LOGIC OF BRANCHING TIME

AXIOMATIZATIONS

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Ockhamist axioms

Ockhamist axiom-checking

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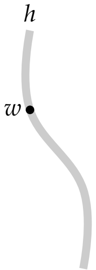

$\mathbf{H}\varphi \rightarrow \mathbf{HH}\varphi$ does the same since the semantics of **P** is the same, this (by the old completeness thm. of K+(4)) implies that we have the validity of $\mathbf{G}\varphi \rightarrow \mathbf{GG}\varphi$ as well.

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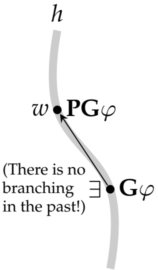

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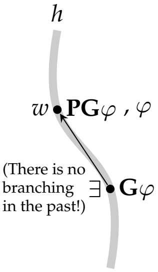

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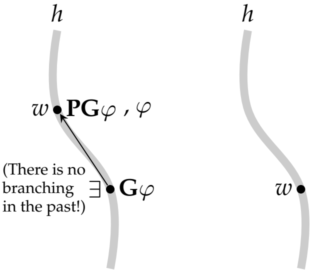

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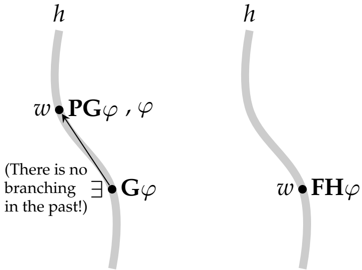

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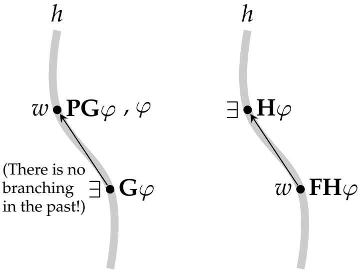

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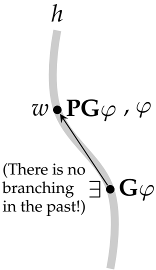

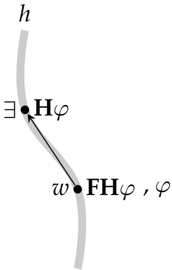

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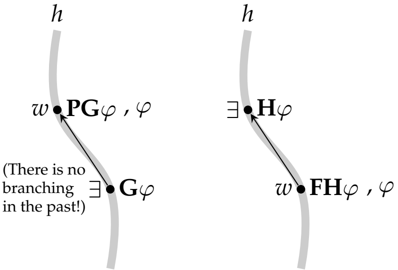

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$$\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \vee \mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi)$$

is O-valid.

Assume indirectly that it's not:

$\exists h$

$\exists \bullet \mathbf{P}(\underline{\mathbf{H}}\varphi \wedge \neg\psi) \wedge \mathbf{P}(\underline{\mathbf{H}}\psi \wedge \neg\varphi)$

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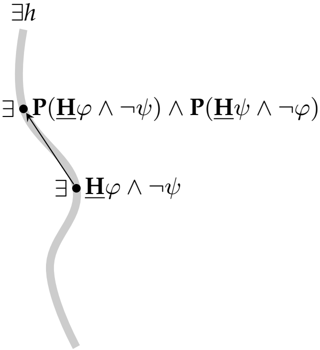

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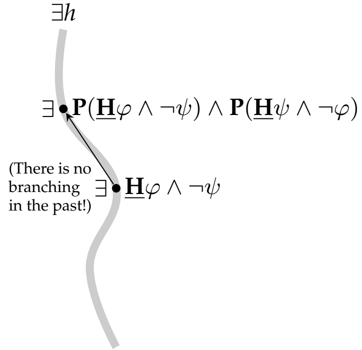

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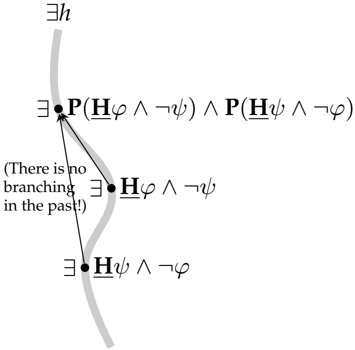

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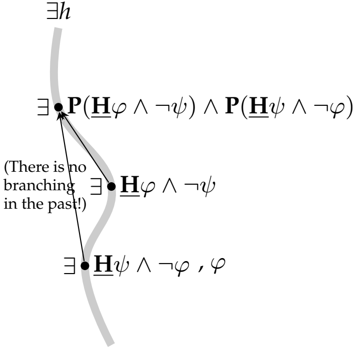

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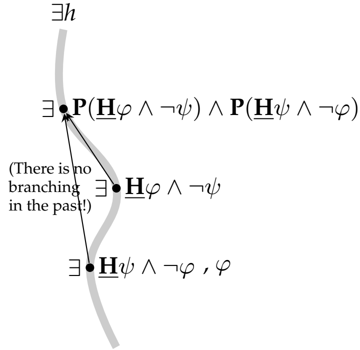

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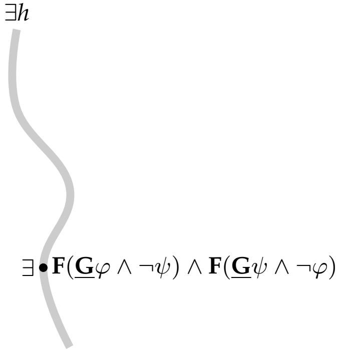

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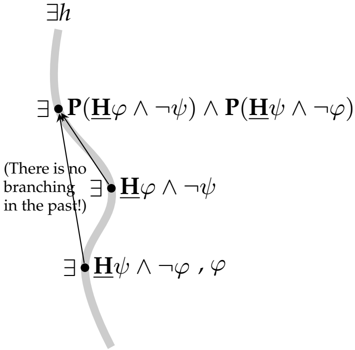

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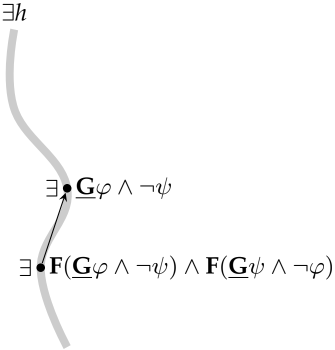

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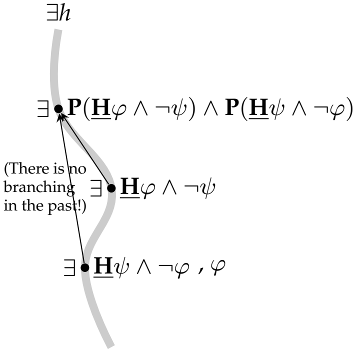

