Is the relativity principle consistent with electrodynamics?

Towards a logico-empiricist reconstruction of a physical theory

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Abstract

It is common in the literature on electrodynamics and relativity theory that the transformation rules for the basic electrodynamical quantities are derived from the hypothesis that the relativity principle (RP) applies for Maxwell's electrodynamics. As it will turn out from our analysis, these derivations raise several problems, and certain steps are logically questionable. This is, however, not our main concern in this paper. Even if these derivations were completely correct, they leave open the following questions: (1) Is (RP) a true law of nature for electrodynamical phenomena? (2) Are, at least, the transformation rules of the fundamental electrodynamical quantities, derived from (RP), true? (3) Is (RP) consistent with the laws of electrodynamics in one single inertial frame of reference? (4) Are, at least, the derived transformation rules consistent with the laws of electrodynamics in one single frame of reference? Obviously, (1) and (2) are empirical questions. In this paper, we will investigate problems (3) and (4).

First we will give a general mathematical formulation of (RP). In the second part, we will deal with the operational definitions of the fundamental electrodynamical quantities. As we will see, these semantic issues are not as trivial as one might think. In the third part of the paper, applying what J. S. Bell calls "Lorentzian pedagogy"—according to which the laws of physics in any one reference frame account for all physical phenomena—we will show that the transformation rules of the electrodynamical quantities are identical with the ones obtained by presuming the covariance of the coupled Maxwell–Lorentz equations, and that the covariance is indeed satisfied.

As to problem (3), the situation is much more complex. As we will see, the relativity principle is actually not a matter of the covariance of the physical equations, but it is a matter of the details of the solutions of the equations, which describe the behavior of moving objects. This raises conceptual problems concerning the meaning of the notion "the same system in a collective motion". In case of electrodynamics, there seems no satisfactory solution to this conceptual problem; thus, contrary to the widespread views, the question we asked in the title has no obvious answer.

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1 Introduction

It is common in the literature on electrodynamics and relativity theory that the transformation rules for the basic electrodynamical quantities are *derived* from the hypothesis that the relativity principle applies for Maxwell's electrodynamics. As it will turn out from our analysis, these derivations raise several problems, and in fact they are logically questionable. This is, however, not our main concern in this paper. Even if these derivations were completely correct, they leave open the following questions:

(Q1) Is the relativity principle a true law of nature for electrodynamical phenomena?

(Q2) Are, at least, the transformation rules of the fundamental electrodynamical quantities, derived from the relativity principle, true?

(Q3) Is the relativity principle consistent with the laws of electrodynamics in one single inertial frame of reference?

(Q4) Are, at least, the derived transformation rules consistent with the laws of electrodynamics in one single frame of reference?

In a typical text book formulation, the relativity principle is the assertion that "All the laws of physics take the same form in any inertial frame of reference." In an earlier paper (Szabó 2004) we were concerned with the question of what this principle actually asserts and concluded with the following more detailed formulation:

(RP) The laws of physics describing the behavior of a system co-moving as a whole with an inertial frame K, expressed in terms of the results of measurements obtainable by means of measuring equipments co-moving with K take the same form as the laws of physics describing the similar behavior of the same system when it is co-moving with another inertial frame K', expressed in terms of the measurements with the same equipments when they are co-moving with K'.

Apparently, to answer question (Q1), that is to verify whether the principle holds for the laws describing electromagnetic phenomena, the following will be needed:

(a) We must be able to tell when two electrodynamical systems are the same except that they are moving, as a whole, relative to each other—one system is at rest relative to K, the other is at rest relative to K'.

(b) We must have proper descriptions of the behaviors of both systems, expressed in terms of two different sets of corresponding variables—one belonging to K the other to K'.

(c) The relativity principle would be completely meaningless if we mixed up different physical quantities, because, in terms of different variables, one and the same physical law in one and the same inertial frame of reference can be expressed in different forms. Consequently, we must be able to tell which variable in K corresponds to which variable in K'; that is, which variables in K and K' correspond to the same physical quantity. This immediately raises the next problem of how the physical quantities defined in the two different inertial frames are identified. Obviously, we identify those physical quantities that have identical empirical definitions.

(d) The empirical definition of a physical quantity is based on standard measuring equipments and standard operational procedures. How do the observers in different reference frames share these standard measuring equipments and operational procedures? Do they all base their definitions on the same standard measuring equipments? On the one hand, they must do something like that, otherwise any comparison between their observations would be meaningless. On the other hand, however, it is quite obvious that the principle is understood in a different way. Consider how Einstein applies the principle:

> Let there be given a stationary rigid rod; and let its length be l as measured by a measuring-rod which is also stationary. We now imagine the axis of the rod lying along the axis of x of the stationary system of coordinates, and that a uniform motion of parallel translation with velocity v along the axis of x in the direction of increasing x is then imparted to the rod. We now inquire as to the length of the moving rod, and imagine its length to be ascertained by the following two operations:
>
> (a) The observer moves together with the given measuring-rod and the rod to be measured, and measures the length of the rod directly by superposing the measuring-rod, in just the same way *as if all three were at rest* [italics added].

(b) By means of stationary clocks set up in the stationary system and synchronizing in accordance with [the light-signal synchronization], the observer ascertains at what points of the stationary system the two ends of the rod to be measured are located at a definite time. The distance between these two points, measured by the measuring-rod already employed, which in this case is at rest, is also a length which may be designated "the length of the rod."

In accordance with the principle of relativity the length to be discovered by the operation (a)—we will call it "the length of the rod in the moving system"—must be equal to the length l of the stationary rod.

The length to be discovered by the operation (b) we will call "the length of the (moving) rod in the stationary system." This we shall determine on the basis of our two principles, and we shall find that it differs from l. (Einstein 1905)

That is to say, if the standard measuring equipment defining a physical quantity X^K is, for example, at rest in K and, therefore, moving in K', then the observer in K' does not define the corresponding $X^{K'}$ as the physical quantity obtainable by means of the original standard equipment—being at rest in K and moving in K'—but rather as the one obtainable by means of the same standard equipment *in another state of motion*, namely when it is at rest in K' and moving in K. Thus, we must be able to tell when two measuring equipments are the same, except that they are moving, as a whole, relative to each other—one is at rest relative to K, the other is at rest relative to K'. Similarly, we must be able to tell when two operational procedures performed by the two observers are the "same"; in spite of the fact that the procedure performed in K' obviously differs from the one performed in K.

(e) Obviously, in order to compare these procedures we must know what the procedures exactly are; that is, we must have precise operational definitions of the quantities in question in one single inertial frame of reference.

All these issues naturally arise if we want to verify *empirically* whether the relativity principle is a true law of nature for electrodynamical phenomena. For, an empirical verification, no doubt, requires that the physicist knows which body of observations corresponds to when statement (RP) is true, and which one corresponds to when (RP) is false. Without entering here into the discussion of verificationism in general, we have only two remarks to make.

First, our approach is entirely compatible with confirmation/semantic holism. The position we are advocating here is essentially holistic. We accept it as true that "our statements about the external world face the tribunal of sense experience not individually but only as a corporate body" (Quine 1951). On the one hand this means that a theory, together with its semantics, as a whole is falsified if any single sentence of its deductive closure is empirically falsified; any part of the theory can be reconsidered—the basic deductive system, the applied mathematical tools, and the semantic rules of correspondence included. On the other hand, contrary to what is often claimed, this kind of holism does not imply that the sentences of a physical theory, at least partly, cannot be provided with empirical meaning by reducing them to a sense-datum language. In our view, on the contrary, what semantic holism implies is that the empirical definition of a physical term must not be regarded in isolation from the empirical definitions of the other terms involved in the definition. For example, as we will see, the empirical definitions of electrodynamical quantities cannot be separated from the notion of mass; in fact, the definitions in the usual electrodynamics and mechanics textbooks, together, constitute an incoherent body of definitions with circularities. This is perhaps a forgivable sin in the textbook literature. But, in philosophy of physics, the recognition of these incoherencies should not lead us to jettison the empirical content of an individual statement; on the contrary, we have to reconstruct our theories on the basis of a sufficiently large coherent body of empirical/operational definitions. In our understanding, this is the real holistic approach—a super-holistic, if you like.

Second, in fact, our arguments in this paper will rely on the verificationist theory of meaning in the following very weak sense: In physics, the meaning of a term standing for a *measurable quantity* which is supposed to characterize an objective feature of physical reality is determined by the empirical operations with which the value of the quantity in question can be ascertained. Such a limited verificationism is widely accepted among physicists; almost all electrodynamics textbooks start with some descriptions of how the basic quantities like electric charge, electric and magnetic field strengths, etc. are empirically interpreted. Our concern is that these empirical definitions do not satisfy the standard of the above mentioned super-holistic coherence, and the solution of the problem is not entirely trivial.

In any event, the demand for precise operational definitions of electrodynamical quantities emerges not from an epistemological context; not from philosophical ideas about the relationship between physical theories, sense-data, and the external reality; not from the empirical questions (Q1) and (Q2). The problem of operational definitions emerges, as a problem of pure theoretical physics, in answering questions (Q3) and (Q4), in the context of the *inner consistency* of our theories—and this inner consistency will be our main concern.

The basic idea is what J. S. Bell (1987, p. 77) calls "Lorentzian pedagogy", according to which "the laws of physics in any one reference frame account for all physical phenomena, including the observations of moving observers". All we will say about "operational" definitions and about "empirical" facts—issues (a)–(e) included—are represented and accounted *within the theory itself*. Therefore, given that all the required concepts are clarified, the laws *in one single* inertial frame of reference determine whether (RP) is true or not.

Thus, accordingly, the paper will consists of the following major parts. First of all we will give a general mathematical formulation of the relativity principle. In the second part, we will clarify the semantic issues addressed in point (e). In the third part, we will derive the transformation rules of the electrodynamical quantities, from the operational definitions and from the laws of electrodynamics in one single inertial frame of reference—independently of the relativity principle; by which we will answer our question (Q4).

As we will see, the relativity principle is actually not a matter of the covariance of the physical equations, but it is a matter of the details of the solutions of the equations, which describe the behavior of moving objects. This raises conceptual problems concerning the meaning of the notion "the same system in a collective motion". As it will be discussed in the last section, in case of electrodynamics, there seems no satisfactory solution to this conceptual problem; thus, contrary to the widespread views, the question we asked in the title has no obvious answer.

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Throughout it will be assumed that space and time coordinates are already defined in all inertial frames of reference; that is, in an arbitrary inertial frame K, space tags $\mathbf{r}(A) = (x(A), y(A), z(A)) \in \mathbb{R}^3$ and a time tag $t(A) \in \mathbb{R}$ are assigned to every event A—by means of some empirical operations.[1] Velocity, acceleration, etc., are defined in the usual way on the basis of the space and time tags. We will denote by c the "velocity of light" or "limiting velocity", etc., that is, the "c" in the Lorentz transformations, in the Lorentz–FitzGerard contraction, or in the time dilatation formula.

We will consider two inertial frames of reference, K and K'. For the sake of simplicity, we will assume the standard situation: the corresponding axises are parallel and K' is moving along the x-axis with velocity $\mathbf{V} = (V, 0, 0)$ relative to K, and the two origins coincide at time 0.[2]

We will also assume that the usual Lorentz transformation rules hold for the *kinematical* quantities. Below we recall the most important formulas we will use.

The connection between the space and time tags of an event A in K and K' is the following:

$$x'(A) = \frac{x(A) - Vt(A)}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (1)$$

$$y'(A) = y(A) \qquad (2)$$

$$z'(A) = z(A) \qquad (3)$$

$$t'(A) = \frac{t(A) - \frac{V}{c^2}x(A)}{\sqrt{1-\frac{V^2}{c^2}}} \qquad (4)$$

Let A be an event on the worldline of a particle. For the velocity of the particle at A we have:

[1] In fact, to give precise empirical definitions of the basic spatio-temporal quantities in physics is not a trivial problem (Szabó 2009).

[2] All "vectors" are meant to be in $\mathbb{R}^3$; boldface letters $\mathbf{r}, \mathbf{v}, \mathbf{E} \ldots$ simply denote vector matrices.

$$v'_x(A) = \frac{v_x(A) - V}{1 - \frac{v_x(A)V}{c^2}} \tag{5}$$

$$v'_y(A) = \frac{v_y(A)\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{v_x(A)V}{c^2}} \tag{6}$$

$$v'_z(A) = \frac{v_z(A)\sqrt{1 - \frac{V^2}{c^2}}}{1 - \frac{v_x(A)V}{c^2}} \tag{7}$$

We shall use the inverse transformation in the following special case:

$$\mathbf{v}'(A) = \left(v', 0, 0\right) \quad \mapsto \quad \mathbf{v}(A) = \left(\frac{v' + V}{1 + \frac{v'V}{c^2}}, 0, 0\right) \tag{8}$$

$$\mathbf{v}'(A) = \left(0, 0, v'\right) \quad \mapsto \quad \mathbf{v}(A) = \left(V, 0, v'\sqrt{1 - \frac{V^2}{c^2}}\right) \tag{9}$$

The transformation rule of acceleration is much more complex, but we need it only for $\mathbf{v}'(A) = (0, 0, 0)$:

$$a'_x(A) = \frac{a_x(A)}{\left(1 - \frac{V^2}{c^2}\right)^{\frac{3}{2}}} \tag{10}$$

$$a'_y(A) = \frac{a_y(A)}{1 - \frac{V^2}{c^2}} \tag{11}$$

$$a'_z(A) = \frac{a_z(A)}{1 - \frac{V^2}{c^2}} \tag{12}$$

We will also need the y-component of acceleration in case of $\mathbf{v}'(A) = (0, 0, v')$:

$$a'_y(A) = \frac{a_y(A)}{1 - \frac{V^2}{c^2}} \tag{13}$$

2 Mathematics of the relativity principle

Let us try to unpack the verbal formulations of the relativity principle (RP) in a more mathematical way. Consider some variables $\xi_1, \xi_2, \ldots \xi_n$ in K, operationally defined by means of measuring equipments at rest in K. Let $\xi'_1, \xi'_2, \ldots \xi'_n$ denote the corresponding variables in K'; that is, the physical quantities obtainable by means of the same operations with the same equipments when they are co-moving with K'. Since, for all $i = 1, 2, \ldots n$, both ξ_i and ξ'_i are measured by the same equipment—although in different physical conditions—with the same pointer scale, it is plausible to assume that $\xi_i, \xi'_i \in \sigma_i \subseteq \mathbb{R}$. We introduce the following notation: $\Sigma = \times_{i=1}^{n} \sigma_i$.

