



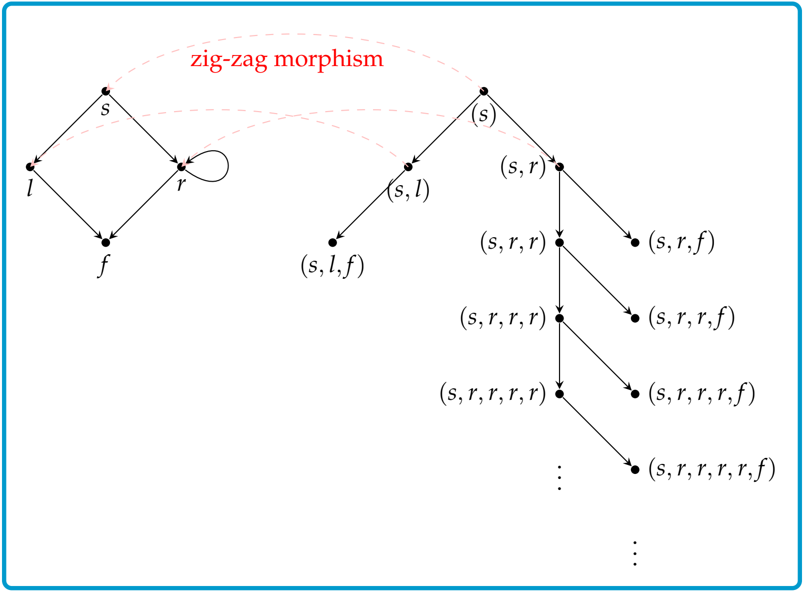
Examples



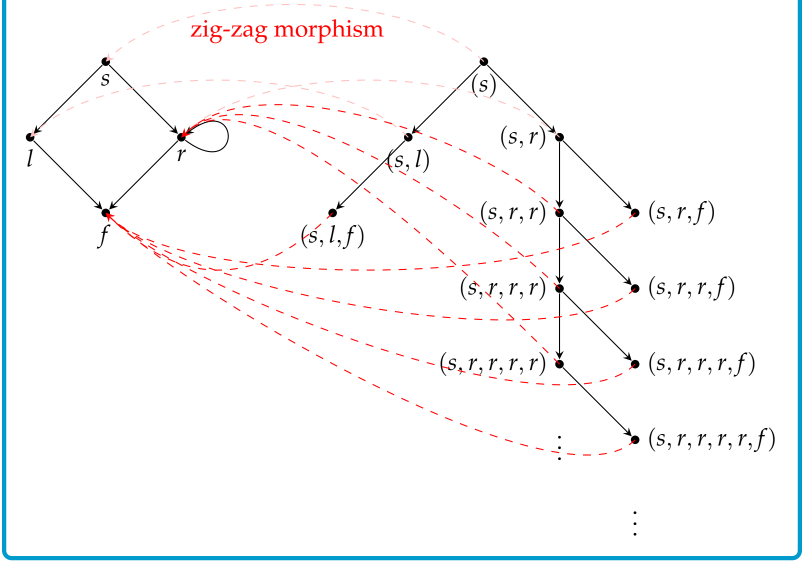
Examples



Examples

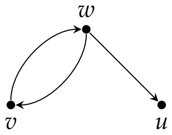

zig-zag morphism
s
l
r
f
(s)
(s, l)
(s, r)
(s, l, f)
(s, r, r)
(s, r, f)
(s, r, r, r)
(s, r, r, f)
(s, r, r, r, r)
(s, r, r, r, f)
(s, r, r, r, r, f)

Examples


zig-zag morphism
s
l
r
f
(s)
(s, l)
(s, r)
(s, l, f)
(s, r, r)
(s, r, f)
(s, r, r, r)
(s, r, r, f)
(s, r, r, r, r)
(s, r, r, r, f)
(s, r, r, r, r, f)

Examples

Unravel this around w:



Logic of transitive forests

COROLLARY: $\mathbf{K} + (4)$ is the logic of **trees** (too).

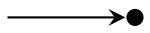
Logic of transitive forests

Corollary: $\mathbf{K} + (4)$ is the logic of **trees** (too).