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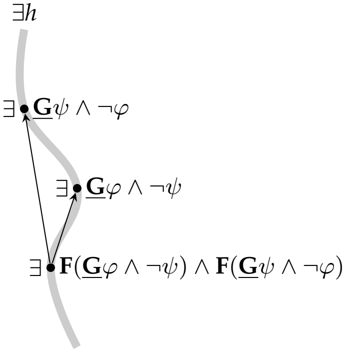

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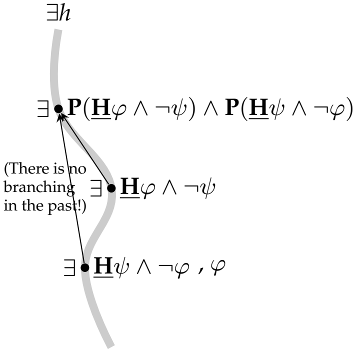

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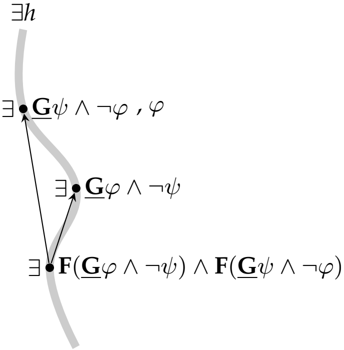

"A SHOT IN THE DARK"

Since $h \overset{w}{\sim} h'$ is an equivalence relation between histories, we should try those axioms for necessity that are valid on equivalence relational frames, i.e., the three axioms of S5:

- $\Box\varphi \to \varphi$ (valid by reflexivity)
- $\Box\varphi \to \Box\Box\varphi$ (valid by transitivity)
- $\varphi \to \Box\Diamond\varphi$ (valid by symmetry)

We try to prove the completeness theorem with these. We reserve the right to take new axioms if encounter an appealing formula/rule.

Trees vs flows

First of all, if we maintain that the canonical worlds are maximally consistent sets, and the alternative relation is $\mathbf{G}^{-}(\Gamma) \subseteq \Gamma$, then the canonical frame is not one or more tree, but a big union of transitive linear flows (by the validity / axiom status of 4, H.3 and G.3). This is not entirely wrong, since these linear flows with some bulldozing can be good candidates for histories. But how could we make a tree from these histories? The idea will be that we will connect these trees with a new alternative relation, the one which will interpret the historical necessity $\Box$. But then of course this will be a bimodal frame, called Kamp-frame, which is not a usual frame (we have two alternative relation instead of one). But we will show that every Kamp-frame determines a normal frame uniquely, and every normal frame determines a Kamp-frame uniquely.

Kamp-frames

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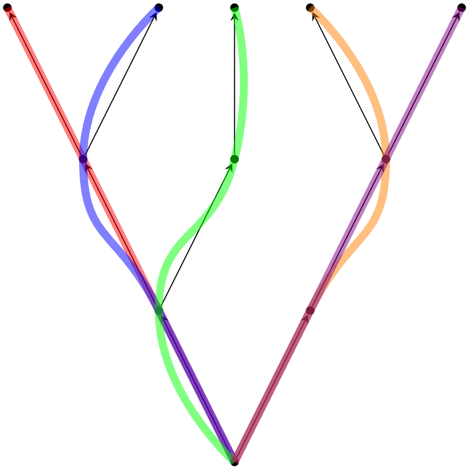
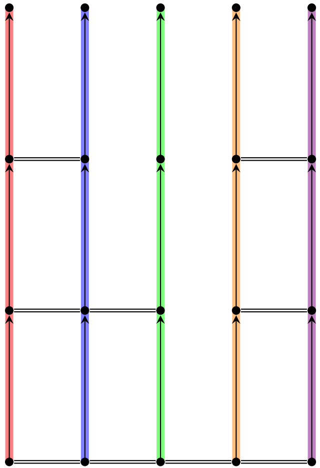
Standard (tree) frame

$(W, <)$



Kamp-frame

$(W, <, \equiv)$

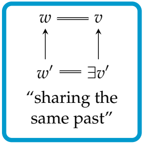


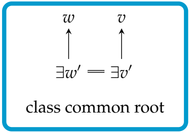
On a Kamp-frame, $<$ is non-branching, while $\equiv$ is an equivalence relation.
Think of $\equiv$ as "is the same as", or as a rope what we use to make the bundle

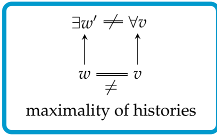
Kamp-frames

A Kamp-frame is a triplet $(W, <, \equiv)$ where

- $<$ is irreflexive, transitive and non-branching:
  - $w \not< w$
  - $(w < v \wedge v < u) \to w < u$
  - $(w < v \wedge w < u) \to (v < u \vee v = u \vee v > u)$
  - $(w > v \wedge w > u) \to (v < u \vee v = u \vee v > u)$
- $\equiv$ is reflexive, transitive and symmetric:
  - $w \equiv w$
  - $(w \equiv v \wedge v \equiv u) \to w \equiv u$
  - $w \equiv v \to v \equiv w$
- $x \equiv y \to x \not< y$ — class irreflexivity
- $(w \equiv v \wedge w' < w) \to (\exists v' < v)\, w' \equiv v'$ — "sharing the same past"
- $(\forall w, v)(\exists w' < w)(\exists v' < v)\, w \equiv v$ — class common root
- $(\forall w, v)(w \equiv v \wedge w \neq v)(\exists w' > w)(\forall v' > v)\, w' \not\equiv v'$ — maximality of histories







Kamp-models

Let $\mathfrak{K} = (W, <, \equiv)$ be a Kamp-frame. A Kamp-valuation is a $V : \mathrm{At} \to \wp W$ for which the following additional property holds:

$$w \in V(p) \Rightarrow (\forall v \equiv w)\ v \in V(p) \qquad \text{for all } p \in \mathrm{At}$$

a Kamp-frame $\mathfrak{K} = (W, <, \equiv)$ together with such a valuation V is a Kamp-model $\mathfrak{M}_K = (\mathfrak{K}, V)$.

$$\begin{array}{lcl}
\mathfrak{M}_K, w \models^{K} p & \overset{\text{def}}{\Leftrightarrow} & w \in V(p) \\
\mathfrak{M}_K, w \models^{K} \neg\varphi & \overset{\text{def}}{\Leftrightarrow} & \text{it is not true that } \mathfrak{M}_K, w \models^{K} \varphi \\
\mathfrak{M}_K, w \models^{K} \varphi \wedge \psi & \overset{\text{def}}{\Leftrightarrow} & \mathfrak{M}_K, w \models^{K} \varphi \text{ and } \mathfrak{M}_K, w \models^{K} \psi \\
\mathfrak{M}_K, w \models^{K} \mathbf{P}\varphi & \overset{\text{def}}{\Leftrightarrow} & (\exists v < w)\ \mathfrak{M}_K, v \models^{K} \varphi \\
\mathfrak{M}_K, w \models^{K} \mathbf{F}\varphi & \overset{\text{def}}{\Leftrightarrow} & (\exists v > w)\ \mathfrak{M}_K, v \models^{K} \varphi \\
\mathfrak{M}_K, w \models^{K} \Diamond\varphi & \overset{\text{def}}{\Leftrightarrow} & (\exists v \equiv w)\ \mathfrak{M}_K, v \models^{K} \varphi
\end{array}$$