Assume that there is an injection $\tilde{T}_{\mathbf{V}} : \Sigma \rightarrowtail \Sigma$, the so-called "transformation rule",

$$\left(\xi'_1, \xi'_2, \ldots \xi'_n\right) = \tilde{T}_{\mathbf{V}}\left(\xi_1, \xi_2, \ldots \xi_n\right) \tag{14}$$

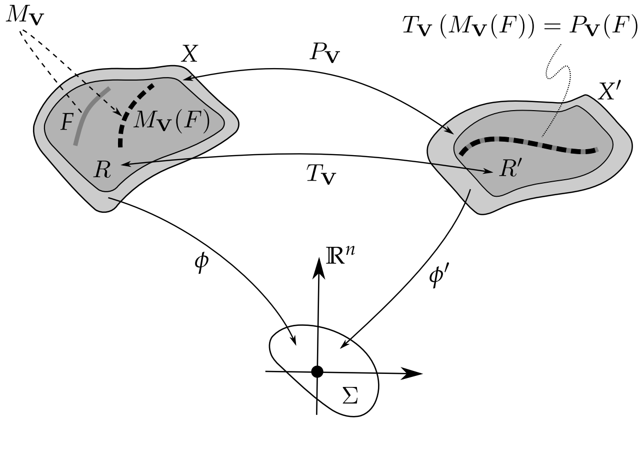

Figure 1: The relativity principle

expressing the variables in K' with the variables in K; in the sense that the paired $(\xi_1, \xi_2, \ldots \xi_n)$ and $\left(\xi'_1, \xi'_2, \ldots \xi'_n\right)$ belong to the same *physical* thing; to the same physical event, or to the same physical object in the same state, etc. Therefore, $\mathsf{Dom}\tilde{T}_{\mathbf{V}}$ and $\mathsf{Ran}\tilde{T}_{\mathbf{V}}$ only contain the physically possible/realizable n-tuples $(\xi_1, \xi_2, \ldots \xi_n)$ and $\left(\xi'_1, \xi'_2, \ldots \xi'_n\right)$ from Σ.

From physical point of view, variables $\xi_1, \xi_2, \ldots \xi_n$ and $\xi'_1, \xi'_2, \ldots \xi'_n$ are generally different physical quantities—as it will be clearly seen, for example, in electrodynamics. Consequently, what is described by $\tilde{T}_{\mathbf{V}}$ must not be regarded as either a passive (coordinate) or active transformation of the *same* "space", but rather as a map between two *different* "spaces", expressing a contingent relationship between different physical quantities. Mathematically, this idea can be expressed by considering two different n-dimensional manifolds X and X', each covered by one *global* coordinate system, ϕ and ϕ' respectively (Fig. 1). The coordinate maps ϕ and ϕ' play distinguished roles among the possible coordinate maps of the two manifolds, by carrying physical meaning: $\phi : X \to \Sigma$ assigns to every point of X one of the possible n-tuples of values of physical quantities $\xi_1, \xi_2, \ldots \xi_n$; and $\phi' : X' \to \Sigma$ has similar physical meaning with $\xi'_1, \xi'_2, \ldots \xi'_n$.

We will also use the bijection $P_{\mathbf{V}} : X \to X'$ that is uniquely determined by the distinguished coordinate maps ϕ and ϕ':

$$P_{\mathbf{V}} \stackrel{def}{=} \left(\phi'\right)^{-1} \circ \phi \tag{15}$$

Let $R \stackrel{def}{=} \phi^{-1}\left(\mathsf{Dom}\tilde{T}_{\mathbf{V}}\right) \subseteq X$ and $R' \stackrel{def}{=} \left(\phi'\right)^{-1}\left(\mathsf{Ran}\tilde{T}_{\mathbf{V}}\right) \subseteq X'$. In coordinates ϕ and ϕ', $\tilde{T}_{\mathbf{V}}$ defines a bijective map $T_{\mathbf{V}} : R \to R'$:

$$T_{\mathbf{V}} \stackrel{def}{=} \phi'^{-1} \circ \tilde{T}_{\mathbf{V}} \circ \phi \tag{16}$$

It is of course difficult to give a formal description of a "behavior" of a physical system in general. But we are probably not far from the truth if we assume that a particular law describing a particular behavior of a system in a given situation is a *relation* between the physical quantities. Let F be such a functional relation between the physical quantities $\xi_1, \xi_2, \ldots \xi_n$. In general, it can be given as a subset of R. Consider the following subsets[3] of X', determined by $F \subset R$:

$P_{\mathbf{V}}(F) \subseteq X'$ which formally is the "primed F", that is the "law" of exactly the same form as F, but in the primed variables. Relation $P_{\mathbf{V}}(F)$ is not necessarily a true physical law, as it can not be realized in nature.

$T_{\mathbf{V}}(F) \subseteq R'$ which is the same physical law as F, but expressed in the primed variables.

In order to formulate the relativity principle we need one more concept. Let the situation described by F be considered as the one in which the system, as a whole, is at rest relative to K. (In principle, arbitrary F allowed by the laws of physics can be considered as describing a situation in which the system is at rest relative to K.) Let $M_{\mathbf{V}}(F) \subset R$ be another relation of the same type, which is supposed to describe the same system in the same situation, except that it is, as a whole, in a collective motion with velocity $\mathbf{V}$ relative to K, together with reference frame K'. As we will see later on, $M_{\mathbf{V}}$ is a vague concept (see also Szabó 2004). Moreover, one may not assume that every $F \subset R$ describing a situation in which the system is, as a whole, stipulated as being at rest relative to K, has a counterpart $M_{\mathbf{V}}(F)$ for arbitrary velocity $\mathbf{V}$; because $M_{\mathbf{V}}(F)$ must describe a *real* physical situation, admitted by the relevant physical laws.[4]

Now, applying these concepts, what (RP) states is the following:

$$T_{\mathbf{V}}\left(M_{\mathbf{V}}(F)\right) = P_{\mathbf{V}}(F) \tag{17}$$

or equivalently,

$$P_{\mathbf{V}}(F) \subset R' \text{ and } M_{\mathbf{V}}(F) = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) \tag{18}$$

for all $F \subset R$ for which there exists a physically admissible $M_{\mathbf{V}}(F)$.

Let us turn to the situation similar to electrodynamics, when the physical system in question is described—in K—by a system of equations $\mathcal{E}$; the functional relation $F \subset R$ describing a particular behavior of the system is now given as a solution of $\mathcal{E}$. In general, $\mathcal{E}$ can be a set of algebraic equations, ordinary and partial integro-differential equations, linear and nonlinear, whatsoever. Without specifying these details, we will identify a system of equations with the set of its solutions; that is, as a set of subsets of R: $\mathcal{E} \subset 2^R$. We only make a physical assumption about $\mathcal{E}$: Let $\mathcal{E}^{\mathbf{V}} \subseteq \mathcal{E}$ denote the subset of those solutions F for which there exists a physically admissible counterpart $M_{\mathbf{V}}(F)$. We assume that $M_{\mathbf{V}}(\mathcal{E}^{\mathbf{V}}) \subseteq \mathcal{E}$; that is to say, the solutions of $\mathcal{E}$ are capable to describe all possible physical situations, in which the system in question is in all physically possible states of motion.

Thus, in this case, the relativity principle can be formulated as a condition for the solutions of $\mathcal{E}$:

$$T_{\mathbf{V}}\left(M_{\mathbf{V}}(F)\right) = P_{\mathbf{V}}(F) \quad \text{for all } F \in \mathcal{E}^{\mathbf{V}} \tag{19}$$

[3]We denote the map of type $X \to X'$ and its direct image maps of type $2^X \to 2^{X'}$ and $2^{2^X} \to 2^{2^{X'}}$ or their restrictions by the same symbol.

[4]For example, let the system in question be consisting of one single particle, and let F be the description of the particle's behavior when it is moving with constant velocity $\mathbf{w}$ relative to K. And let $M_{\mathbf{V}}(F)$ be understood as the relation describing the motion of a similar particle with a constant velocity $\tilde{\mathbf{w}}$, such that the relative velocity of the two particles is $\tilde{\mathbf{w}} - \mathbf{w} = \mathbf{V}$. (All velocities are relative to K.) Now, $M_{\mathbf{V}}(F)$ represents a possible physical situation only if $|\tilde{\mathbf{w}}| < c$.

or, in the more often used equivalent [5] form,

$$P_{\mathbf{V}}(F) \subset R' \text{ and } M_{\mathbf{V}}(F) = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) \quad \text{for all } F \in \mathcal{E}^{\mathbf{V}} \tag{20}$$

Now we have a strict mathematical formulation of the relativity principle for a physical system described by a system of equations $\mathcal{E}$. Remarkably, however, we still have not encountered the concept of "covariance" of equations $\mathcal{E}$. The reason is that the relativity principle and the covariance of equations $\mathcal{E}$ are not equivalent—in contrast to what many believe. In fact, the logical relationship between the two conditions is much more complex. To see this relationship in more details, we previously need to clarify a few things.

Consider the following two sets: $P_{\mathbf{V}}(\mathcal{E}) = \{P_{\mathbf{V}}(F)|F \in \mathcal{E}\}$ and $T_{\mathbf{V}}(\mathcal{E}) = \{T_{\mathbf{V}}(F)|F \in \mathcal{E}\}$. Since a system of equations can be identified with its set of solutions, $P_{\mathbf{V}}(\mathcal{E}) \subset 2^{X'}$ and $T_{\mathbf{V}}(\mathcal{E}) \subset 2^{R'}$ can be regarded as two systems of equations for functional relations between $\xi_1', \xi_2', \ldots \xi_n'$. In the primed variables, $P_{\mathbf{V}}(\mathcal{E})$ has "the same form" as $\mathcal{E}$. Nevertheless, it can be the case that $P_{\mathbf{V}}(\mathcal{E})$ does not express a true physical law, in the sense that its solutions do not necessarily describe true physical situations. In contrast, $T_{\mathbf{V}}(\mathcal{E})$ is nothing but $\mathcal{E}$ expressed in variables $\xi_1', \xi_2', \ldots \xi_n'$.

Now, covariance intuitively means that equations $\mathcal{E}$ "preserve their forms against the transformation $T_{\mathbf{V}}$". That is, in terms of the formalism we developed:

$$T_{\mathbf{V}}(\mathcal{E}) = P_{\mathbf{V}}(\mathcal{E}) \tag{21}$$

or, equivalently,

$$P_{\mathbf{V}}(\mathcal{E}) \subset 2^{R'} \text{ and } \mathcal{E} = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\mathcal{E})\right) \tag{22}$$

The first thing we have to make clear is that—even if we know or presume that it holds—covariance (22) is obviously *not sufficient* for the relativity principle (20). For, (22) only guarantees the invariance of the set of solutions, $\mathcal{E}$, against $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}$, but it says nothing about which solution of $\mathcal{E}$ corresponds to which solution; while it is the very essence of the relativity principle that the solution $M_{\mathbf{V}}(F)$, describing the system *in motion* relative to K, corresponds to solution $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}(F)$. [6]

What makes the matter more complex is that covariance is not only not sufficient for the relativity principle, but it is *not even necessary* (Fig. 2). The relativity principle only implies that

$$T_{\mathbf{V}}(\mathcal{E}) \supseteq T_{\mathbf{V}}\left(M_{\mathbf{V}}(\mathcal{E}^{\mathbf{V}})\right) = P_{\mathbf{V}}(\mathcal{E}^{\mathbf{V}})$$

(19) implies (21) only if we have some extra conditions; for example

$$\mathcal{E}^{\mathbf{V}} = \mathcal{E} \tag{23}$$

[5] For example, the usual way in which we derive the electromagnetic field of a moving point charge is that we take the Coulomb-field in the primed variables and then apply an inverse Lorentz transformation.

[6] See the application of the principle in the derivation of electromagnetic field of a uniformly moving point charge (footnote 5). What we use in the derivation is not the covariance of the Maxwell–Lorentz equations, but statement (20), that is, what the relativity principle claims about the solutions of the equations in details. (With respect to our conclusion at the end of section 8, we must note here that in this particular case the relativity principle is indeed a meaningful statement.)

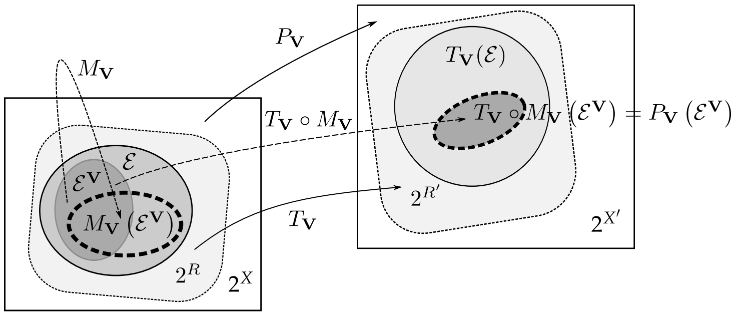

Figure 2: The relativity principle only implies that $T_{\mathbf{V}} \circ M_{\mathbf{V}}(\mathcal{E}^{\mathbf{V}}) = P_{\mathbf{V}}(\mathcal{E}^{\mathbf{V}})$. Covariance of $\mathcal{E}$ would require that $T_{\mathbf{V}}(\mathcal{E}) = P_{\mathbf{V}}(\mathcal{E})$, which is generally not the case, except if $M_{\mathbf{V}} : \mathcal{E}^{\mathbf{V}} \to \mathcal{E}$ is surjective

$$M_{\mathbf{V}}(\mathcal{E}) = \mathcal{E} \tag{24}$$

We will return to the problem of how little we can say about $M_{\mathbf{V}}$ in general; what we have to see here is that the relativity principle in itself does not imply the covariance of the physical equations.

What is the situation in electrodynamics?

