The Metaphysical Basis of Mathematics

A Physicalist Approach

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Contents

1

Introduction

1 . If you are an incorrigible empiricist like me, you might encounter the following difficulties in connection with the truths of formal logic and mathematics. In Ayer's words:

> For whereas a scientific generalization is readily admitted to be fallible, the truths of mathematics and logic appear to everyone to be necessary and certain. But if empiricism is correct no proposition which has a factual content can be necessary or certain. Accordingly the empiricist must deal with the truths of logic and mathematics in one of the following ways: he must say either that they are not necessary truths, in which case he must account for the universal conviction that they are; or he must say that they have no factual content, and then he must explain how a proposition which is empty of all factual content can be true and useful and surprising. ...
>
> If neither of these courses proves satisfactory, we shall be obliged to give way to rationalism. We shall be obliged to admit that there are some

> truths about the world which we can know independently of experience; ... (Ayer, 1952, p. 72.)

The aim of this book is to develop a complete resolution of this problem, but without the least surrendering of a rabid empiricist position.

I call my approach a "physicalist" metaphysical foundation of mathematics, rather than an "empiricist" one. I shall use the term "physicalism" for the commitment to the following two metaphysical positions:

- Genuine information about the world must be acquired by *a posteriori* means. (empiricism)
- The process of experiencing itself, as any other mental phenomena, including the mental processing the experiences, can be wholly explained in terms of physical properties, states, and events in the physical world. (physicalism in philosophy of mind)

As we will see, all embarrassing features of logical and mathematical truths can be easily accounted for if we consistently remain within the framework of this physicalist view of the world. We will show that if you are, like me, an incorrigible reductionist you can also be sure of not being challenged by any rationalistic claim about logic and mathematics.

2 . I readily admit that my account of mathematics will be philosophically/scientifically rather than "mathematically" motivated.[1] Moreover, I do not have any scruples about admitting that these philosophical/scientific motivations are *ex-*

[1] I mean "philosophical" and "scientific" from the point of view of "deans and librarians"; it doesn't make to much difference—in my understanding, Quine's continuity argument must work in both directions. When I say "sciences" I mean sciences like physics, economics, etc., but not mathematics. (This is just a terminological decision without any deeper message.)

ternal to mathematics.[2] The reason is very simple. Although, in our final physicalist conclusion, we will show that mathematical and logical truths do have contingent content in a very sophisticated sense, I reject the idea, as this thesis is usually understood, that mathematics is about the real world. In the physicalist world-view I am proposing here—which does not admit any sort of platonism—, I actually reject the idea that mathematics is about anything. Consequently, any metamathematical theory, that is, any claim *about* mathematics must be philosophical/scientific—since it cannot be mathematical. A practicing mathematician can have his or her own claims about mathematics. But these claims are not (cannot be) mathematical. And I do not see any reason for believing that a practicing mathematician can have more valid philosophical/scientific claims about mathematics than a practicing philosopher or scientist.

3 . My philosophical analysis will be based on contemporary logic and mathematics, but will be very poor from a historical point of view. The basic methodology of the formal sciences has undergone radical changes in the last, relatively short, historic period since the emergence of non-Euclidean geometries. The main trend of this development can be characterized by the shift, at least in the methodological declarations, from poorly formalized quasi-intuitive mathematical reasonings to more and more strictly formalized mathematical proves. Contrary to Lakatos, in my view, it is pointless to look at mathematics preceding these radical changes.

4 . In addition, contemporary mathematics will be examined with a critical eye. As I have already admitted, my analysis will be philosophically/scientifically motivated. It is not my aim to understand how certain views which are common to many contemporary mathematicians emerged and developed,

[2]Cf. Maddy 1997, Part III.

and I do not want to enter in a long discussion of the various philosophical frameworks with which some of these views could be compatible. My only concern is whether these views are tenable from the point of view of the basic epistemological and ontological commitments of physicalism, or not. I cannot—as many philosophers of mathematics do[3]—"take it as 'data' that most contemporary mathematics is correct" before first determining what this "correctness" actually means.

Thus, with respect to the "philosophy first" and "philosophy last if at all" dichotomy, I support the philosophy(-and-science)-first principle. This is not a privileged preference among academic disciplines. The reason is that—contrary the widely accepted view—a certain part of the corpus of contemporary mathematics is *not* invariant over the possible philosophical positions. In other words, if Gödel is right by saying that

> after sufficient clarification of the concepts in question it will be possible to conduct these discussions with mathematical rigor and that the result then will be that (under certain assumptions which can hardly be denied [in particular the assumption that there exists at all something like mathematical knowledge] the platonistic view is the only one tenable[4]

then it follows that you cannot avoid questioning certain parts of mathematics from an empiricist/physicalist standpoint. Of course, I do not suggest to question in any way any of the *strict* formal derivations of mathematics. However, I do not have scruples about questioning those claims of mathematicians that are not based on strict proofs but based on some naive ideas or intuition.[5] What is particularly striking is how much

[3]Shapiro 1997, p. 4.
[4]Gödel 1951, p. 322.
[5]Cf. Lewis 1998, p. 218.

of contemporary mathematics is affected.