Models and Kamp-models

We can transform every Kamp-model $\mathfrak{M}_K$ into a standard tree-model $\mathrm{str}(\mathfrak{M}_K)$. Let $\mathfrak{M}_K = \{W, <, \equiv, V\}$ be a Kamp-model. The **standard transformation** of a Kamp-model $\mathfrak{M}_K$ will be

$$\mathrm{str}(\mathfrak{M}_K) \stackrel{\text{def}}{=} (W/\equiv, <^{\mathrm{str}}, V^{\mathrm{str}})$$

where

- $W/\equiv$ is the set of all $\equiv$-equivalence classes, i.e.,

  $$W/\equiv \stackrel{\text{def}}{=} \{w/\equiv : w \in W\},$$

  where $w/\equiv \stackrel{\text{def}}{=} \{v : w \equiv v\}$.
- $<^{\mathrm{str}}$ is defined as

  $$w/\equiv <^{\mathrm{str}} v/\equiv \stackrel{\text{def}}{\Leftrightarrow} (\exists w' \in w/\equiv)(\exists v' \in v/\equiv)\ w < v$$

- The valuation of the standard transformed model will be

  $$w/\equiv \in V^{\mathrm{str}}(p) \stackrel{\text{def}}{\Leftrightarrow} w \in V(p)$$

  This definition is correct (does not depend on the choice of w) since we used Kamp-valuations, for which it is true that the same atomic sentences are true in equivalent worlds.

MODELS AND KAMP-MODELS

PROPOSITION: $(W/\equiv, <^{\text{str}})$ is a tree.

COROLLARY: $\text{str}(\mathfrak{M}_K)$ is a tree model.

MODELS AND KAMP-MODELS

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$$\frac{w/\equiv <^{\text{str}} v/\equiv \qquad v/\equiv <^{\text{str}} u/\equiv}{w/\equiv <^{\text{str}} u/\equiv}$$

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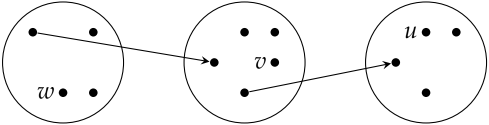

- $<^{str}$ is rooted:

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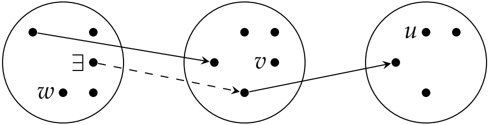

"sharing the same past"

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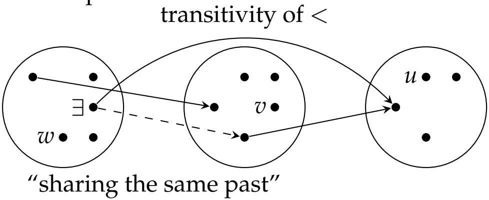

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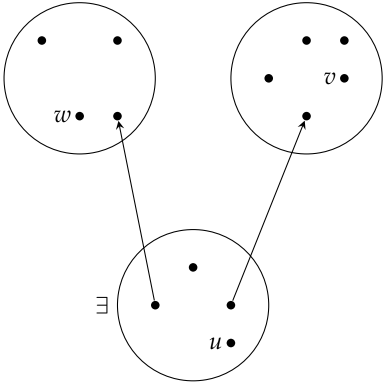


This is equivalent with the Kamp-frame constraint we labelled with "there is a common root".

Models and Kamp-models

Now we should prove something like this:

$$\mathfrak{M}_K, w \models \varphi \iff \mathrm{str}(\mathfrak{M}_K), h_w, w/\equiv \models^{O} \varphi$$

where h_w is defined to be the set of those equivalence classes that contains an element related with w:

$$h_w \stackrel{\text{def}}{=} \{v/\equiv \, : \, (\exists u \in v/\equiv)(w < u \vee w > u)\}$$

h_w will be linear subset because the Kamp-frame's $<$ is always non-branching. It is maximally linear by the way we defined it.

- $w \equiv v$ implies $h_w \stackrel{w/\equiv}{\sim} h_v$
- $w \equiv v$ is not implied by $h_w \stackrel{w/\equiv}{\sim} h_v$
- there is an $u \equiv w$, s.t. $u < v$ or $u > v$, if $h_w \stackrel{w/\equiv}{\sim} h_v$

Truth in Models and Kamp-models

Now the problem will be that

$$\mathfrak{M}_K, w \models \varphi \quad \Longleftrightarrow \quad \mathrm{str}(\mathfrak{M}_K), h_w, w/\equiv \models^{O} \varphi$$

is not true in general.

It is true when the Kamp-frame is made from a real tree's all histories, but not every Kamp-frame is can be gained from a tree's all histories. Correspondingly, Kamp-validity does not correspond to Ockhamist validity. That is not entirely surprising, since Kamp-frames **cheat** in the interpretation of $\Diamond$: Originally $\Diamond$ quantified over histories, i.e., **sets** of possible worlds. But in a Kamp-frame, $\Diamond$ quantifies over only possible worlds. And it can be the case that there are more histories than worlds. (Although this was not the case in our **finite** examples; in finite examples, the validity of the two are the same.)

But before we introduce the corresponding structure for Kamp-frames, let us find out how far we can get by the proof of the statement above.