- As we will see later, the very concept of $M_{\mathbf{V}}$ is problematic in electrodynamics, and this fact will raise further difficulties. Consequently, there is no guarantee that conditions (23)–(24) are satisfied.
- In any event, we will show the covariance of the Maxwell–Lorentz equations, *independently of the relativity principle*; in the sense that we will determine the transformation of the electrodynamical quantities, independently of the relativity principle—and without presuming the covariance, of course—and will see that the equations are covariant against these transformations.
- The covariance of the Maxwell–Lorentz equations, on the other hand, is not sufficient; whether the relativity principle holds in electrodynamics will remain a question we will discuss in section 8.

Let us finally consider the situation, similar to electrodynamics, when the solutions of a system of equations $\mathcal{E}$ are specified by (initial and/or boundary value) extra conditions. In our general formalism, an extra condition for $\mathcal{E}$ is a system of equations $\psi \subset 2^X$ such that there exists exactly one solution $[\psi]_{\mathcal{E}}$ satisfying both $\mathcal{E}$ and ψ. That is, $\mathcal{E} \cap \psi = \{[\psi]_{\mathcal{E}}\}$, where $\{[\psi]_{\mathcal{E}}\}$ is a singleton set. Since $\mathcal{E} \subset 2^R$, without loss of generality we may assume that $\psi \subset 2^R$.

Since $P_{\mathbf{V}}$ and $T_{\mathbf{V}}$ are injective, $P_{\mathbf{V}}(\psi)$ and $T_{\mathbf{V}}(\psi)$ are extra conditions for equations $P_{\mathbf{V}}(\mathcal{E})$ and $T_{\mathbf{V}}(\mathcal{E})$ respectively, and we have

$$P_{\mathbf{V}}([\psi]_{\mathcal{E}}) = [P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \tag{25}$$

$$T_{\mathbf{V}}([\psi]_{\mathcal{E}}) = [T_{\mathbf{V}}(\psi)]_{T_{\mathbf{V}}(\mathcal{E})} \tag{26}$$

for all extra conditions ψ for $\mathcal{E}$. Similarly, if $P_{\mathbf{V}}(\mathcal{E}), P_{\mathbf{V}}(\psi) \subset 2^{R'}$ then

$T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\psi)\right)$ is an extra condition for $T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\mathcal{E})\right)$, and

$$\left[T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\psi)\right)\right]_{T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(\mathcal{E}))} = T_{\mathbf{V}}^{-1}\left([P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})}\right) \tag{27}$$

Consider now a set of extra conditions $\mathcal{C} \subset 2^{2^R}$. Assume that $\mathcal{C}$ is a *parametrizing set of extra conditions* for $\mathcal{E}$; by which we mean that for all $F \in \mathcal{E}$ there exists exactly one $\psi \in \mathcal{C}$ such that $F = [\psi]_{\mathcal{E}}$; in other words,

$$\psi \in \mathcal{C} \mapsto [\psi]_{\mathcal{E}} \in \mathcal{E}$$

is a bijection.

Let us introduce the following notation:

$$\mathcal{C}^{\mathbf{V}} \stackrel{def}{=} \left\{\psi \in \mathcal{C} \middle| [\psi]_{\mathcal{E}} \in \mathcal{E}^{\mathbf{V}}\right\}$$

$M_{\mathbf{V}} : \mathcal{E}^{\mathbf{V}} \subseteq \mathcal{E} \to \mathcal{E}$ was introduced as a map between solutions of $\mathcal{E}$. Now, as there is a one-to-one correspondence between the elements of $\mathcal{C}$ and $\mathcal{E}$, it generates a map $M_{\mathbf{V}} : \mathcal{C}^{\mathbf{V}} \subseteq \mathcal{C} \to \mathcal{C}$, such that

$$[M_{\mathbf{V}}(\psi)]_{\mathcal{E}} = M_{\mathbf{V}}\left([\psi]_{\mathcal{E}}\right) \tag{28}$$

Thus, from (25) and (28), the relativity principle, that is (19), has the following form:

$$T_{\mathbf{V}}\left([M_{\mathbf{V}}(\psi)]_{\mathcal{E}}\right) = [P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \quad \text{for all } \psi \in \mathcal{C}^{\mathbf{V}} \tag{29}$$

or, equivalently, (20) reads

$$[P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \subset R' \text{ and } [M_{\mathbf{V}}(\psi)]_{\mathcal{E}} = T_{\mathbf{V}}^{-1}\left([P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})}\right) \tag{30}$$

We will make use of the following theorem:

Theorem 1. *Assume that the system of equations $\mathcal{E} \subset 2^R$ is covariant, that is, (21) is satisfied. Then,*

(i) for all $\psi \in \mathcal{C}^{\mathbf{V}}$, $T_{\mathbf{V}}\left(M_{\mathbf{V}}(\psi)\right)$ is an extra condition for the system of equations $P_{\mathbf{V}}(\mathcal{E})$, and, (29) is equivalent to the following condition:

$$\left[T_{\mathbf{V}}\left(M_{\mathbf{V}}(\psi)\right)\right]_{P_{\mathbf{V}}(\mathcal{E})} = [P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \tag{31}$$

(ii) for all $\psi \in \mathcal{C}^{\mathbf{V}}$, $P_{\mathbf{V}}(\psi) \subset 2^{R'}$, $T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\psi)\right)$ is an extra condition for the system of equations $\mathcal{E}$ and (30) is equivalent to the following condition:

$$[M_{\mathbf{V}}(\psi)]_{\mathcal{E}} = \left[T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\psi)\right)\right]_{\mathcal{E}} \tag{32}$$

Proof. (i) Obviously, $T_{\mathbf{V}}(\mathcal{E}) \cap T_{\mathbf{V}}\left(M_{\mathbf{V}}(\psi)\right)$ exists and is a singleton; and, due to (21), it is equal to $P_{\mathbf{V}}(\mathcal{E}) \cap T_{\mathbf{V}}\left(M_{\mathbf{V}}(\psi)\right)$; therefore this latter is a singleton, too. Applying (26) and (21), we have

$$T_{\mathbf{V}}\left([M_{\mathbf{V}}(\psi)]_{\mathcal{E}}\right) = \left[T_{\mathbf{V}}\left(M_{\mathbf{V}}(\psi)\right)\right]_{T_{\mathbf{V}}(\mathcal{E})} = \left[T_{\mathbf{V}}\left(M_{\mathbf{V}}(\psi)\right)\right]_{P_{\mathbf{V}}(\mathcal{E})}$$

therefore, (31) implies (30).

(ii) Similarly, due to $P_{\mathbf{V}}(\psi) \subset 2^{R'}$ and (22), $\mathcal{E} \cap T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\psi)\right)$ exists and is a singleton. Applying (27) and (22), we have

$$T_{\mathbf{V}}^{-1}\left([P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})}\right) = \left[T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\psi)\right)\right]_{T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(\mathcal{E}))} = \left[T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(\psi)\right)\right]_{\mathcal{E}}$$

that is, (32) implies (30). □

Remark 1. As we see, $M_{\mathbf{V}}$ plays a crucial role. Formally, one could say, the relativity principle is relative to a given definition of $M_{\mathbf{V}}$. Therefore, the physical content of the relativity principle depends on how $M_{\mathbf{V}}(F)$ is physically understood. But, what does it mean to say that a physical system is the same and of the same behavior as the one described by F, except that it is, as a whole, in a collective motion with velocity $\mathbf{V}$ relative to K? Without answering this crucial question the relativity principle is meaningless. On the other hand, the answer is not at all obvious. The vagueness of $M_{\mathbf{V}}$ leads serious problems to which we will return in section 8.

In fact, the same ambiguities are present in the definitions of quantities $\xi_1', \xi_2', \ldots \xi_n'$—and, therefore, in the meanings of $T_{\mathbf{V}}$ and $P_{\mathbf{V}}$. For, $\xi_1', \xi_2', \ldots \xi_n'$ are not simply arbitrary variables assigned to reference frame K', in one-to-one relations with $\xi_1, \xi_2, \ldots \xi_n$, but the physical quantities obtainable by means of the same operations with the same measuring equipments as in the operational definitions of $\xi_1, \xi_2, \ldots \xi_n$, except that everything is in a collective motion with velocity $\mathbf{V}$. Therefore, we should know what we mean by "the same measuring equipment but in collective motion". From this point of view, it does not matter whether the system in question is the object to be observed or a measuring equipment involved in the observation.

It is sometimes claimed that $M_{\mathbf{V}}(F)$, describing the moving system, is equal to the "Lorentz boosted solution" *by definition*:

$$M_{\mathbf{V}}(F) \quad \stackrel{def}{=} \quad T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) \tag{33}$$

At first sight this suggestion seems to resolve all troubles around $M_{\mathbf{V}}$. But a little reflection will show that it is, in fact, untenable.

(a) In this case, (20) would read

$$T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) = T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right) \tag{34}$$

That is, the relativity principle would become a tautology; a statement which is always true, independently of any contingent fact of nature; independently of the actual behavior of moving physical objects; and independently of the actual empirical meanings of physical quantities $\xi_1', \xi_2', \ldots \xi_n'$. But, the relativity principle supposed to be a fundamental *law of nature*. Note that a tautology is entirely different from a fundamental principle, even if the principle is used as a fundamental hypothesis or fundamental premise of a theory, from which one derives further physical statements. For, a fundamental premise, as expressing a contingent fact of nature, is potentially falsifiable by testing its consequences; a tautology is not.

(b) Even if accepted, (33) can provide physical meaning to $M_{\mathbf{V}}(F)$, only if we know the meanings of $T_{\mathbf{V}}$ and $P_{\mathbf{V}}$, that is, if we know the empirical meanings of the quantities denoted by $\xi_1', \xi_2', \ldots \xi_n'$. But, the physical meaning of $\xi_1', \xi_2', \ldots \xi_n'$ are obtained from the operational definitions: they are the quantities obtained by "the same measurements with the same equipments when they are, as a whole, co-moving with K' with velocity $\mathbf{V}$ relative to K". Symbolically, we need, priory, the concepts of $M_{\mathbf{V}}(\xi_i\text{-}equipment\ at\ rest)$. And this is a conceptual circularity: in order to have the concept of what it is to be an $M_{\mathbf{V}}(brick\ at\ rest)$ the (size)′ of which we would like to ascertain, we need to have the concept of what it is to be an $M_{\mathbf{V}}(measuring\ rod\ at\ rest)$—which is exactly the same conceptual problem.

(c) It is often claimed that we do not need to specify the concepts of $M_{\mathbf{V}}(\xi_i\text{-}equipment\ at\ rest)$ in order to know the *values* of quantities $\xi_1', \xi_2', \ldots \xi_n'$ we obtain by the measurements with the moving equipments, given that we can know the transformation rule $T_{\mathbf{V}}$ independently of knowing the operational definitions of $\xi_1', \xi_2', \ldots \xi_n'$. Typically, $T_{\mathbf{V}}$ is thought to be derived from the assumption that the relativity principle (20) holds. If however $M_{\mathbf{V}}$ is, by definition, equal to $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}$, then in place of (20) we have the tautology (34), which does not determine $T_{\mathbf{V}}$.

(d) Therefore, unsurprisingly, it is not the relativity principle from which transformation rule $T_{\mathbf{V}}$ is routinely deduced, but the covariance (22). As we have seen, however, covariance is, in general, neither sufficient nor necessary for the relativity principle. Whether (20) implies (22) hinges on physical facts, namely, whether $M_{\mathbf{V}}(\mathcal{E}) = \mathcal{E}$. But, if $M_{\mathbf{V}}$ is taken to be $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}$ by definition, the relativity principle becomes true—in the form of tautology (34)—but does not imply covariance $T_{\mathbf{V}}^{-1} \circ P_{\mathbf{V}}(\mathcal{E}) = \mathcal{E}$.

(e) Even if we assume that a transformation rule $T_{\mathbf{V}}$ were derived from some independent premises—from the independent assumption of covariance, for example—how do we know that the $T_{\mathbf{V}}$ we obtained and the quantities of values $T_{\mathbf{V}}(\xi_1, \xi_2, \ldots \xi_n)$ are correct plugins for the relativity principle? How could we verify that $T_{\mathbf{V}}(\xi_1, \xi_2, \ldots \xi_n)$ are indeed the values measured by a moving observer applying the same operations with the same measuring equipments, etc.?—without having an independent concept of $M_{\mathbf{V}}$, at least for the measuring equipments?

(f) One could argue that we do not need such a verification; $T_{\mathbf{V}}(\xi_1, \xi_2, \ldots \xi_n)$ can be regarded *as the empirical definition* of the primed quantities:

$$\xi_1', \xi_2', \ldots \xi_n' \stackrel{def}{=} T_{\mathbf{V}}(\xi_1, \xi_2, \ldots \xi_n) \tag{35}$$

This is of course logically possible. The operational definition of the primed quantities would say: ask the observer at rest in K to measure $\xi_1, \xi_2, \ldots \xi_n$ with the measuring equipments at rest in K, and then perform the mathematical operation (35). In this way, however, even the transformation rules would become tautologies; they would be true, no matter how the things are in the physical world.

Thus, we have to reject the view that $M_{\mathbf{V}}(F)$, describing the moving system, is *by definition* equal to the "Lorentz boosted solution" $T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(F))$. The definition of $M_{\mathbf{V}}(F)$ is a matter of convention, to be sure; but, whether it is equal to $T_{\mathbf{V}}^{-1}\left(P_{\mathbf{V}}(F)\right)$ should be a matter of contingent facts of the world.

3 Operational definitions of electrodynamical quantities in K

Now we turn to the operational definitions of the fundamental electrodynamical quantities in one single reference frame K and to the basic observational facts about these quantities.

The operational definition of a physical quantity requires the specification of *etalon* physical objects and standard physical processes by means of which the value of the quantity is ascertained. In case of electrodynamical quantities the only "device" we need is a point-like test particle, and the standard measuring procedures by which the kinematical properties of the test particle are ascertained.