Prove that statement

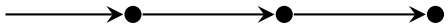
Bulldozing

Examples

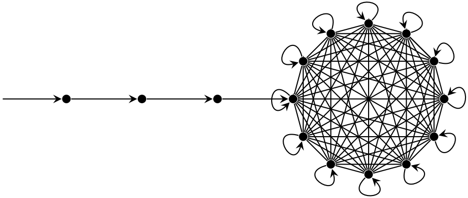


Examples

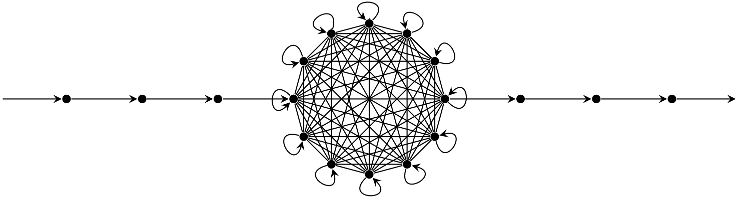
Examples



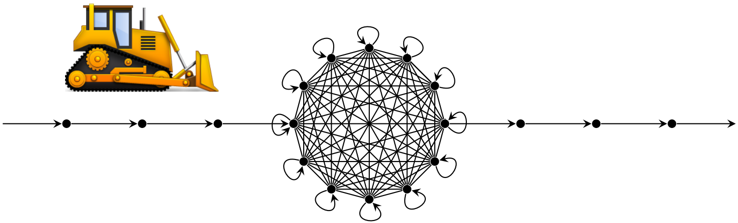
EXAMPLES



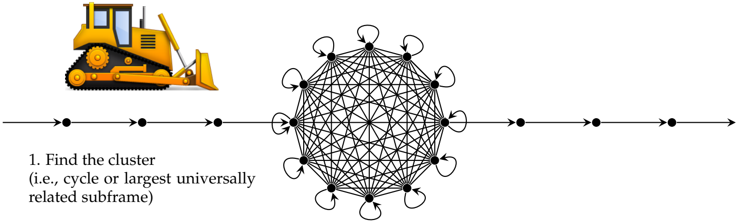
Examples



Examples

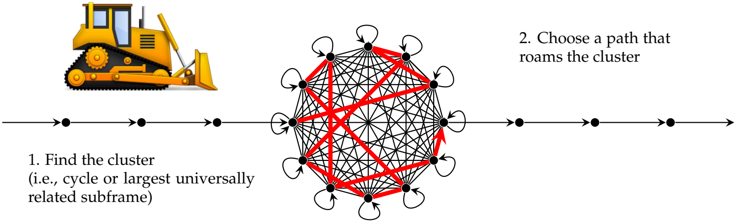


Examples

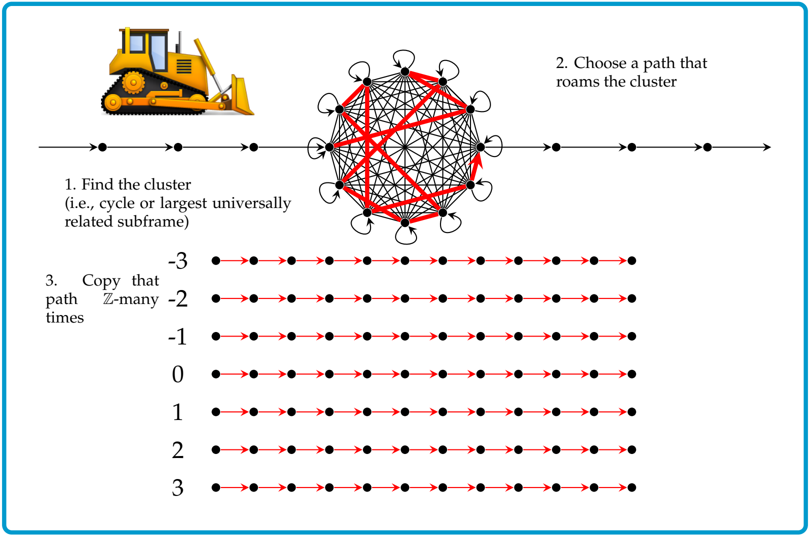


1. Find the cluster
(i.e., cycle or largest universally related subframe)

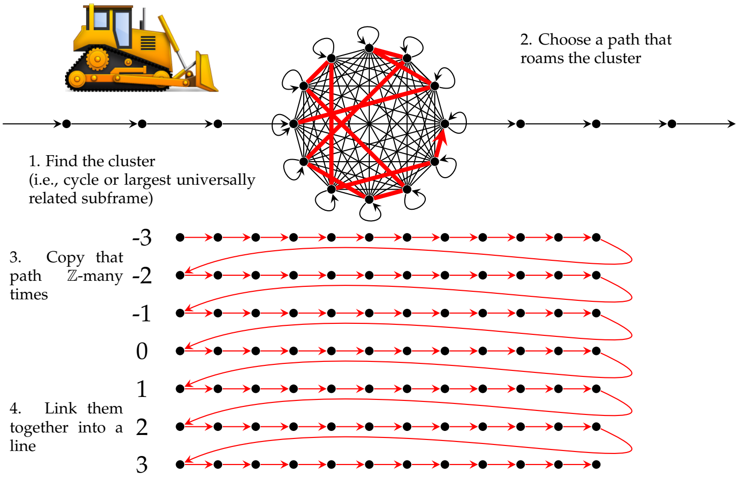
Examples


1. Find the cluster
(i.e., cycle or largest universally
related subframe)
2. Choose a path that
roams the cluster

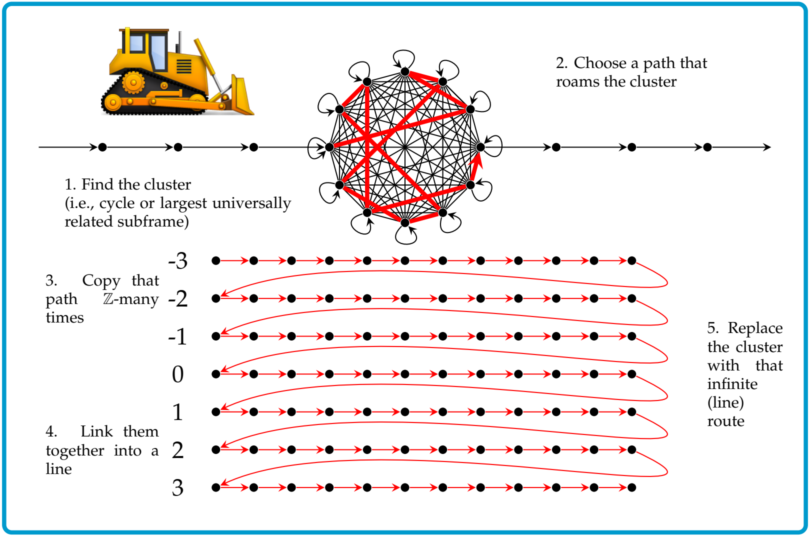
Examples


2. Choose a path that roams the cluster
1. Find the cluster (i.e., cycle or largest universally related subframe)
3. Copy that path ℤ-many times
-3
-2
-1
0
1
2
3

Examples


2. Choose a path that roams the cluster
1. Find the cluster (i.e., cycle or largest universally related subframe)
3. Copy that path ℤ-many times
4. Link them together into a line
-3
-2
-1
0
1
2
3

Examples



2. Choose a path that roams the cluster
1. Find the cluster (i.e., cycle or largest universally related subframe)
3. Copy that path ℤ-many times
4. Link them together into a line
5. Replace the cluster with that infinite (line) route
-3
-2
-1
0
1
2
3

Bulldozing

THEOREM: Every K4.3 model is a zig-zag image of its bulldozed model.

PROPOSITION: Every bulldozed model of a K4.3 model is acyclic (i.e., irreflexive and is free from any equivalence related subframes).

COROLLARY: Every point-generated subframe of a bulldozed model of a K4.3 model is trichotomic.

COROLLARY: K4.3 is the temporal logic of flow of times (= STO's = strict total orders).

Workshop topic: Formalize the idea of bulldozing and prove that theorem above.
(Source: Blackburn–de Rijke–Venema: Thm 4.56. though I think it is easier to develop this formalism by yourself – it is quite hard to read a formalism like that unless you were the one who wrote it.)