Models and Kamp-models

THEOREM: $\mathfrak{M}_K, w \models \varphi \iff \mathrm{str}(\mathfrak{M}_K), h_w, w/{\equiv} \models^{O} \varphi$

---

the atomic case is

$$\begin{aligned} \mathfrak{M}_K, w \models p &\iff w \in V(p) && \text{def of Kamp-}\models \\ &\iff w/{\equiv} \in V^{\mathrm{str}}(p) && \text{def of } V^{\mathrm{tr}} \\ &\iff \mathrm{str}(\mathfrak{M}_K), h_w, w/{\equiv} \models^{O} \varphi && \text{def of } \models^{O} \end{aligned}$$

the $\wedge$ and $\neg$ are trivial by induction, the modal cases are

$$\begin{aligned} \mathfrak{M}_K, w \models \mathbf{F}\varphi &\iff (\exists v > w)\mathfrak{M}_K, v \models \varphi && \text{def of Kamp-}\models \\ &\iff (\exists v > w)\mathrm{str}(\mathfrak{M}_K), h_v, v/{\equiv} \models^{O} \varphi && \text{ind.hip} \\ &\iff (\exists v > w)\mathrm{str}(\mathfrak{M}_K), h_w, v/{\equiv} \models^{O} \varphi && h_w = h_v \text{ by } w < v \\ &\iff (\exists v/{\equiv} >^{\mathrm{str}} w/{\equiv})\mathrm{str}(\mathfrak{M}_K), h_w, v/{\equiv} \models^{O} \varphi && \text{def.of } <^{\mathrm{str}} \\ &\iff \mathrm{str}(\mathfrak{M}_K), h_w, w/{\equiv} \models^{O} \mathbf{F}\varphi && \text{def of } \models^{O} \end{aligned}$$

here the left direction needs some explanation: From $h_w, w/{\equiv} \models^{O} \mathbf{F}\varphi$ we know that there is an $u/{\equiv} \in h_w$ where φ is true, hence by the def. of h_w, $(\exists v \in u/{\equiv})\ w < u \equiv v$ and we arrived to the third line.

Models and Kamp-models

THEOREM: $\mathfrak{M}_K, w \models \varphi \iff \mathrm{str}(\mathfrak{M}_K), h_w, w/{\equiv} \models^{O} \varphi$

$$
\begin{aligned}
\mathfrak{M}_K, w \models \mathbf{P}\varphi &\iff (\exists v < w)\mathfrak{M}_K, v \models \varphi && \text{def of Kamp-}\models\\
&\iff (\exists v < w)\mathrm{str}(\mathfrak{M}_K), h_v, v/{\equiv} \models^{O} \varphi && \text{ind.hip}\\
&\iff (\exists v < w)\mathrm{str}(\mathfrak{M}_K), h_w, v/{\equiv} \models^{O} \varphi && h_w = h_v \text{ by } w < v\\
&\iff (\exists v/{\equiv} <^{\mathrm{str}} w/{\equiv})\mathrm{str}(\mathfrak{M}_K), h_w, v/{\equiv} \models^{O} \varphi && \text{def.of } <^{\mathrm{str}}\\
&\iff \mathrm{str}(\mathfrak{M}_K), h_w, w/{\equiv} \models^{O} \mathbf{P}\varphi && \text{def of } \models^{O}
\end{aligned}
$$

here the left direction needs the dual argumentation as was presented in the previous slide.

$$
\begin{aligned}
\mathfrak{M}_K, w \models \Diamond\varphi &\iff (\exists v \equiv w)\mathfrak{M}_K, v \models \varphi && \text{def of Kamp-}\models\\
&\iff (\exists v \equiv w)\mathrm{str}(\mathfrak{M}_K), h_v, v/{\equiv} \models^{O} \varphi && \text{ind.hip}\\
&\iff (\exists v \equiv w)\mathrm{str}(\mathfrak{M}_K), h_v, w/{\equiv} \models^{O} \varphi && \text{by } \equiv\\
&\iff (\exists h_v \overset{w/\equiv}{\sim} h_w)\mathrm{str}(\mathfrak{M}_K), h_v, w/{\equiv} \models^{O} \varphi && \text{using the HW.}\\
&\Rightarrow \mathrm{str}(\mathfrak{M}_K), h_w, w/{\equiv} \models^{O} \Diamond\varphi && \text{def of } \models^{O}
\end{aligned}
$$

For the other direction we can start with

$$(\exists h \overset{w/\equiv}{\sim} h_w)\mathrm{str}(\mathfrak{M}_K), h, w/{\equiv} \models^{O} \varphi.$$

Now to proceed further, we have to show that this h was already there in the original Kamp-frame. Well, that is not always true.

Counterexample: Infinite binary trees

$W \overset{\text{def}}{=} \{w : w \text{ is a route to a point}\}$
$= \{\langle w_1, \ldots, w_n \rangle : n \in \omega, (\forall i \leq n) w_i \in \{U, R\}\}$

$w \sqsubseteq v \overset{\text{def}}{\Leftrightarrow}$ v is a continuation of w, i.e.,
iff $(\forall i \leq n) w_i = v_i$ where n is the length of w.

Note that histories correspond to infinite routes!

Also note we can not name the histories by worlds (as was the case in the finite cases)! There are (infinitely) many infinite continutations of finite routes.

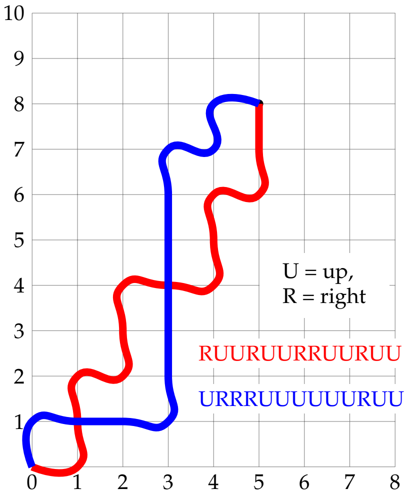
A set of histories $B \subseteq \mathrm{H}(\mathfrak{F})$ is called a **bundle** iff

$$\bigcup B = W,$$

that is, for every $w \in W$ there is a history $h \in B$ s.t. $w \in h$.

We can find a proper bundle, which is in fact can be named by worlds:

$$\{h \in \mathrm{H}(\mathfrak{F}) : \exists w (\forall v > w) v = \langle w, U, \ldots, U \rangle\}$$



Bundled trees

<u>Definition</u>: A bundled tree is a triplet $(W, <, B)$ where $(W, <)$ is a tree and $B \subseteq \mathrm{H}(W, <)$ is a bundle.

<u>Proposition</u>: Every bundled tree can be "turned into" a Kamp-frame.

<u>Proposition</u>: Validity on Kamp-frames correspond to the validity on bundled trees.

Workshop project

Invent a transformation rts that goes in the reverse direction: that transforms an arbitrary standard model into a Kamp-model:

- prove that the resulting frame is always a Kamp-frame, and the resulting valuation is a Kamp-valuation.
- prove that this transformation preserves the truth.
- this transformation is indeed the reverse of str, by proving the following statement:

$$\mathrm{str}(\mathrm{rts}(\mathfrak{M})) \simeq \mathfrak{M} \quad \text{and} \quad \mathrm{rts}(\mathrm{str}(\mathfrak{M}_K)) \simeq \mathfrak{M}_K$$

where $\mathfrak{M} \simeq \mathfrak{M}'$ means that there is an f bijection between W and W' s.t.

$$wRv \iff f(w)R'f(v) \text{ and } w \in V(p) \iff f(w) \in V'(p)$$

Ockham completeness

Canonical model

We introduced the Kamp-models because it is easier to create a canonical Kamp-model than a standard tree model, and we have two reasons:

- If we want to maintain our definition of the canonical alternative relation to be $\mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$, then the presence of the axioms H.3 and G.3 will force the canonical relation to be non-branching. But we need a tree for a standard model. But in a Kamp model $<$ is nonbranching!
- Now we have this thing called history. This history should be present in the truth lemma, so we have to formulate a new lemma. Also the notion of history needs syntactical construction. What should that be? These questions do not arise in Kamp-frames since there are no histories there.