So, assume we have chosen an *etalon* test particle, and let $\mathbf{r}^{etalon}(t)$, $\mathbf{v}^{etalon}(t)$, $\mathbf{a}^{etalon}(t)$ denote its position, velocity and acceleration at time t. It is assumed that we are able to set the *etalon* test particle into motion with arbitrary velocity $\mathbf{v}^{etalon} < c$ at arbitrary location. We will need more "copies" of the *etalon* test particle:

Definition (D0) A particle e is called *test particle* if for all $\mathbf{r}$ and t

$$\mathbf{v}^{e}\left(t\right)\Big|_{\mathbf{r}^{e}(t)=\mathbf{r}} = \mathbf{v}^{etalon}\left(t\right)\Big|_{\mathbf{r}^{etalon}(t)=\mathbf{r}}$$

implies

$$\mathbf{a}^{e}\left(t\right)\Big|_{\mathbf{r}^{e}(t)=\mathbf{r}} = \mathbf{a}^{etalon}\left(t\right)\Big|_{\mathbf{r}^{etalon}(t)=\mathbf{r}}$$

(The "restriction signs" will refer to *physical* situations; for example, $|_{\mathbf{r}^{e}(t)=\mathbf{r}}$ indicates that the test particle e is at point $\mathbf{r}$ at time t.)

Note, that some of the definitions and statements below require the existence of many test particles; which is, of course, a matter of empirical fact, and will be provided by (E0) below.

First we define the electric and magnetic field strengths. The only measuring device we need is a test particle being at rest relative to K.

Definition (D1) *Electric field strength* at point $\mathbf{r}$ and time t is defined as the acceleration of an arbitrary test particle e, such that $\mathbf{r}^{e}(t) = \mathbf{r}$ and $\mathbf{v}^{e}(t) = 0$:

$$\mathbf{E}\left(\mathbf{r}, t\right) \stackrel{def}{=} \mathbf{a}^{e}(t)|_{\mathbf{r}^{e}(t)=\mathbf{r};\, \mathbf{v}^{e}(t)=0} \tag{36}$$

Magnetic field strength is defined by means of how the acceleration $\mathbf{a}^{e}$ of the rest test particle changes with an infinitesimal perturbation of its state of rest, that is, if an infinitesimally small velocity $\mathbf{v}^{e}$ is imparted to the particle. Of course, we cannot perform various small perturbations simultaneously on one and the same rest test particle, therefore we perform the measurements on many rest test particles with various small perturbations. Let $\delta \subset \mathbb{R}^3$ be an arbitrary infinitesimal neighborhood of $0 \in \mathbb{R}^3$. First we define the following function:

$$\begin{aligned} \mathbf{U}^{\mathbf{r},t} \quad &: \quad \delta \subset \mathbb{R}^3 \to \mathbb{R}^3 \\ &\mathbf{U}^{\mathbf{r},t}(\mathbf{v}) \stackrel{def}{=} \mathbf{a}^e(t)|_{\mathbf{r}^e(t)=\mathbf{r};\,\mathbf{v}^e(t)=\mathbf{v}} \end{aligned} \tag{37}$$

Definition (D2) *Magnetic field strength* at point $\mathbf{r}$ and time t is

$$\mathbf{B}(\mathbf{r},t) \stackrel{def}{=} \begin{pmatrix} \left.\frac{\partial U_y^{\mathbf{r},t}}{\partial v_z}\right|_{\mathbf{v}=0} \\ \left.\frac{\partial U_z^{\mathbf{r},t}}{\partial v_x}\right|_{\mathbf{v}=0} \\ \left.\frac{\partial U_x^{\mathbf{r},t}}{\partial v_y}\right|_{\mathbf{v}=0} \end{pmatrix} \tag{38}$$

Practically it means that one can determine the value of $\mathbf{B}(\mathbf{r},t)$, with arbitrary precision, by means of measuring the accelerations of a few test particles of velocity $\mathbf{v}^e \in \delta$.

Next we introduce the concepts of source densities:

Definition (D3)

$$\varrho(\mathbf{r},t) \stackrel{def}{=} \mathrm{div}\mathbf{E}(\mathbf{r},t) \tag{39}$$

$$\mathbf{j}(\mathbf{r},t) \stackrel{def}{=} c^2\mathrm{rot}\mathbf{B}(\mathbf{r},t) - \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \tag{40}$$

are called *active electric charge density* and *active electric current density*, respectively.

A simple consequence of the *definitions* is that a continuity equation holds for ϱ and $\mathbf{j}$:

Theorem 2.

$$\frac{\partial \varrho(\mathbf{r},t)}{\partial t} + \mathrm{div}\mathbf{j}(\mathbf{r},t) = 0 \tag{41}$$

Remark 2. In our construction, the two Maxwell equations (39)–(40), are mere *definitions* of the concepts of active electric charge density and active electric current density. They do not contain information whatsoever about how "matter produces electromagnetic field". And it is not because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are, of course, "unspecified distributions" in these "general laws", but because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ cannot be specified prior to or at least independently of the field strengths $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$. It is not the case—as is often claimed—that equations (39)–(40) express contingent physical laws about the relationship between the charge and current distributions $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ and the electromagnetic field. Again, because $\varrho(\mathbf{r},t)$ and $\mathbf{j}(\mathbf{r},t)$ are just abbreviations, standing for the expressions on the right hand sides of (39)–(40). In other words, any statement about the "charge distribution" will be a statement about $\mathrm{div}\mathbf{E}$, and any statement about the "current distribution" will be a statement about $c^2\mathrm{rot}\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t}$.

The operational definitions of the field strengths and the source densities are based on the kinematical properties of the test particles. The following definition describes the concept of a charged point-like particle, in general.

Definition (D4) A particle b is called *charged point-particle* of *specific passive electric charge* π^b and of *active electric charge* α^b if the following is true:

1. It satisfies the relativistic Lorentz equation,

$$\begin{aligned} \frac{\mathbf{a}^b(t)}{\pi^b\sqrt{1-\frac{\left(\mathbf{v}^b(t)\right)^2}{c^2}}} &= \mathbf{E}\left(\mathbf{r}^b(t), t\right) \\ &\quad -\frac{1}{c^2}\mathbf{v}^b(t)\left(\mathbf{v}^b(t)\cdot\mathbf{E}\left(\mathbf{r}^b(t), t\right)\right) \\ &\quad +\mathbf{v}^b(t)\times\mathbf{B}\left(\mathbf{r}^b(t), t\right) \end{aligned} \tag{42}$$

2. If it is the only particle whose worldline intersects a given space-time region Ω, then for all $(\mathbf{r}, t) \in \Omega$ the source densities are of the following form:

$$\varrho(\mathbf{r}, t) = \alpha^b\delta\left(\mathbf{r}-\mathbf{r}^b(t)\right) \tag{43}$$

$$\mathbf{j}(\mathbf{r}, t) = \alpha^b\delta\left(\mathbf{r}-\mathbf{r}^b(t)\right)\mathbf{v}^b(t) \tag{44}$$

where $\mathbf{r}^b(t)$, $\mathbf{v}^b(t)$ and $\mathbf{a}^b(t)$ are the particle's position, velocity and acceleration. The ratio $\mu^b \overset{def}{=} \frac{\alpha^b}{\pi^b}$ is called the *electric inertial rest mass* of the particle.

Remark 3. Of course, (42) is equivalent to the standard form of the Lorentz equation:

$$\frac{d}{dt}\left(\frac{\frac{1}{\pi}\mathbf{v}(t)}{\sqrt{1-\frac{\mathbf{v}(t)^2}{c^2}}}\right) = \mathbf{E}(\mathbf{r}(t), t) + \mathbf{v}(t)\times\mathbf{B}(\mathbf{r}(t), t)$$

with $\pi = \frac{q}{m}$ in the usual terminology, where q is the passive electric charge and m is the inertial (rest) mass of the particle—that is why we call π *specific* passive electric charge. Nevertheless, it must be clear that for all charged point-particles we introduced *two independent*, empirically meaningful and experimentally testable quantities: specific passive electric charge π and active electric charge α. There is no universal law-like relationship between these two quantities: the ratio between them varies from particle to particle. In the traditional sense, this ratio is, however, nothing but the particle's rest mass.

We must emphasize that the concept of mass so obtained, as defined by only means of electrodynamical quantities, is essentially related to electrodynamics, that is to say, to electromagnetic interaction. There seems no way to give a consistent and non-circular operational definition of inertial mass in general, independently of the context of a particular type of physical interaction. Without entering here into the detailed discussion of the problem, we only mention that, for example, Weyl's commonly accepted definition (Jammer 2000) and all similar definitions based on the conservation of momentum in particle collisions suffer from the following difficulty. There is no "collision" as a purely "mechanical" process. During a collision the particles are moving in a physical field—or fields—of interaction. Therefore: 1) the system of particles, separately, cannot be regarded as a closed system; 2) the inertial properties of the particles, in fact, reveal themselves in the interactions with the field. Thus, the concepts of inertial rest mass belonging to different interactions differ from each other; whether they are equal (proportional) to each other is a matter of contingent fact of nature.

Remark 4. The choice of the *etalon* test particle is, of course, a matter of convention, just as the definitions (D0)–(D4) themselves. It is important to note that all these conventional factors play constitutive role in the fundamental concepts of electrodynamics (Reichenbach 1965). With these choices we not only make semantic conventions determining the meanings of the terms, but also make a decision about the body of concepts by means of which we grasp physical reality. There are a few things, however, that must be pointed out:

(a) This kind of conventionality does not mean that the physical quantities defined in (D0)–(D4) cannot describe *objective* features of physical reality. It only means that we make a decision which objective features of reality we are dealing with. With another body of conventions we have another body of physical concepts/physical quantities and another body of empirical facts.

(b) On the other hand, it does not mean either that our knowledge of the physical world would not be objective but a product of our conventions. If two theories obtained by starting with two different bodies of conventions are complete enough accounts of the physical phenomena, then they describe the same reality, expressed in terms of different physical quantities. Let us spell out an example: Definition (40) is entirely conventional—no objective fact of the world determines the formula on the right hand side. Therefore, we could make another choice, say,

$$\mathbf{j}_\Theta(\mathbf{r},t) \stackrel{def}{=} \Theta^2 \mathrm{rot}\mathbf{B}(\mathbf{r},t) - \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \tag{45}$$

with some $\Theta \neq c$. At first sight, one might think that this choice will alter the speed of electromagnetic waves. This is however not the case. It will be an empirical fact *about* $\mathbf{j}_\Theta(\mathbf{r},t)$ that if a particle b is the only one whose worldline intersects a given space-time region

Ω, then for all $A \in \Omega$

$$\mathbf{j}_{\Theta}\left(\mathbf{r}(A), t(A)\right) = \alpha^b \delta\left(\mathbf{r}(A) - \mathbf{r}^b(t)\right)\mathbf{v}^b(t) + \left(\Theta^2 - c^2\right)\mathsf{rot}\mathbf{B}(\mathbf{r}(A), t(A)) \qquad (46)$$

Now, consider a region where there is no particle. Taking into account (46), we have (47)–(48) and

$$\mathsf{div}\mathbf{E}(\mathbf{r}, t) = 0$$

$$\Theta^2 \mathsf{rot}\mathbf{B}(\mathbf{r}, t) - \frac{\partial \mathbf{E}(\mathbf{r}, t)}{\partial t} = \left(\Theta^2 - c^2\right)\mathsf{rot}\mathbf{B}(\mathbf{r}, t)$$

which lead to the usual wave equation with propagation speed c. (Of course, in this particular example, one of the possible choices, namely $\Theta = c$, is distinguished by its simplicity. Note, however, that simplicity is not an epistemologically conceivable notion.)

4 Empirical facts of electrodynamics

Both "empirical" and "fact" are used in different senses. Statements (E0)–(E4) below are universal generalizations, rather than statements of particular observations. Nevertheless we call them "empirical facts", by which we simply mean that they are truths which can be acquired by *a posteriori* means. Normally, they can be considered as laws obtained by inductive generalization; statements the truths of which can be, in principle, confirmed empirically.

On the other hand, in the context of the consistency questions (Q3) and (Q4), it is not important how these statements are empirically confirmed. (E0)–(E4) can be regarded as axioms of the Maxwell–Lorentz theory in K. What is important for us is that from these *axioms*, in conjunction with the theoretical representations of the measurement operations, there follow assertions about what the moving observer in K' observes. Section 6 will be concerned with these consequences.

(E0) There exist many enough test particles and we can settle them into all required positions and velocities.

Consequently, (D1)–(D4) are sound definitions. From observations about **E**, **B** and the charged point-particles, we have further empirical facts:

(E1) In all situations, the electric and magnetic field strengths satisfy the following two Maxwell equations:

$$\mathsf{div}\mathbf{B}(\mathbf{r}, t) = 0 \qquad (47)$$

$$\mathsf{rot}\mathbf{E}(\mathbf{r}, t) + \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} = 0 \qquad (48)$$

(E2) Each particle is a charged point-particle, satisfying (D4) with some specific passive electric charge π and active electric charge α. This is also true for the test particles, with—as follows from the definitions—specific passive electric charge $\pi = 1$.

(E3) If b_1, b_2,..., b_n are the only particles whose worldline intersects a given space-time region Ω, then for all $(\mathbf{r}, t) \in \Omega$ the source densities are:

$$\varrho\left(\mathbf{r},t\right) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}\left(t\right)\right) \tag{49}$$

$$\mathbf{j}\left(\mathbf{r},t\right) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}\left(t\right)\right) \mathbf{v}^{b_i}\left(t\right) \tag{50}$$

Putting facts (E1)–(E3) together, we have the coupled Maxwell–Lorentz equations:

$$\mathsf{div}\mathbf{E}\left(\mathbf{r},t\right) = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}\left(t\right)\right) \tag{51}$$

$$c^2 \mathsf{rot}\mathbf{B}\left(\mathbf{r},t\right) - \frac{\partial \mathbf{E}\left(\mathbf{r},t\right)}{\partial t} = \sum_{i=1}^{n} \alpha^{b_i} \delta\left(\mathbf{r} - \mathbf{r}^{b_i}\left(t\right)\right) \mathbf{v}^{b_i}\left(t\right) \tag{52}$$

$$\mathsf{div}\mathbf{B}\left(\mathbf{r},t\right) = 0 \tag{53}$$

$$\mathsf{rot}\mathbf{E}\left(\mathbf{r},t\right) + \frac{\partial \mathbf{B}\left(\mathbf{r},t\right)}{\partial t} = 0 \tag{54}$$

$$\begin{aligned} \frac{\mathbf{a}^{b_i}(t)}{\pi^{b_i}\sqrt{1 - \frac{\mathbf{v}^{b_i}(t)^2}{c^2}}} = {} & \mathbf{E}\left(\mathbf{r}^{b_i}\left(t\right), t\right) - \frac{1}{c^2}\mathbf{v}^{b_i}\left(t\right)\left(\mathbf{v}^{b_i}\left(t\right) \cdot \mathbf{E}\left(\mathbf{r}^{b_i}\left(t\right), t\right)\right) \\ & + \mathbf{v}^{b_i}\left(t\right) \times \mathbf{B}\left(\mathbf{r}^{b_i}\left(t\right), t\right) \\ & (i = 1, 2, \ldots n) \end{aligned} \tag{55}$$

These are the fundamental equations of electrodynamics, describing an interacting system of n particles and the electromagnetic field.

We mention only one important theorem which can be derived from empirical fact (E3).

Theorem 3. *Consider a spatial space-time region, that is a region Γ in a space-like hypersurface H. Let $b_1, b_2, \ldots$ be the charged point-particles the wordlines of which intersect Γ. Then the following holds for the "active charge of Γ":*

$$\int_{\Gamma} \varrho d^3 h = \sum_i \alpha^{b_i} \tag{56}$$

where d^3h denotes the induced volume measure on hypersurface H.

Proof. We omit the general proof here. (It needs only (49)–(50) and some Minkowskian kinematics.) In the particular case of $\Gamma \subseteq H = \mathbb{R}^3 \times \{t\}$, (56) trivially follows from (49). □

5 Operational definitions of electrodynamical quantities in K'

So far we have only considered electrodynamics in one single frame of reference K. Now we turn to the question of how a moving observer describes the same phenomena in K'. The observed phenomena are the same, but the measuring equipments by means of which the phenomena are observed are not entirely the same; instead of being at rest in K, they are co-moving with K'.

Accordingly, we will repeat the operational definitions (D0)–(D4) with the following differences:

1. The "rest test particles" will be at rest relative to reference frame K', that is, *in motion with velocity* $\mathbf{V}$ relative to K.

2. The measuring equipments by means of which the kinematical quantities are ascertained—say, the measuring rods and clocks—will be at rest relative to K', that is, *in motion with velocity* $\mathbf{V}$ relative to K. In other words, kinematical quantities $t, \mathbf{r}, \mathbf{v}, \mathbf{a}$ in definitions (D0)–(D4) will be *replaced with*—not expressed in terms of— $t', \mathbf{r}', \mathbf{v}', \mathbf{a}'$.

Definition (D0′) Particle e is called *(test particle)′* if for all $\mathbf{r}'$ and t'

$$\mathbf{v}'^{e}\left(t'\right)\Big|_{\mathbf{r}'^{e}(t')=\mathbf{r}'} = \mathbf{v}'^{etalon}\left(t'\right)\Big|_{\mathbf{r}'^{etalon}(t')=\mathbf{r}'}$$

implies

$$\mathbf{a}'^{e}\left(t'\right)\Big|_{\mathbf{r}'^{e}(t')=\mathbf{r}'} = \mathbf{a}'^{etalon}\left(t'\right)\Big|_{\mathbf{r}'^{etalon}(t')=\mathbf{r}'}$$

A (test particle)′ e moving with velocity $\mathbf{V}$ relative to K is at rest relative to K', that is, $\mathbf{v}'^{e} = 0$. Accordingly:

Definition (D1′) *(Electric field strength)′* at point $\mathbf{r}'$ and time t' is defined as the acceleration of an arbitrary (test particle)′ e, such that $\mathbf{r}'^{e}(t) = \mathbf{r}'$ and $\mathbf{v}'^{e}(t') = 0$:

$$\mathbf{E}'\left(\mathbf{r}', t'\right) \stackrel{def}{=} \mathbf{a}'^{e}(t')\big|_{\mathbf{r}'^{e}(t')=\mathbf{r}';\, \mathbf{v}'^{e}(t')=0} \tag{57}$$

Similarly, (magnetic field strength)′ is defined by means of how the acceleration $\mathbf{a}'^{e}$ of a rest (test particle)′—rest, of course, relative to K'—changes with a small perturbation of its state of motion, that is, if an infinitesimally small velocity $\mathbf{v}'^{e}$ is imparted to the particle. Just as in (D2), let $\delta' \subset \mathbb{R}^3$ be an arbitrary infinitesimal neighborhood of $0 \in \mathbb{R}^3$. We define the following function:

$$\begin{aligned} \mathbf{U}'^{\mathbf{r}',t'} \quad &: \quad \delta' \subset \mathbb{R}^3 \to \mathbb{R}^3 \\ &\mathbf{U}'^{\mathbf{r}',t'}(\mathbf{v}') \stackrel{def}{=} \mathbf{a}'^{e}(t')\big|_{\mathbf{r}'^{e}(t')=\mathbf{r}';\, \mathbf{v}'^{e}(t')=\mathbf{v}'} \end{aligned} \tag{58}$$

Definition (D2′) *(Magnetic field strength)′* at point $\mathbf{r}'$ and time t' is

$$\mathbf{B}'(\mathbf{r}',t') \stackrel{def}{=} \begin{pmatrix} \left.\frac{\partial U'^{\mathbf{r}',t'}_y}{\partial v'_z}\right|_{\mathbf{v}'=0} \\ \left.\frac{\partial U'^{\mathbf{r}',t'}_z}{\partial v'_x}\right|_{\mathbf{v}'=0} \\ \left.\frac{\partial U'^{\mathbf{r}',t'}_x}{\partial v'_y}\right|_{\mathbf{v}'=0} \end{pmatrix} \tag{59}$$

Definition (D3')

$$\varrho'\left(\mathbf{r}',t'\right) \stackrel{def}{=} \mathsf{div}\mathbf{E}'\left(\mathbf{r}',t'\right) \tag{60}$$

$$\mathbf{j}'\left(\mathbf{r}',t'\right) \stackrel{def}{=} c^2\mathsf{rot}\mathbf{B}'\left(\mathbf{r}',t'\right) - \frac{\partial \mathbf{E}'\left(\mathbf{r}',t'\right)}{\partial t'} \tag{61}$$

are called *(active electric charge density)'* and *(active electric current density)'*, respectively.

Of course, we have:

Theorem 4.

$$\frac{\partial \varrho'\left(\mathbf{r}',t'\right)}{\partial t'} + \mathsf{div}\mathbf{j}'\left(\mathbf{r}',t'\right) = 0 \tag{62}$$

Definition (D4') A particle is called *(charged point-particle)'* of *(specific passive electric charge)'* π'^b and of *(active electric charge)'* α'^b if the following is true:

1. It satisfies the relativistic Lorentz equation,

$$\begin{aligned} \frac{\mathbf{a}'^b(t')}{\pi'^b\sqrt{1-\frac{\left(\mathbf{v}'^b(t')\right)^2}{c^2}}} &= \mathbf{E}'\left(\mathbf{r}'^b\left(t'\right),t'\right) \\ &\quad -\frac{1}{c^2}\mathbf{v}'^b\left(t'\right)\left(\mathbf{v}'^b\left(t'\right)\cdot\mathbf{E}'\left(\mathbf{r}'^b\left(t'\right),t'\right)\right) \\ &\quad +\mathbf{v}'^b\left(t'\right)\times\mathbf{B}'\left(\mathbf{r}'^b\left(t'\right),t'\right) \end{aligned} \tag{63}$$

2. If it is the only particle whose worldline intersects a given space-time region Ω, then for all $\left(\mathbf{r}',t'\right)\in\Omega$ the (source densities)' are of the following form:

$$\varrho'\left(\mathbf{r}',t'\right) = \alpha'^b\delta\left(\mathbf{r}'-\mathbf{r}'^b\left(t'\right)\right) \tag{64}$$

$$\mathbf{j}'\left(\mathbf{r}',t'\right) = \alpha'^b\delta\left(\mathbf{r}'-\mathbf{r}'^b\left(t'\right)\right)\mathbf{v}'^b\left(t'\right) \tag{65}$$

where $\mathbf{r}'^b\left(t'\right)$, $\mathbf{v}'^b\left(t'\right)$ and $\mathbf{a}'^b\left(t'\right)$ is the particle's position, velocity and acceleration in K'. The ratio $\mu'^b \stackrel{def}{=} \frac{\alpha'^b}{\pi'^b}$ is called the *(electric inertial rest mass)'* of the particle.

Remark 5. It is worthwhile to make a few remarks about some epistemological issues:

(a) The physical quantities defined in (D1)–(D4) *differ* from the physical quantities defined in (D1′)–(D4′), simply because the physical situation in which a test particle is at rest relative to K differs from the one in which it is co-moving with K' with velocity $\mathbf{V}$ relative to K; and, as we know *from the laws of electrodynamics in* K, this difference really matters.

Someone might object that if this is so then any two instances of the same measurement must be regarded as measurements of different physical quantities. For, if the difference in the test particle's velocity is enough reason to say that the two operations determine two different quantities, then, by the same token, two operations must be regarded as different operations—and the corresponding quantities as different physical quantities—if the test particle is at different points of space, or the operations simply happen at different moments of time. And this consequence, the objection goes, seems to be absurd: if it were true, then science would not be possible, because we would not have the power to make law-like assertions at all; therefore we must admit that empiricism fails to explain how natural laws are possible, and, as many argue, science cannot do without metaphysical pre-assumptions.

Our response to such an objections is the following. First, concerning the general epistemological issue, we believe, nothing disastrous follows from admitting that two phenomena observed at different place or at different time *are* distinct. And if they are stated as instances of the same phenomenon, this statement is not a logical or metaphysical necessity—derived from some logical/metaphysical pre-assumptions—but an ordinary scientific hypothesis obtained by induction and confirmed or disconfirmed together with the *whole* scientific theory. In fact, this is precisely the case with respect to the definitions of the fundamental electrodynamical quantities. For example, definition (D1) is in fact a family of definitions each belonging to a particular situation individuated by the space-time locus $(\mathbf{r}, t)$.

Second, in this paper, we must emphasize again, the question of operational definitions of electrodynamical quantities first of all emerges not from an epistemological context, but from the context of the *inner consistency* of our theories, in answering questions (Q3) and (Q4). In the next section, all the results of the measurement operations defined in (D1′)–(D4′) will be predicted from the laws of electrodynamics in K. And, electrodynamics itself says that some differences in the conditions are relevant from the point of view of the measured acceleration of the a test particle, some others are not; some of the originally distinct quantities are contingently equal, some others not.

(b) From mathematical point of view, both (D0)–(D4) and (D0′)–(D4′) are definitions. However, while the choice of the *etalon* test particle and definitions (D0)–(D4) are entirely *conventional*, no additional conventionality in (D0′)–(D4′). The way in which we define the electrodynamical quantities in inertial frame K', automatically follows from (D0)–(D4) and from the question we would like to answer, namely, whether the relativity principle holds for electrodynamics; since the principle is about "quantities obtained by the same operational procedures with the same measuring equipments when they are co-moving with K'".

(c) In fact, one of the constituents of the concepts defined in K' is not determined by the operational definitions in K. Namely, the notion of "the same operational procedures with the same measuring equipments when they are co-

moving with K'", that is, the notion of $M_{\mathbf{V}}$ applied for the measuring operation and the measuring equipments. This is however not an additional freedom of conventionality, but a simple vagueness in our physical theories in K. In any event, in our case, the notion of the only moving measuring device, that is, the notion of "a test particle at rest relative to K'" is quite clear.

6 Observations of moving observer

Now we have another collection of operationally defined notions, $\mathbf{E}', \mathbf{B}', \varrho', \mathbf{j}'$, the concept of (charged point-particle)′ defined in the primed terms, and its properties π', α' and μ'. Normally, one should investigate these quantities experimentally and collect new empirical facts about both the relationships between the primed quantities and about the relationships between the primed quantities and the ones defined in (D1)–(D4). In contrast, we will continue our analysis in another way; following the "Lorentzian pedagogy", we will determine from the laws of physics in K what an observer co-moving with K' should observe. In fact, with this method, we will answer our question (Q4), whether the textbook transformation rules, derived from the relativity principle, are compatible with the laws of electrodynamics in one single frame of reference. We will also see whether the the basic equations (51)–(55) are covariant against these transformations.

Throughout the theorems below, it is important that when we compare, for example, $\mathbf{E}(\mathbf{r}, t)$ with $\mathbf{E}'(\mathbf{r}', t')$, we compare the values of the fields *in one and the same space-time point*, that is, we compare $\mathbf{E}(\mathbf{r}(A), t(A))$ with $\mathbf{E}'(\mathbf{r}'(A), t'(A))$. For the sake of brevity, however, we omit the indication of this fact.

The first theorem trivially follows from the fact that the Lorentz transformations of the kinematical quantities are one-to-one:

Theorem 5. *A particle is a (test particle)′ if and only if it is a test particle.*

Consequently, we have many enough (test particles)′ for definitions (D1′)–(D4′); and each is a charged point-particle satisfying the Lorentz equation (42) with specific passive electric charge $\pi = 1$.