ANOTHER SHOT IN THE DARK

So we have the hunch that (Ockhamist) Kamp-frames have the same logic as normal (Ockhamist) tree models. Then we can conjuncture some more axioms.

Remember that the Kamp-valuation had this important defining property:

$$w \in V(p) \Rightarrow \forall (w' \equiv w)\ w' \in V(p)$$

The object language can notice this by the validity of the formulas

$$p \to \Box p \qquad \text{where } p \in \mathrm{At}$$

Note that of course cannot be true for any formula, only for the atomic ones. (The future tenses make the histories different!) So we have an axiom scheme that is true only for the atomic sentences, but not all sentences. This will cause that some very popular modal properties will fail, e.g. the substitutivity:
PROPOSITION: It is not true that if φ is valid (true on every model's every history's every world) then $\varphi[p/\psi]$, the formula which is resulted by substituting every occurrences of p by ψ in φ, is valid as well.

Workshop project:

- Prove that this theorem was in true in the determinist flow of time logics! (You can find it in the literature, but my opinion is that it is easier to prove it again than find such a proof.)
- Under what restrictions is that theorem is true? (like "φ (or ψ) has no **F**-s (or **P**-s) (or $\Diamond$-s), or has at most only the **F** (or **P**) (or $\Diamond$) nonclassical operators in it" – that is 12 option!)
- Try to formalize the most general statement(s) using 12 cases.

Canonical model

Our starting axiom system is

$$\mathbf{OBT} + (\text{F4}) + (\text{H.3}) + (\text{G.3}) + (\text{T}) + (4) + (\text{B}) + (\text{aTriv})$$

(PC1) $\varphi \to .\psi \to \varphi$

(PC2) $\varphi \to (\psi \to \chi) \to .(\varphi \to \psi) \to .\varphi \to \chi$

(PC3) $\varphi \to \psi \to .\neg\psi \to \neg\varphi$

(CP) $\mathbf{PG}\varphi \to \varphi$

(CF) $\mathbf{FH}\varphi \to \varphi$

(AP) $(\mathbf{G}\varphi \wedge \mathbf{G}\psi) \to \mathbf{G}(\varphi \wedge \psi)$

(AF) $(\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi)$

(F4) $\mathbf{G}\varphi \to \mathbf{GG}\varphi$

(H.3) $\mathbf{H}(\underline{\mathbf{H}}\varphi \to \psi) \vee \mathbf{H}(\underline{\mathbf{H}}\psi \to \varphi)$

(G.3) $\mathbf{G}(\underline{\mathbf{G}}\varphi \to \psi) \vee \mathbf{G}(\underline{\mathbf{G}}\psi \to \varphi)$

(T) $\Box\varphi \to \varphi$

(4) $\Box\varphi \to \Box\Box\varphi$

(B) $\Diamond\Box\varphi \to \varphi$

(aTriv) $p \to \Box p$ where p is atomic

(MP) $\dfrac{\varphi \quad \varphi \to \psi}{\psi}$

(Lem) $\dfrac{\varphi \to \psi}{\mathbf{H}\varphi \to \mathbf{H}\psi} \quad \dfrac{\varphi \to \psi}{\mathbf{G}\varphi \to \mathbf{G}\psi}$

It is very likely that we are still missing some axioms/rules since we have no axioms/rules about how the mixed temporal-alethic formulas behaves, i.e., about the $\Diamond$-**P**-**F** interplay!

A CANONICAL MODEL OF **K**

$$\mathfrak{M}_{\mathbf{OBT}} \stackrel{\text{def}}{=} (W_{\mathbf{OBT}}, <_{\mathbf{OBT}}, \equiv_{\mathbf{OBT}}, V_{\mathbf{OBT}})$$

where

- $W_{\mathbf{OBT}} \stackrel{\text{def}}{=} \{\Gamma : \Gamma$ is a maximally **OBT**-consistent set$\}$, i.e.,
- $\Gamma <_{\mathbf{OBT}} \Gamma'$ iff Γ' contains φ whenever Γ contains $\mathbf{G}\varphi$, formally:

  $\Gamma <_{\mathbf{OBT}} \Gamma' \stackrel{\text{def}}{\Leftrightarrow} \mathbf{G}^{-}(\Gamma) \subseteq \Gamma'$ where $\mathbf{G}^{-}(\Gamma) \stackrel{\text{def}}{=} \{\varphi : \mathbf{G}\varphi \in \Gamma\}$

- $\Gamma \equiv_{\mathbf{OBT}} \Gamma'$ iff Γ' contains φ whenever Γ contains $\Box\varphi$, formally:

  $\Gamma \equiv_{\mathbf{OBT}} \Gamma' \stackrel{\text{def}}{\Leftrightarrow} \Box^{-}(\Gamma) \subseteq \Gamma'$ where $\Box^{-}(\Gamma) \stackrel{\text{def}}{=} \{\varphi : \Box\varphi \in \Gamma\}$

- $\Gamma \in V_{\mathbf{OBT}}(p) \stackrel{\text{def}}{\Leftrightarrow} p \in \Gamma$

A CANONICAL MODEL OF **K**

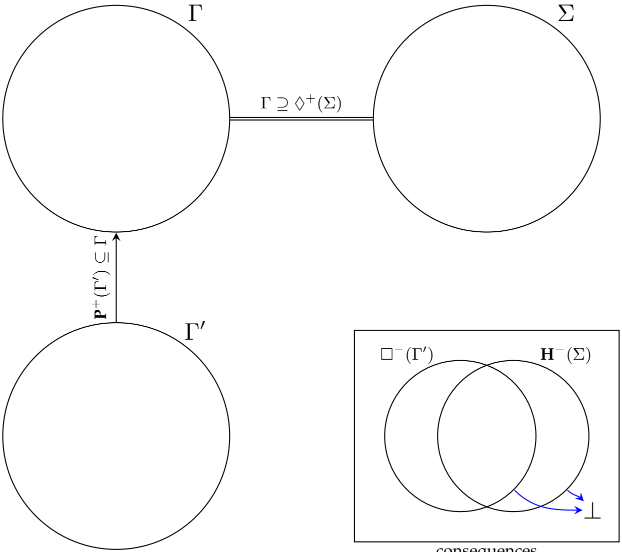
THEOREM: $\mathfrak{M}_{\mathbf{OBT}}$ is a model.

- The valuation is Kampian if we have the truth lemma: if $w \in V_{\mathbf{OBT}}(p)$ then $p \in w$, but then since we have the axioms $p \to \Box p$ for atomic formulas, and axioms are contained in every canonical worlds, and canonical worlds are closed under the derivation rules, $\Box p \in w$, then by the truth lemma, $\forall v \equiv w\ p \in v$.
- $<_{\mathbf{OBT}}$ is transitive by the canonicity of $F4$.
- $<_{\mathbf{OBT}}$ is non-branching by the canonicity of $G.3$ and $H.3$
- $<_{\mathbf{OBT}}$ is not irreflexive, so we have to bulldoze the clusters later.
- $\equiv_{\mathbf{OBT}}$ is reflexive by the canonicity of T.
- $\equiv_{\mathbf{OBT}}$ is transitive by the canonicity of 4.
- $\equiv_{\mathbf{OBT}}$ is symmetric by the canonicity of B.

SHARING THE SAME PAST

Suppose that $\Gamma' <_{\mathbf{OBT}} \Gamma$ and $\Gamma \equiv_{\mathbf{OBT}} \Sigma$. We have to show that there is a Σ' s.t. $\Sigma' <_{\mathbf{OBT}} \Sigma$ and $\Gamma' \equiv_{\mathbf{OBT}} \Sigma'$, that is, the situation that is depicted on the right.

To do so, we will show that the set $\Box^{-}(\Gamma) \cup \mathbf{H}^{-}(\Sigma)$ is consistent, therefore we can extend it to a maximally consistent set Σ', which satisfy the properties above since by containing $\Box^{-}(\Gamma)$ and $\mathbf{H}^{-}(\Sigma)$, it satisfies the (necessary and) sufficient conditions to be connected by the sets Γ' and Σ.



consequences

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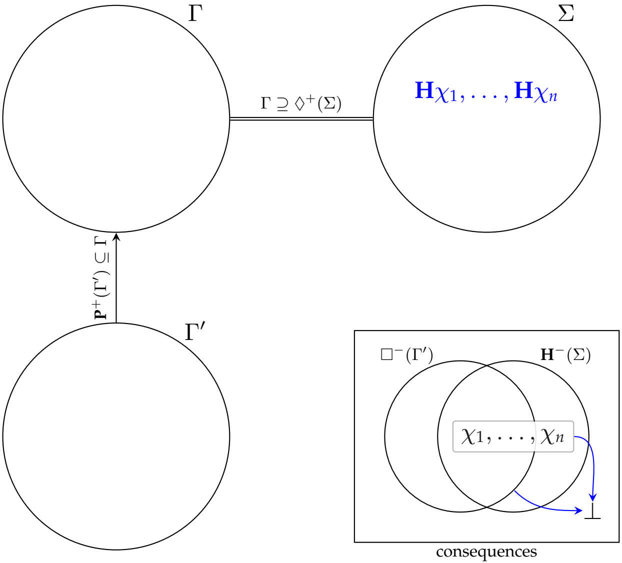

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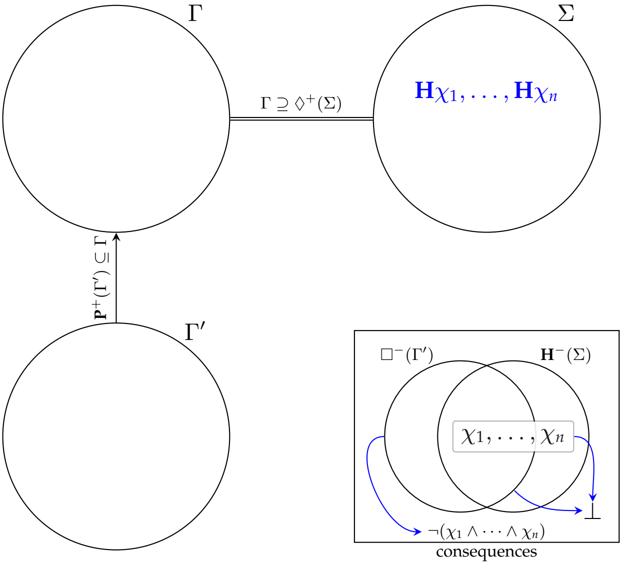

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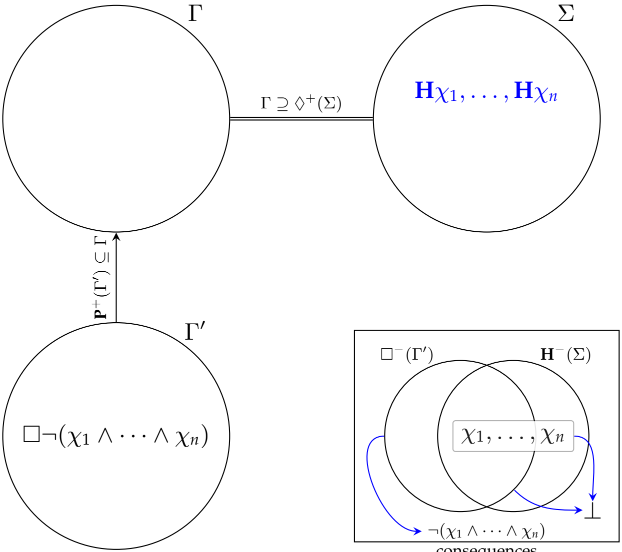

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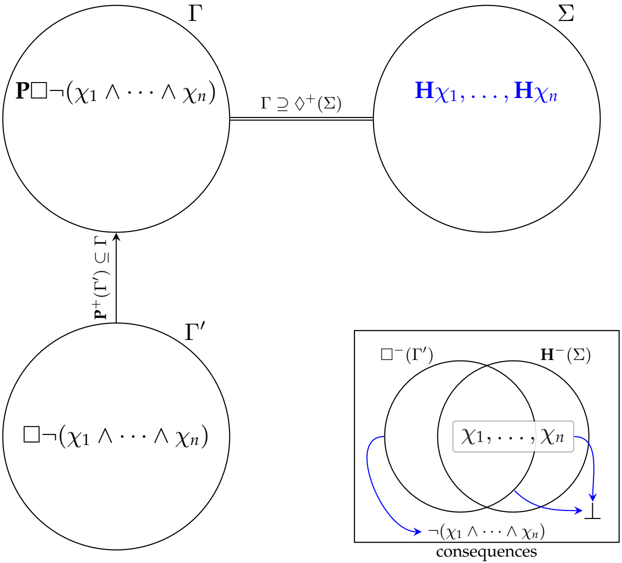