Theorem 6.

$$E'_x = E_x \tag{66}$$

$$E'_y = \frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}} \tag{67}$$

$$E'_z = \frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}} \tag{68}$$

Proof. When the (test particle)′ is at rest relative to K', it is moving with velocity $\mathbf{v}^e = (V, 0, 0)$ relative to K. From (42) (with $\pi = 1$) we have

$$a^e_x = \left(1-\frac{V^2}{c^2}\right)^{\frac{3}{2}} E_x \tag{69}$$

$$a^e_y = \sqrt{1-\frac{V^2}{c^2}}\,(E_y - VB_z) \tag{70}$$

$$a^e_z = \sqrt{1-\frac{V^2}{c^2}}\,(E_z + VB_y) \tag{71}$$

Applying (10)–(12), we can calculate the acceleration $\mathbf{a}'^e$ in K', and, accordingly, we find

$$E'_x = a'^e_x = \frac{a^e_x}{\left(1-\frac{V^2}{c^2}\right)^{\frac{3}{2}}} = E_x$$

$$E'_y = a'^e_y = \frac{a^e_y}{1-\frac{V^2}{c^2}} = \frac{E_y - VB_z}{\sqrt{1-\frac{V^2}{c^2}}}$$

$$E'_z = a'^e_z = \frac{a^e_z}{1-\frac{V^2}{c^2}} = \frac{E_z + VB_y}{\sqrt{1-\frac{V^2}{c^2}}}$$

□

Theorem 7.

$$B'_x = B_x \tag{72}$$

$$B'_y = \frac{B_y + \frac{V}{c^2}E_z}{\sqrt{1-\frac{V^2}{c^2}}} \tag{73}$$

$$B'_z = \frac{B_z - \frac{V}{c^2}E_y}{\sqrt{1-\frac{V^2}{c^2}}} \tag{74}$$

Proof. Consider for instance B'_x. By definition,

$$B'_x = \left.\frac{\partial U'^{\mathbf{r}',t'}_y}{\partial v'_z}\right|_{\mathbf{v}'=0} \tag{75}$$

According to (58), the value of $U'^{\mathbf{r}',t'}_y(\mathbf{v}')$ is equal to

$$a'^e_y\Big|_{\mathbf{r}'^e(t')=\mathbf{r}';\,\mathbf{v}'^e(t')=\mathbf{v}'}$$

that is, the y-component of the acceleration of a (test particle)′ e in a situation in which $\mathbf{r}'^e(t') = \mathbf{r}'$ and $\mathbf{v}'^e(t') = \mathbf{v}'$. Accordingly, in order to determine the partial derivative (75) we have to determine

$$\left.\frac{d}{dw}\right|_{w=0}\left(a'^e_y\Big|_{\mathbf{r}'^e(t')=\mathbf{r}';\,\mathbf{v}'^e(t')=(0,0,w)}\right)$$

Now, according to (9), condition $\mathbf{v}'^e = (0,0,w)$ corresponds to

$$\mathbf{v}^e = \left(V, 0, w\sqrt{1-\frac{V^2}{c^2}}\right)$$

Substituting this velocity into (42), we have:

$$a^e_y = \sqrt{1-\frac{V^2+w^2\left(1-\frac{V^2}{c^2}\right)}{c^2}}\left(E_y + w\sqrt{1-\frac{V^2}{c^2}}B_x - VB_z\right) \tag{76}$$

Applying (13), one finds:

$$\begin{aligned} a'^e_y &= \frac{a^e_y}{1-\frac{V^2}{c^2}} = \frac{\sqrt{1-\frac{V^2+w^2\left(1-\frac{V^2}{c^2}\right)}{c^2}}}{1-\frac{V^2}{c^2}}\left(E_y + w\sqrt{1-\frac{V^2}{c^2}}B_x - VB_z\right) \\ &= \sqrt{\frac{1-\frac{w^2}{c^2}}{1-\frac{V^2}{c^2}}}\left(E_y + w\sqrt{1-\frac{V^2}{c^2}}B_x - VB_z\right) \end{aligned} \tag{77}$$

Differentiating with respect to w at $w = 0$, we obtain

$$B'_x = B_x$$

The other components can be obtained in the same way. □

Theorem 8.

$$\varrho' = \frac{\varrho - \frac{V}{c^2}j_x}{\sqrt{1-\frac{V^2}{c^2}}} \tag{78}$$

$$j'_x = \frac{j_x - V\varrho}{\sqrt{1-\frac{V^2}{c^2}}} \tag{79}$$

$$j'_y = j_y \tag{80}$$

$$j'_z = j_z \tag{81}$$

Proof. Substituting $\mathbf{E}'$ and $\mathbf{B}'$ with (66)–(68) and (72)–(74), $\mathbf{r}$ and t with the inverse of (1)–(4), then differentiating the composite function and taking into account (39)–(40), we get (78)–(81). □

Theorem 9. *A particle b is charged point-particle of specific passive electric charge π^b and of active electric charge α^b if and only if it is a (charged point-particle)′ of (specific passive electric charge)′ π'^b and of (active electric charge)′ α'^b, such that $\pi'^b = \pi^b$ and $\alpha'^b = \alpha^b$.*

Proof. First we will prove (63). For the sake of simplicity, we will verify this in case of $\mathbf{v}'^b = (0, 0, w)$. We can use (76):

$$a_y^b = \pi^b \sqrt{1 - \frac{V^2 + w^2\left(1 - \frac{V^2}{c^2}\right)}{c^2}} \left(E_y + w\sqrt{1 - \frac{V^2}{c^2}} B_x - V B_z\right)$$

From (13), (67), (72), and (74) we have

$$\begin{aligned} a_y'^b &= \pi^b \sqrt{1 - \frac{w^2}{c^2}} \left(E_y' + w B_x'\right) \\ &= \left.\left[\pi^b \sqrt{1 - \frac{\left(\mathbf{v}'^b\right)^2}{c^2}} \left(\mathbf{E}' - \frac{1}{c^2}\mathbf{v}'^b \left(\mathbf{v}'^b \cdot \mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_y \right|_{\mathbf{v}'^b = (0,0,w)} \end{aligned}$$

Similarly,

$$\begin{aligned} a_x'^b &= \pi^b \sqrt{1 - \frac{w^2}{c^2}} \left(E_x' - w B_y'\right) \\ &= \left.\left[\pi^b \sqrt{1 - \frac{\left(\mathbf{v}'^b\right)^2}{c^2}} \left(\mathbf{E}' - \frac{1}{c^2}\mathbf{v}'^b \left(\mathbf{v}'^b \cdot \mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_x \right|_{\mathbf{v}'^b = (0,0,w)} \\ a_z'^b &= \pi^b \left(1 - \frac{w^2}{c^2}\right)^{\frac{3}{2}} E_z' \\ &= \left.\left[\pi^b \sqrt{1 - \frac{\left(\mathbf{v}'^b\right)^2}{c^2}} \left(\mathbf{E}' - \frac{1}{c^2}\mathbf{v}'^b \left(\mathbf{v}'^b \cdot \mathbf{E}'\right) + \mathbf{v}'^b \times \mathbf{B}'\right)\right]_z \right|_{\mathbf{v}'^b = (0,0,w)} \end{aligned}$$

That is, (63) is satisfied, indeed.

In the second part, we will show that (64)–(65) are nothing but (43)–(44) expressed in terms of $\mathbf{r}', t', \varrho'$ and $\mathbf{j}'$, with $\alpha'^b = \alpha^b$.

It will be demonstrated for a particle of trajectory $\mathbf{r}'^b(t') = (wt', 0, 0)$. Applying (8), (43)–(44) have the following forms:

$$\begin{aligned} \varrho(\mathbf{r}, t) &= \alpha^b \delta\left(x - \frac{w + V}{1 + \frac{wV}{c^2}} t\right) \delta(y)\, \delta(z) \\ \mathbf{j}(\mathbf{r}, t) &= \alpha^b \delta\left(x - \frac{w + V}{1 + \frac{wV}{c^2}} t\right) \delta(y)\, \delta(z) \begin{pmatrix} \frac{w+V}{1+\frac{wV}{c^2}} \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

$\mathbf{r}, t, \varrho$ and $\mathbf{j}$ can be expressed with the primed quantities by applying the inverse of (1)–(4) and (78)–(81):

$$\begin{aligned}
\frac{\varrho'\left(\mathbf{r}',t'\right)+\frac{V}{c^2}j'_x\left(\mathbf{r}',t'\right)}{\sqrt{1-\frac{V^2}{c^2}}} &= \alpha^b\delta\left(\frac{x'+Vt'}{\sqrt{1-\frac{V^2}{c^2}}}-\frac{w+V}{1+\frac{wV}{c^2}}\frac{t'+\frac{V}{c^2}x'}{\sqrt{1-\frac{V^2}{c^2}}}\right)\delta\left(y'\right)\delta\left(z'\right)\\
\frac{j'_x\left(\mathbf{r}',t'\right)+V\varrho'\left(\mathbf{r}',t'\right)}{\sqrt{1-\frac{V^2}{c^2}}} &= \alpha^b\delta\left(\frac{x'+Vt'}{\sqrt{1-\frac{V^2}{c^2}}}-\frac{w+V}{1+\frac{wV}{c^2}}\frac{t'+\frac{V}{c^2}x'}{\sqrt{1-\frac{V^2}{c^2}}}\right)\\
&\quad\times\delta\left(y'\right)\delta\left(z'\right)\frac{w+V}{1+\frac{wV}{c^2}}\\
j'_y\left(\mathbf{r}',t'\right) &= 0\\
j'_z\left(\mathbf{r}',t'\right) &= 0
\end{aligned}$$

One can solve this system of equations for ϱ' and j'_x:

$$\varrho'\left(\mathbf{r}',t'\right) = \alpha^b\delta\left(x'-wt'\right)\delta\left(y'\right)\delta\left(z'\right) \tag{82}$$

$$\mathbf{j}'\left(\mathbf{r}',t'\right) = \alpha^b\delta\left(x'-wt'\right)\delta\left(y'\right)\delta\left(z'\right)\begin{pmatrix} w\\ 0\\ 0\end{pmatrix} \tag{83}$$

□

Theorem 10.

$$\mathsf{div}\mathbf{B}'\left(\mathbf{r}',t'\right) = 0 \tag{84}$$

$$\mathsf{rot}\mathbf{E}'\left(\mathbf{r}',t'\right)+\frac{\partial\mathbf{B}'\left(\mathbf{r}',t'\right)}{\partial t'} = 0 \tag{85}$$

Proof. Expressing (47)–(48) in terms of $\mathbf{r}'$, t', $\mathbf{E}'$ and $\mathbf{B}'$ by means of (1)–(4), (66)–(68) and (72)–(74), we have

$$\begin{aligned}
\mathsf{div}\mathbf{B}'-\frac{V}{c^2}\left(\mathsf{rot}\mathbf{E}'+\frac{\partial\mathbf{B}'}{\partial t'}\right)_x &= 0\\
\left(\mathsf{rot}\mathbf{E}'+\frac{\partial\mathbf{B}'}{\partial t'}\right)_x-V\mathsf{div}\mathbf{B}' &= 0\\
\left(\mathsf{rot}\mathbf{E}'+\frac{\partial\mathbf{B}'}{\partial t'}\right)_y &= 0\\
\left(\mathsf{rot}\mathbf{E}'+\frac{\partial\mathbf{B}'}{\partial t'}\right)_z &= 0
\end{aligned}$$

which is equivalent to (84)–(85), indeed. □

Theorem 11. *If b_1, b_2,..., b_n are the only particles whose worldline intersects a given space-time region Ω', then for all $\left(\mathbf{r}',t'\right)\in\Omega'$ the (source densities)′ are:*

$$\varrho'\left(\mathbf{r}',t'\right) = \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right) \tag{86}$$

$$\mathbf{j}'\left(\mathbf{r}',t'\right) = \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)\mathbf{v}'^{b_i}\left(t'\right) \tag{87}$$

Proof. Due to Theorem 9, each (charged point-particle)′ is a charged point-particle with $\alpha'^{b} = \alpha^{b}$. Therefore, we only need to prove that equations (86)–(87) amount to (49)–(50) expressed in the primed variables. On the left hand side of (49)–(50), ϱ and $\mathbf{j}$ can be expressed by means of the inverse of (78)–(81); on the right hand side, we take $\alpha'^{b} = \alpha^{b}$, and apply the inverse of (1)–(4), just as in the derivation of (82)–(83). From the above, we obtain:

$$\begin{aligned}
\varrho'\left(\mathbf{r}',t'\right)+\frac{V}{c^{2}}j'_{x}\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)\\
&\quad +\frac{V}{c^{2}}\sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v'^{b_i}_{x}\left(t'\right)\\
j'_{x}\left(\mathbf{r}',t'\right)+V\varrho'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v'^{b_i}_{x}\left(t'\right)\\
&\quad +V\sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)\\
j'_{y}\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v'^{b_i}_{y}\left(t'\right)\\
j'_{z}\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)v'^{b_i}_{z}\left(t'\right)
\end{aligned}$$

Solving these linear equations for ϱ' and $\mathbf{j}'$ we obtain (86)–(87). □

Combining the results we obtained in Theorems 9–11, we have

$$\begin{aligned}
\mathrm{div}\mathbf{E}'\left(\mathbf{r}',t'\right) &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right) && (88)\\
c^{2}\mathrm{rot}\mathbf{B}'\left(\mathbf{r}',t'\right)-\frac{\partial\mathbf{E}'\left(\mathbf{r}',t'\right)}{\partial t'} &= \sum_{i=1}^{n}\alpha^{b_i}\delta\left(\mathbf{r}'-\mathbf{r}'^{b_i}\left(t'\right)\right)\mathbf{v}'^{b_i}\left(t'\right) && (89)\\
\mathrm{div}\mathbf{B}'\left(\mathbf{r}',t'\right) &= 0 && (90)\\
\mathrm{rot}\mathbf{E}'\left(\mathbf{r}',t'\right)+\frac{\partial\mathbf{B}'\left(\mathbf{r}',t'\right)}{\partial t'} &= 0 && (91)\\
\frac{\mathbf{a}'^{b_i}\left(t'\right)}{\pi^{b_i}\sqrt{1-\frac{\mathbf{v}'^{b_i}\left(t'\right)^{2}}{c^{2}}}} &= \mathbf{E}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)\\
&\quad -\frac{1}{c^{2}}\mathbf{v}'^{b_i}\left(t'\right)\left(\mathbf{v}'^{b_i}\left(t'\right)\cdot\mathbf{E}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)\right)\\
&\quad +\mathbf{v}'^{b_i}\left(t'\right)\times\mathbf{B}'\left(\mathbf{r}'^{b_i}\left(t'\right),t'\right)\\
&\quad (i=1,2,\ldots n) && (92)
\end{aligned}$$

7 Are the textbook transformation rules consistent with the laws of electrodynamics in one single frame of reference?

Now, everything is at hand to declare that the textbook transformation rules for electrodynamical quantities, routinely derived from the *presumed* covariance of the Maxwell equations, are in fact true, at least in the sense that they are derivable from the laws of electrodynamics in one single frame of reference, including—it must be emphasized—the precise operational definitions of the quantities in question. For, Theorems 6 and 7 show the well-known transformation rules for the field variables. What Theorem 8 asserts is nothing but the well-known transformation rule for charge density and current density. Finally, Theorem 9 shows that a particle's electric specific passive charge, active charge and electric rest mass are invariant scalars.