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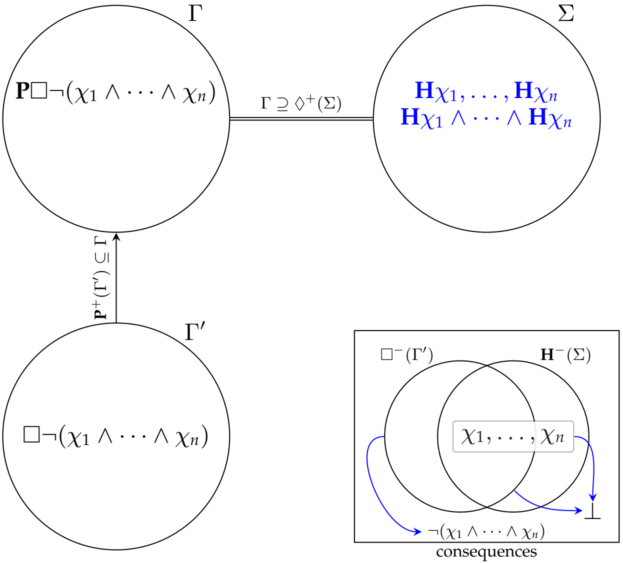

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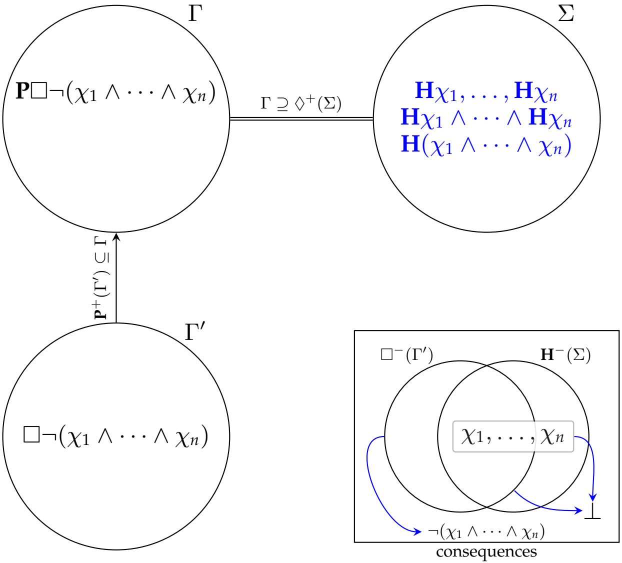

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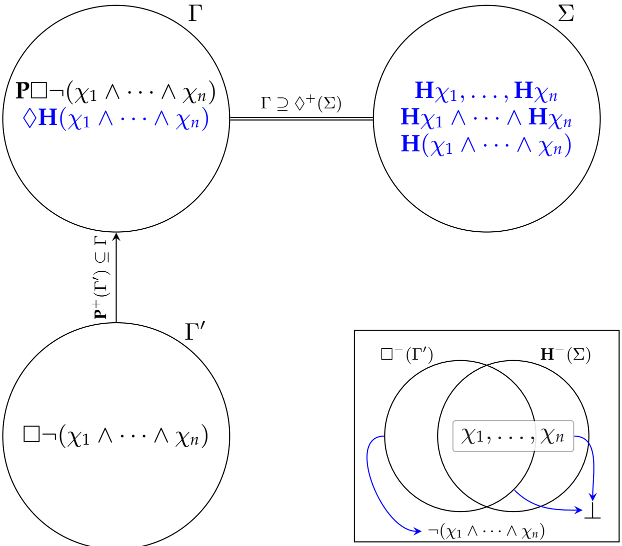

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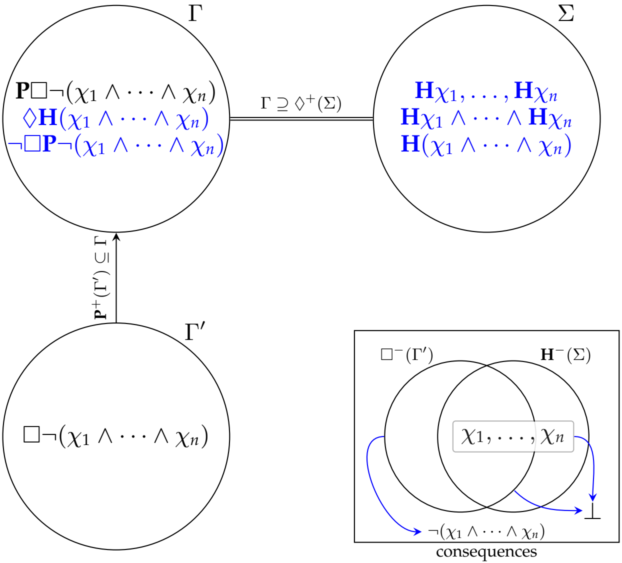

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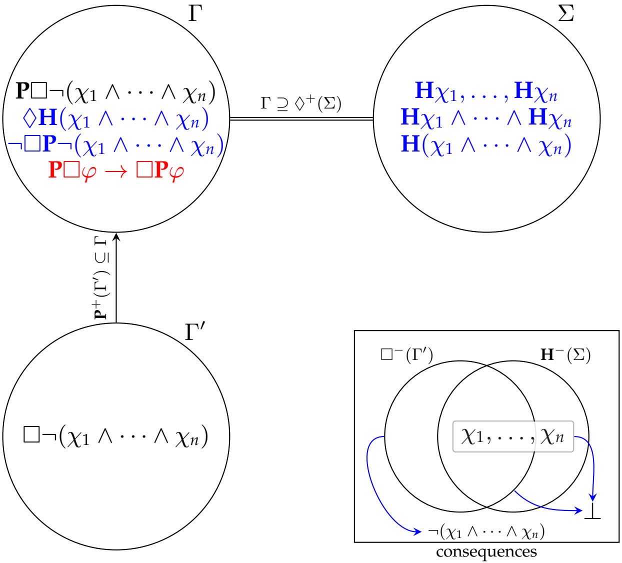

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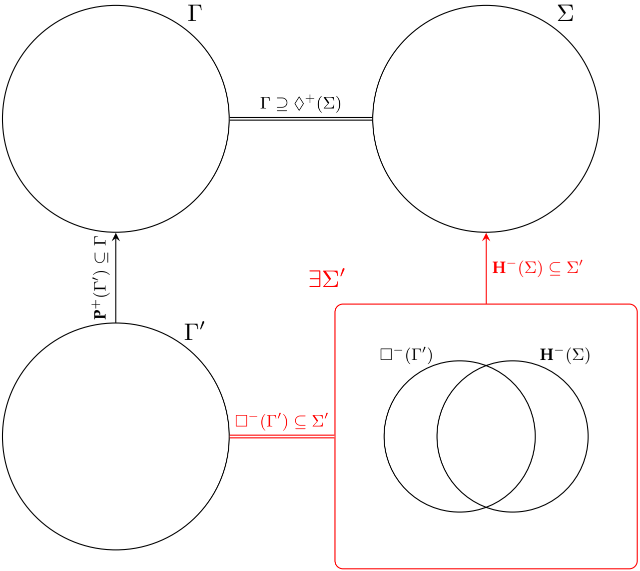

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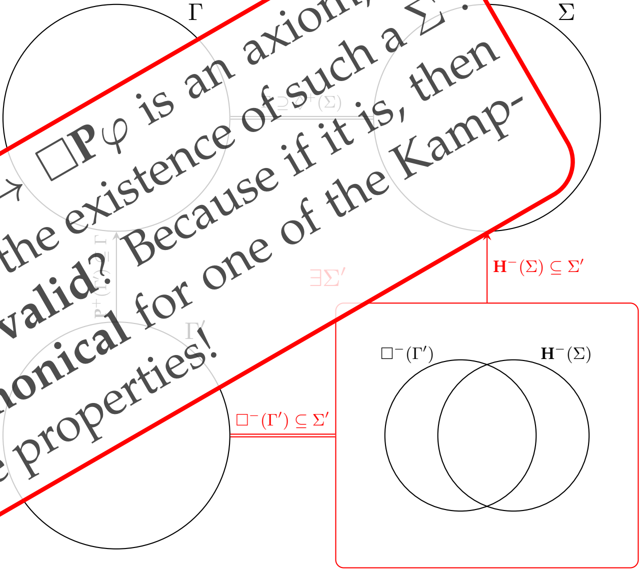


So if $\mathbf{P}\Box\varphi \to \Box\mathbf{P}\varphi$ is an axiom, we can prove the existence of such a Σ'. But **is it valid**? Because if it is, then it is **canonical** for one of the Kamp-frame properties!

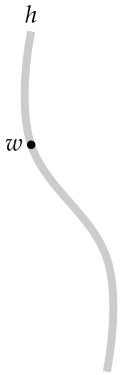
SHARING THE SAME PAST

The formal derivation:

$\Box^-(\Gamma') \cup \mathbf{H}^-(\Sigma)$	$\vdash$	$\bot$	indirect assumption
$\Box^-(\Gamma')$	$\vdash$	$\neg(\chi_1 \wedge \cdots \wedge \chi_n)$	where $\Box\chi_i$-s are all in Σ
Σ	$\vdash$	$\Box\neg(\chi_1 \wedge \cdots \wedge \chi_n)$	$\Box^-(\Gamma) \vdash \varphi \Leftrightarrow \Gamma \vdash \Box\varphi$
Γ	$\vdash$	$\mathbf{P}\Box\neg(\chi_1 \wedge \cdots \wedge \chi_n)$	$\mathbf{H}^-(\Gamma) \subseteq \Gamma' \Leftrightarrow \mathbf{P}^+(\Gamma') \subseteq \Gamma$
Γ	$\vdash$	$\Box\mathbf{P}\neg(\chi_1 \wedge \cdots \wedge \chi_n)$	THAT WOULD BE GREAT, BECAUSE...
Σ	$\vdash$	$\mathbf{P}\neg(\chi_1 \wedge \cdots \wedge \chi_n)$	$\Box^-(\Gamma) \subseteq \Sigma$
Σ	$\vdash$	$\neg\mathbf{H}(\chi_1 \wedge \cdots \wedge \chi_n)$	duality
Σ	$\vdash$	$\neg(\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n)$	$(\mathbf{H}\varphi \wedge \mathbf{H}\psi) \to \mathbf{H}(\varphi \wedge \psi)$
Σ	$\vdash$	$\mathbf{H}\chi_1 \wedge \cdots \wedge \mathbf{H}\chi_n$	$\Box\chi_i$-s are all in Σ!!

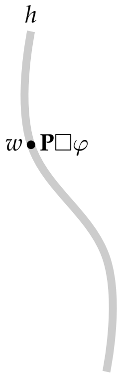
Validity of (HN)

$\mathbf{P}\Box\varphi \to \Box\mathbf{P}\varphi$ is O-valid



VALIDITY OF (HN)

$\mathbf{P}\Box\varphi \rightarrow \Box\mathbf{P}\varphi$ is O-valid



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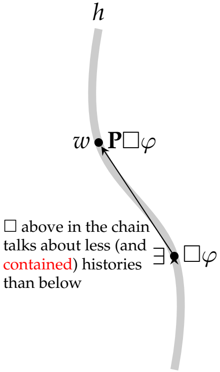

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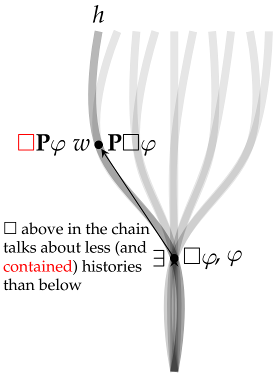

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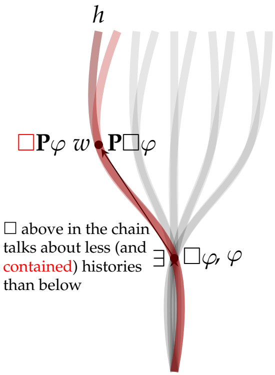

Validity of (HN)

From now on we refer to $\mathbf{P}\Box\varphi \to \Box\mathbf{P}\varphi$ as the axiom of historical necessity, or (HN). Now we prove that it is valid on every tree model.

<u>THEOREM</u>: (HN) is valid on every tree.

<u>PROOF</u>: Suppose that it is not, i.e., there is a tree model $\mathfrak{M}$ and a world w on a history h s.t.

$$\mathfrak{M}, h, w \models^{O} \mathbf{P}\Box\varphi \qquad \text{but not} \qquad \mathfrak{M}, h, w \models^{O} \Box\mathbf{P}\varphi.$$

So $\mathfrak{M}, h, w \models^{O} \neg\Box\mathbf{P}\varphi$, i.e., $\mathfrak{M}, h, w \models^{O} \Diamond\mathbf{H}\neg\varphi$. This means that there is a history $h' \overset{w}{\sim} h$ s.t. $\mathfrak{M}, h', w \models^{O} \mathbf{H}\neg\varphi$, and by that we have

$$\text{for all } v < w \qquad \mathfrak{M}, h', v \models^{O} \neg\varphi \tag{1}$$

But our assumption was that $\mathfrak{M}, h, w \models^{O} \mathbf{P}\Box\varphi$, which means that there is an $u < w$ s.t. $\mathfrak{M}, h, u \models^{O} \Box\varphi$, which implies that $\mathfrak{M}, h', u \models^{O} \varphi$. But this contradicts to (1).

<u>THEOREM</u>: (HN) is valid on every Kamp-model.