At this point, having ascertained the transformation rules, we can declare that equations (88)–(92) are nothing but $T_{\mathbf{V}}(\mathcal{E})$, where $\mathcal{E}$ stands for the equations (51)–(55). At the same time, (88)–(92) are manifestly equal to $P_{\mathbf{V}}(\mathcal{E})$. Therefore, we proved that the Maxwell–Lorentz equations are covariant against the transformations of the kinematical and electrodynamical quantities.

Remark 6. The fact that the proper calculation of the transformation rules for the field strengths and for the source densities leads to the familiar textbook transformation rules hinges on the *relativistic* version of the Lorentz equation, in particular, on the "relativistic mass-formula". Without factor $\sqrt{1-\frac{\left(\mathbf{v}^{b}\right)^{2}}{c^{2}}}$ in (55), the proper transformation rules were different and the Maxwell equations were not covariant—against the proper transformations.

Remark 7. This is not the place to review the various versions of the textbook derivation of the transformation rules for electrodynamical quantities, nevertheless, a few remarks seem necessary. Among those with which we are acquainted, there are basically two major branches, and both are problematic. The first version follows Einstein's 1905 paper:

(1a) The transformation rules of electric and magnetic field strengths are derived from the presumption of the covariance of the homogeneous (with no sources) Maxwell equations.

(1b) The transformation rules of source densities are derived from the transformations of the field variables.

(1c) From the transformation rules of charge and current densities, it is derived that electric charge is an invariant scalar.

The second version is this:

(2a) The transformation rules of the charge and current densities are derived from some additional assumptions; typically from one of the followings:

(2a1) the invariance of electric charge (Jackson 1999, pp. 553–558)

(2a2) the current density is of form $\varrho\mathbf{u}(\mathbf{r},t)$, where $\mathbf{u}(\mathbf{r},t)$ is a velocity field (Tolman 1934, p. 85; Møller 1955, p. 140).

(2b) The transformation of the field strengths are derived from the transformation of ϱ and $\mathbf{j}$ and form the presumption of the covariance of the inhomogeneous Maxwell equations.

Unfortunately, with the only exception of (1b), none of the above steps is completely correct. Without entering into the details, let us mention that (2a1) and (2a2) both involve some further empirical information about the world, which does not follow from the simple assumption of covariance. Even in case of (1a) we must have the tacit assumption that zero charge and current densities go to zero charge and current densities during the transformation—otherwise the homogeneous equation would not be covariant separately. One encounters the next major difficulty in both (1a) and (2b): neither the homogeneous nor the inhomogeneous Maxwell equations determine the transformation rules of the field variables uniquely; $\mathbf{E}'$ and $\mathbf{B}'$ are only determined by $\mathbf{E}$ and $\mathbf{B}$ up to an arbitrary solution of the homogeneous equations. Finally, let us mention a conceptual confusion that seems to be routinely overlooked in (1c), (2a1) and (2a2). There is no such a simplified relation between the scalar invariance of charge and the transformation of charge and current densities, as is usually claimed. For example, it is meaningless to say that

$$Q = \varrho\Delta W = Q' = \varrho'\Delta W'$$

where ΔW denotes a volume element, and

$$\Delta W' = \frac{\Delta W}{\sqrt{1-\frac{V^2}{c^2}}}$$

Whose charge is Q, which remains invariant? Whose volume is ΔW and in what sense is that volume Lorentz contracted? In another form, in (2a2), whose velocity is $\mathbf{u}(\mathbf{r},t)$?

Remark 8. In the previous remark we pointed out typical problems in the derivations of the transformation rules *from the covariance* of the electrodynamical equations. There is however a more fundamental problem: How do we arrive at the covariance itself? Obviously, it would be a completely mistaken idea to regard covariance as a "known/verifiable property of the equations", because we cannot verify that the equations are covariant against the transformations of electrodynamical quantities, *prior to* we know the transformations themselves against which the equations must be covariant. Therefore, the usual claim is that the covariance of the equations of electrodynamics against the transformations of electrodynamical quantities—whatever these transformations are—is *implied* by the assumption that the relativity principle holds. Now, the problem is that this implication is, as we have seen in section 2, not true. Covariance follows from the relativity principle only if $M_{\mathbf{V}}$ is surjective; which is a questionable assumption, and, as far as we know, it has never been shown. Thus, disregarding the minor flaws mentioned in Remark 7, in the absence of the proof of this implication, one is not entitled to say that either the covariance of the Maxwell–Lorentz equations or the transformation rules of electrodynamical quantities are *derived* from the principle of relativity.

In contrast, we have indeed calculated the transformation rules from the proper operational definitions of the basic electrodynamical quantities, and have shown that the Maxwell–Lorentz equations are indeed covariant against these transformations—*independently* of the principle of relativity. In fact, the question whether the principle of relativity holds for electrodynamics has been left open.

8 Is the relativity principle consistent with the laws of electrodynamics in one single frame of reference?

One might think, we simply have to verify whether the solutions of equations (51)–(55) satisfy condition (19) in section 2. However, we still have some vagueness in the relativity principle; namely, the vagueness of $M_{\mathbf{V}}(F)$. For, when can we say that a solution describes the same behavior of the same system, except that it is in an additional collective motion at velocity $\mathbf{V}$? While there is unambiguous meaning of $M_{\mathbf{V}}(F)$ in the Galileo covariant classical mechanics, one can show simple situations in relativistic physics, in which a solution of the equations describing the system in question doubtlessly corresponds to the concept of $M_{\mathbf{V}}(F)$ relative to another solution F, but still $M_{\mathbf{V}}(F) \neq T_{\mathbf{V}}^{-1}(P_{\mathbf{V}}(F))$ (Szabó 2004). Unfortunately, the concept of $M_{\mathbf{V}}(F)$ is especially problematic in case of a coupled particles + electromagnetic field system, as the following considerations will demonstrate.

As is known, a solution of the coupled Maxwell–Lorentz equations is uniquely determined by a set of Cauchy data along a $t = t_0$ Cauchy surface. The Cauchy data are the values of the particles' positions and velocities, and the values of the electric and magnetic field strengths along the Cauchy surface. The corresponding extra conditions are of the following form:

$$\psi \quad \begin{cases} \mathbf{r}^{b_1}(t_0) = \mathbf{r}_0^{b_1} \\ \mathbf{v}^{b_1}(t_0) = \mathbf{v}_0^{b_1} \\ \qquad \vdots \\ \mathbf{r}^{b_n}(t_0) = \mathbf{r}_0^{b_n} \\ \mathbf{v}^{b_n}(t_0) = \mathbf{v}_0^{b_n} \\ \mathbf{E}(\mathbf{r}, t_0) = \mathbf{E}_0(\mathbf{r}) \\ \mathbf{B}(\mathbf{r}, t_0) = \mathbf{B}_0(\mathbf{r}) \end{cases} \tag{93}$$

Due to the fact that there is a one-to-one correspondence between the Cauchy data along the $t = t_0$ Cauchy surface and the solutions of the equations, extra conditions of the form (93) constitute a parametrizing set of extra conditions for the Maxwell–Lorentz equations, defined in section 2.

We have proved, independently of the relativity principle, that the Maxwell–Lorentz equations are covariant; therefore, we can apply Theorem 1. That is, the relativity principle for electrodynamics is equivalent to

$$[T_{\mathbf{V}}\left(M_{\mathbf{V}}(\psi)\right)]_{P_{\mathbf{V}}(\mathcal{E})} = [P_{\mathbf{V}}(\psi)]_{P_{\mathbf{V}}(\mathcal{E})} \tag{94}$$

for all $\psi \in \mathcal{C}^{\mathbf{V}}$, where $\mathcal{E}$ stands for the Maxwell–Lorentz equations, $\mathcal{C}$ denotes the parametrizing set of extra conditions of the form (93), $\mathcal{E}^{\mathbf{V}} \subseteq \mathcal{E}$ denotes the set of solutions for which $M_{\mathbf{V}}\left([\psi]_{\mathcal{E}}\right)$ is physically admissible. So, the question is: what can we say about condition (94) from the laws of electrodynamics? In order to answer this question, we should be able to tell what $M_{\mathbf{V}}(\psi)$ *exactly* means in electrodynamics. Thus, the basic question we have to answer, in order to answer question (Q3) in the Introduction, is the following:

(Q5) What does it exactly mean that a coupled particles + field system is in such a state at time t_0, that is, the Cauchy date along the $t = t_0$ surface are such, that the corresponding time evolution of the system is the same as the one belonging to ψ, except that the whole system is in an additional *collective motion* with velocity $\mathbf{V}$?

If there were an answer to this question it would trivially imply the answer to the following more modest question:

(Q6) What does it exactly mean that a coupled particles + field system is in such a state at time t_0 that the corresponding time evolution of the system is the same as the one belonging to ψ, except that the whole system is in an additional collective motion with velocity $\mathbf{V}$, at least *in an infinitesimally small time-window* $(t_0 - \varepsilon, t_0 + \varepsilon)$?

However, as we will see below, even this latter question has no reasonable answer. For, it is perhaps easy to tell when the particles are initiated in this way. For example,

$$M_{\mathbf{V}}(\psi) \begin{cases} \mathbf{r}^{b_1}(t_0) = \mathbf{r}_0^{b_1} \\ \mathbf{v}^{b_1}(t_0) = \mathbf{v}_0^{b_1} + \mathbf{V} \\ \quad\vdots \\ \mathbf{r}^{b_n}(t_0) = \mathbf{r}_0^{b_n} \\ \mathbf{v}^{b_n}(t_0) = \mathbf{v}_0^{b_n} + \mathbf{V} \\ \quad ? \\ \quad ? \end{cases} \tag{95}$$

can be a reasonable definition, if each particle remains in a physically admissible state of motion, that is, $\left|\mathbf{v}^{b_i}(t_0) + \mathbf{V}\right| < c$. But, we also have to tell when the electromagnetic field is initiated with an additional velocity $\mathbf{V}$ relative to K.

It might be thought that it is enough to set into motion the particles, and we do not need to "set into motion" the field; because we can govern only the sources of the field but not the field itself; and because there are supposedly no "wandering waves" in nature, which are traversing across the universe but did not arise originally from moving charges (see Jánossy 1971, p. 171). This can be true from some particular aspect of electrodynamics. However:

- We cannot govern the particles better than the field, at least not within the theory we are concerned with, described by the Maxwell–Lorentz equations; any constraint on the motion of the particles would come from outside of the Maxwell–Lorentz theory (see footnote 7).

- In any event, the field configurations $\mathbf{E}(\mathbf{r},t_0)$ and $\mathbf{B}(\mathbf{r},t_0)$ are parts of the Cauchy data, therefore one cannot avoid to specify them in order to specify a unique solution of the equations.

Thus,

(Q7) What rational meaning can be attached to the words "the electromagnetic field is *in (an additional) collective motion* with velocity $\mathbf{V}$"?

If this question is meaningful at all, if it is meaningful to talk about an "additional and/or collective motion" of the field, then it must be meaningful to talk about the original and not necessarily collective instantaneous motion of the local parts of the field. That is, we must have a clear answer to the following primary question:

(Q8) What rational meaning can be attached to the words "the electromagnetic field at point $\mathbf{r}$ and time t *is in motion with some local and instant velocity* $\mathbf{v}(\mathbf{r},t)$"?

To sum up: the relativity principle is meaningful for electrodynamics only if we have a clear answer to question (Q5), which implies that we must have an answer to question (Q6), consequently to (Q7) and finally to (Q8). So let us make the first step towards providing meaning to the relativity principle in electrodynamics, by trying to answer the most primary question (Q8).

We can rely on what seems to be commonly accepted: Whatever is the answer to question (Q5), according to the *application* of the relativity principle in the derivation of electromagnetic field of a uniformly moving point charge, the system of the moving charged particle + its electromagnetic field *qualifies* as the system of the charged particle + its field in collective motion (Fig. 3). If so, one might think, we can read off the general answer to question (Q7): the electromagnetic field in collective motion with the point charge of velocity $\mathbf{V}$ can be characterized by the following condition:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r}-\mathbf{V}\delta t, t-\delta t) \tag{96}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}(\mathbf{r}-\mathbf{V}\delta t, t-\delta t) \tag{97}$$

that is

$$-\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} = \mathsf{D}\mathbf{E}(\mathbf{r},t)\mathbf{V} \tag{98}$$

$$-\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = \mathsf{D}\mathbf{B}(\mathbf{r},t)\mathbf{V} \tag{99}$$

where $\mathsf{D}\mathbf{E}(\mathbf{r},t)$ and $\mathsf{D}\mathbf{B}(\mathbf{r},t)$ denote the spatial derivative operators (Jacobians

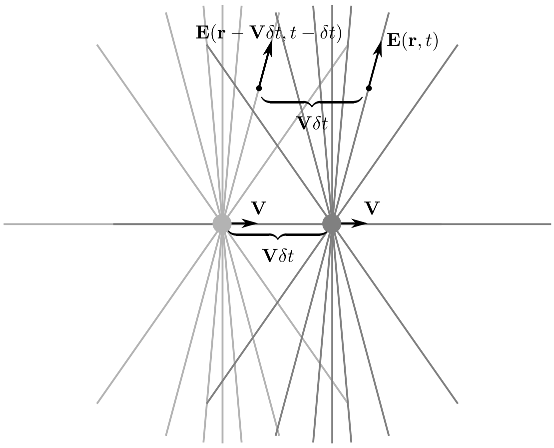

Figure 3: The stationary field of a uniformly moving point charge is in collective motion together with the point charge

for variables x, y and z); that is, in components:

$$-\frac{\partial E_x(\mathbf{r},t)}{\partial t} = \frac{\partial E_x(\mathbf{r},t)}{\partial x}V_x + \frac{\partial E_x(\mathbf{r},t)}{\partial y}V_y + \frac{\partial E_x(\mathbf{r},t)}{\partial z}V_z \tag{100}$$

$$-\frac{\partial E_y(\mathbf{r},t)}{\partial t} = \frac{\partial E_y(\mathbf{r},t)}{\partial x}V_x + \frac{\partial E_y(\mathbf{r},t)}{\partial y}V_y + \frac{\partial E_y(\mathbf{r},t)}{\partial z}V_z \tag{101}$$

$$\vdots$$

$$-\frac{\partial B_z(\mathbf{r},t)}{\partial t} = \frac{\partial B_z(\mathbf{r},t)}{\partial x}V_x + \frac{\partial B_z(\mathbf{r},t)}{\partial y}V_y + \frac{\partial B_z(\mathbf{r},t)}{\partial z}V_z \tag{102}$$

Of course, if conditions (98)–(99) hold for all $(\mathbf{r}, t)$ then the general solution of the partial differential equations (98)–(99) has the following form:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0(\mathbf{r}-\mathbf{V}t) \tag{103}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}_0(\mathbf{r}-\mathbf{V}t) \tag{104}$$

with some time-independent $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{B}_0(\mathbf{r})$. In other words, the field must be a stationary one, that is, a translation of a static field with velocity $\mathbf{V}$. This is correct in the case of a single moving point charge, provided that $\mathbf{E}_0(\mathbf{r})$ and $\mathbf{B}_0(\mathbf{r})$ are the electric and magnetic parts of the "flattened" Coulomb field at time t_0.[7] But, (103)–(104) is certainly not the case in general; the field is not necessarily stationary.

So, this example does not help to find a general answer to question (Q7), but it may help to find the answer to question (Q8). For, from (96)–(97), it is quite

[7] Here we can observe that we need, indeed, to "set into motion" the electromagnetic field too: if

$$\psi \begin{cases} \mathbf{r}(t_0) = \mathbf{r}_0 \\ \mathbf{v}(t_0) = 0 \\ \mathbf{E}(\mathbf{r},t_0) = \mathbf{E}_0^C(\mathbf{r},t_0) \\ \mathbf{B}(\mathbf{r},t_0) = 0 \end{cases} \tag{105}$$

natural to say that the electromagnetic field at point $\mathbf{r}$ and time t is moving with *local* and *instant* velocity $\mathbf{v}(\mathbf{r},t)$ if and only if

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}\left(\mathbf{r}-\mathbf{v}(\mathbf{r},t)\delta t, t-\delta t\right) \tag{106}$$

$$\mathbf{B}(\mathbf{r},t) = \mathbf{B}\left(\mathbf{r}-\mathbf{v}(\mathbf{r},t)\delta t, t-\delta t\right) \tag{107}$$

are satisfied *locally*, in an *infinitesimally* small space and time region at $(\mathbf{r},t)$, for infinitesimally small δt. In other words, the equations (98)–(99) must be satisfied *locally* at point $(\mathbf{r},t)$ with a local and instant velocity $\mathbf{v}(\mathbf{r},t)$:

$$-\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} = \mathsf{D}\mathbf{E}(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) \tag{108}$$

$$-\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = \mathsf{D}\mathbf{B}(\mathbf{r},t)\mathbf{v}(\mathbf{r},t) \tag{109}$$

Now, if the relativity principle, as it is believed, applies for all physically admissible situations, that is for all solutions from $\mathcal{E}^{\mathbf{V}}$, then it must be meaningful for all solutions in $\mathcal{E}^{\mathbf{V}}$; consequently, the concept of "electromagnetic field moving with velocity $\mathbf{v}(\mathbf{r},t)$ at point $\mathbf{r}$ and time t" must be meaningful, in other words, there must exist a local instant velocity field $\mathbf{v}(\mathbf{r},t)$ satisfying (108)–(109), for all possible solutions of the Maxwell–Lorentz equations, belonging to $\mathcal{E}^{\mathbf{V}}$. That is, substituting an arbitrary solution of (51)–(55), belonging to $\mathcal{E}^{\mathbf{V}}$, into (108)–(109), the overdetermined system of equations must have a solution for $\mathbf{v}(\mathbf{r},t)$.

Since we do not know exactly what $M_{\mathbf{V}}$ is, it is hardly possible to say anything definite about the content of $\mathcal{E}^{\mathbf{V}}$. Nevertheless, it seems quite plausible to assume that $\mathsf{int}\left(\mathcal{E}^{\mathbf{V}}\right) \neq \emptyset$—in the topology induced by the topology on the manifold of the basic quantities. Otherwise the relativity principle could apply only for some "isolated" solutions of the Maxwell–Lorentz equations; but, it would become inapplicable by an infinitesimally small variation of the solution. In this case, however, one encounters the following difficulty:

Theorem 12. *There exist a solution of the coupled Maxwell–Lorentz equations (51)–(55) which belongs to $\mathcal{E}^{\mathbf{V}}$ but for which there cannot exist a local instant velocity field $\mathbf{v}(\mathbf{r},t)$ satisfying (108)–(109).*

Proof. The proof is almost trivial for a locus $(\mathbf{r},t)$ where there is a charged point particle. However, in order to avoid the eventual difficulties concerning the

is the initial state of the rest system, where $\mathbf{E}_0^C(\mathbf{r},t_0)$ stands for the Coulomb field, then

$$M_{\mathbf{V}}(\psi)\quad \begin{cases} \mathbf{r}(t_0) = \mathbf{r}_0 \\ \mathbf{v}(t_0) = \mathbf{V} \\ \mathbf{E}(\mathbf{r},t_0) = \mathbf{E}_0^{FC}(\mathbf{r},t_0) \\ \mathbf{B}(\mathbf{r},t_0) = \mathbf{B}_0^{FC}(\mathbf{r},t_0) \end{cases}$$

where $\mathbf{E}_0^{FC}(\mathbf{r},t_0)$ and $\mathbf{B}_0^{FC}(\mathbf{r},t_0)$ stand for the well-known "flattened" fields (that is the electric and magnetic fields of the moving charge at time t_0). Within the framework of the Maxwell–Lorentz theory we cannot describe how the system has been brought into such a state; or we cannot prescribe, by hand, a constraint for the particle to be at rest or to move along a given trajectory—as is the case in many practical applications. The Coulomb field, for example, there appears among the solutions of the Maxwell–Lorentz equations as the one determined by the initial condition (105); and it is a fact about this solution that the particle remains at rest and the field remains the static Coulomb field.

physical interpretation, we are providing a proof for a point $(\mathbf{r}_*, t_*)$ where there is assumed no source at all.

Consider a solution $\left(\mathbf{r}^{b_1}(t), \ldots \mathbf{r}^{b_n}(t), \mathbf{E}(\mathbf{r},t), \mathbf{B}(\mathbf{r},t)\right)$ of the coupled Maxwell–Lorentz equations (51)–(55), which belongs to $\mathsf{int}\left(\mathcal{E}^{\mathbf{V}}\right)$ and which satisfies (108)–(109). At point $(\mathbf{r}_*, t_*)$, the following equations hold:

$$-\frac{\partial \mathbf{E}(\mathbf{r}_*, t_*)}{\partial t} = \mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) \tag{110}$$

$$-\frac{\partial \mathbf{B}(\mathbf{r}_*, t_*)}{\partial t} = \mathsf{D}\mathbf{B}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) \tag{111}$$

$$\frac{\partial \mathbf{E}(\mathbf{r}_*, t_*)}{\partial t} = c^2 \mathsf{rot}\mathbf{B}(\mathbf{r}_*, t_*) \tag{112}$$

$$-\frac{\partial \mathbf{B}(\mathbf{r}_*, t_*)}{\partial t} = \mathsf{rot}\mathbf{E}(\mathbf{r}_*, t_*) \tag{113}$$

$$\mathsf{div}\mathbf{E}(\mathbf{r}_*, t_*) = 0 \tag{114}$$

$$\mathsf{div}\mathbf{B}(\mathbf{r}_*, t_*) = 0 \tag{115}$$

Without loss of generality we can assume—at point $\mathbf{r}_*$ and time t_*—that operators $\mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)$ and $\mathsf{D}\mathbf{B}(\mathbf{r}_*, t_*)$ are invertible and $v_z(\mathbf{r}_*, t_*) \neq 0$.

Now, consider a 3×3 matrix J such that

$$J = \begin{pmatrix} \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial x} & J_{xy} & J_{xz} \\ \frac{\partial E_y(\mathbf{r}_*, t_*)}{\partial x} & \frac{\partial E_y(\mathbf{r}_*, t_*)}{\partial y} & \frac{\partial E_y(\mathbf{r}_*, t_*)}{\partial z} \\ \frac{\partial E_z(\mathbf{r}_*, t_*)}{\partial x} & \frac{\partial E_z(\mathbf{r}_*, t_*)}{\partial y} & \frac{\partial E_z(\mathbf{r}_*, t_*)}{\partial z} \end{pmatrix} \tag{116}$$

with

$$J_{xy} = \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial y} + \lambda \tag{117}$$

$$J_{xz} = \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial z} - \lambda \frac{v_y(\mathbf{r}_*, t_*)}{v_z(\mathbf{r}_*, t_*)} \tag{118}$$

by virtue of which

$$J_{xy} v_y(\mathbf{r}_*, t_*) + J_{xz} v_z(\mathbf{r}_*, t_*) = \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial y} v_y(\mathbf{r}_*, t_*) + \frac{\partial E_x(\mathbf{r}_*, t_*)}{\partial z} v_z(\mathbf{r}_*, t_*)$$

Therefore, $J\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*)$. There always exists a vector field $\mathbf{E}^{\#}_{\lambda}(\mathbf{r})$ such that its Jacobian matrix at point $\mathbf{r}_*$ is equal to J. Obviously, from (114) and (116), $\mathsf{div}\mathbf{E}^{\#}_{\lambda}(\mathbf{r}_*) = 0$. Therefore, there exists a solution of the Maxwell–Lorentz equations, such that the electric and magnetic fields $\mathbf{E}_{\lambda}(\mathbf{r}, t)$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t)$ satisfy the following conditions:[8]

$$\mathbf{E}_{\lambda}(\mathbf{r}, t_*) = \mathbf{E}^{\#}_{\lambda}(\mathbf{r})$$
$$\mathbf{B}_{\lambda}(\mathbf{r}, t_*) = \mathbf{B}(\mathbf{r}, t_*)$$

[8] $\mathbf{E}^{\#}_{\lambda}(\mathbf{r})$ and $\mathbf{B}_{\lambda}(\mathbf{r}, t_*)$ can be regarded as the initial configurations at time t_*; we do not need to specify a particular choice of initial values for the sources.

At $(\mathbf{r}_*, t_*)$, such a solution obviously satisfies the following equations:

$$\frac{\partial \mathbf{E}_\lambda(\mathbf{r}_*, t_*)}{\partial t} = c^2 \mathsf{rot}\mathbf{B}(\mathbf{r}_*, t_*) \tag{119}$$

$$-\frac{\partial \mathbf{B}_\lambda(\mathbf{r}_*, t_*)}{\partial t} = \mathsf{rot}\mathbf{E}^{\#}_\lambda(\mathbf{r}_*) \tag{120}$$

therefore

$$\frac{\partial \mathbf{E}_\lambda(\mathbf{r}_*, t_*)}{\partial t} = \frac{\partial \mathbf{E}(\mathbf{r}_*, t_*)}{\partial t} \tag{121}$$

As a little reflection shows, if $\mathsf{D}\mathbf{E}^{\#}_\lambda(\mathbf{r}_*)$, that is J, happened to be not invertible, then one can choose a *smaller* λ such that $\mathsf{D}\mathbf{E}^{\#}_\lambda(\mathbf{r}_*)$ becomes invertible (due to the fact that $\mathsf{D}\mathbf{E}(\mathbf{r}_*, t_*)$ is invertible), and, at the same time,

$$\mathsf{rot}\mathbf{E}^{\#}_\lambda(\mathbf{r}_*) \neq \mathsf{rot}\mathbf{E}(\mathbf{r}_*, t_*) \tag{122}$$

Consequently, from (121) , (118) and (110) we have

$$-\frac{\partial \mathbf{E}_\lambda(\mathbf{r}_*, t_*)}{\partial t} = \mathsf{D}\mathbf{E}_\lambda(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{E}^{\#}_\lambda(\mathbf{r}_*)\mathbf{v}(\mathbf{r}_*, t_*)$$

and $\mathbf{v}(\mathbf{r}_*, t_*)$ is uniquely determined by this equation. On the other hand, from (120) and (122) we have

$$-\frac{\partial \mathbf{B}_\lambda(\mathbf{r}_*, t_*)}{\partial t} \neq \mathsf{D}\mathbf{B}_\lambda(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*) = \mathsf{D}\mathbf{B}(\mathbf{r}_*, t_*)\mathbf{v}(\mathbf{r}_*, t_*)$$

because $\mathsf{D}\mathbf{B}(\mathbf{r}_*, t_*)$ is invertible, too. That is, for $\mathbf{E}_\lambda(\mathbf{r}, t)$ and $\mathbf{B}_\lambda(\mathbf{r}, t)$ there is no local and instant velocity at point $\mathbf{r}_*$ and time t_*. At the same time, λ can be arbitrary small, and

$$\lim_{\lambda \to 0} \mathbf{E}_\lambda(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t)$$

$$\lim_{\lambda \to 0} \mathbf{B}_\lambda(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t)$$

Therefore solution $\left(\mathbf{r}^{b_1}_\lambda(t), \ldots \mathbf{r}^{b_n}_\lambda(t), \mathbf{E}_\lambda(\mathbf{r}, t), \mathbf{B}_\lambda(\mathbf{r}, t)\right)$ can fall into an arbitrary small neighborhood of $\left(\mathbf{r}^{b_1}(t), \ldots \mathbf{r}^{b_n}(t), \mathbf{E}(\mathbf{r}, t), \mathbf{B}(\mathbf{r}, t)\right)$ in $\mathsf{int}\left(\mathcal{E}^{\mathbf{V}}\right)$, consequently it belongs to $\mathcal{E}^{\mathbf{V}}$. □

Thus, the meaning of the concept of "electromagnetic field moving with velocity $\mathbf{v}(\mathbf{r}, t)$ at point $\mathbf{r}$ and time t", that we obtained by generalizing the example of the stationary field of a uniformly moving charge, is untenable. Perhaps there is no other available rational meaning of this concept. In any event, lacking a better suggestion, we must conclude that the question whether the relativity principle generally holds in electrodynamics remains not only unanswered, but even ununderstood.

Acknowledgment

The research was partly supported by the OTKA Foundation, No. K 68043.

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