5 . According to *mathematical platonism,* substantive existence can be attributed to the classical concepts of mathematics, independently of whether or not anybody has these concepts in mind. A truth about a mathematical concept can be, like any other truth about any other existing thing, *discovered.* The specific way of discovery in which a true mathematical proposition can be obtained is the rational analysis of these concepts.

Intuitionists do not ascribe any existence to mathematical objects independent of their construction by the basic intuition. Instead, they believe in the existence of Intuition, something which is *a priori* given to the universal human apprehension, something which, in this way, guarantees the *objectivity* and *usefulness* of mathematics.

Physical realism is the view that mathematical propositions are true insofar as they correspond with our physical environment.[6] In other words, mathematical propositions express the most general features of physical reality. Although this view played an important role in the history of mathematical sciences, it has become less and less important in modern mathematics. Current thinking assumes there is no such direct correspondence between mathematical notions and the elements of physical reality. Contemporary mathematics is full of complex and abstract constructions which have no application to the physical world. So, those who still insist on a physical realist view of mathematics usually have weaker claims now. They claim that there are some basic mathematical concepts and some basic mathematical propositions (say, the axioms) which reflect some elementary features of the real (physical) world. The rest of mathematical propositions are

[6] I shall use the term "realism" in a broader sense, as a view that mathematical propositions are true insofar as they correspond with the facts of the world, platonic world included.

already derived from these basic ones by applying the usual deductive machinery of formal mathematics.

Physical realists, platonists and intuitionists jointly believe, however, that mathematical concepts and propositions have *meanings*, and when we formalize the language of mathematics, these meanings are meant to be reflected in a more precise and more concise form.

6 . According to the *formalist* understanding of mathematics (at least, according to the radical version of formalism I am proposing here), the opposite is true: *the truth is that mathematical objects carry no meanings.* "The formulas are not about anything; they are just strings of symbols".[7] Hilbert characterized mathematics as a game played according to certain simple rules with *meaningless* marks on paper.[8] That's all. Mathematics has nothing to do with the metaphysical concept of infinity. Mathematics does not produce and does not solve Zeno paradoxes. According to the formalist view (see Heyting, 1983, p. 71), one can write down a sign, say α, and call it the cardinal number of the integers. After that, one can fix rules for its manipulation. The whole finitist struggle is unnecessary. Such a sign as $10^{10^{10}}$ has no other meaning than as a figure on the paper with which we operate according to certain rules, just like any other symbols. Mathematical structures are totally indifferent to our intuition about space, time, probability or continuity. The words in a formal system have no meaning other than that which may be given to them by the axioms. As Hilbert—allegedly—expressed this idea in a famous aphorism about Euclidean geometry: "One must be able to say at all times—instead of points, straight lines, and planes—tables, chairs, and beer mugs."[9]

The complete elimination of intuition, i.e. full reduction

[7]Davis and Hersh 1981, p. 319.

[8]Bell, 1951, p. 38.

[9]Fang, 1970, p. 81.

to a list of axioms and mechanical rules of inference, is possible. The work initiated by Frege, Russell, and Hilbert showed how this could be achieved even with the most complicated mathematical theories. According to the formalist standards, no step of reasoning can be taken without a reference to an exactly formulated list of axioms and rules of inference. Even the most "self-evident" logical principles must be explicitly formulated in the list of axioms and rules. Thus, a precisely formalized mathematical derivation, making a mathematical proposition true, is like a "machinery of cogwheels", rather than the discovery of the "rational order in the world" by an "uncomputable consciousness" in its "clear and distinct intuitions".

7 . Thus the central question I am going to deal with is 'What is mathematical truth, that is, what makes a mathematical proposition true?'. I shall investigate the epistemological and ontological aspects of the problem. According to these two aspects I shall consistently ask the following two test questions:

Q1: What leads us to the knowledge of the truth of a mathematical proposition?

Q2: In what respects is the world different if a given mathematical proposition is true or false?

Investigation of the first question will lead us to a radical formalist position. Answering the second question, we will arrive at the physicalist account of formal systems.

2

Mathematical objects have no meanings

8. When I claim that mathematical objects and mathematical propositions have no meanings, I must make few things clear:

- Of course, the mathematician may have some intuition about the various mathematical objects. What I claim is that these associated ideas are actually *irrelevant* from the point of view of the truth of mathematical propositions.

- The more complex mathematical concepts have mathematical definitions, in the sense that they are defined within the corresponding formal theory. When I say that they have no meanings, I mean that they *do not refer to the world* outside of mathematics, more exactly to the world outside of the formal system in which they are defined.

- According to physical realism, intuitionism, and platonism, this world, outside of the formal system, is

> the physical world, the realm of mental/psychological phenomena, or some platonic conceptual realm, respectively. So, in this sense, the formalist's thesis that 'mathematical objects have no meanings' is a simultaneous rejection of physical realism, intuitionism, and platonism.

9 . Consider the case of the physical realist view of mathematics. My argument will be based on the epistemological test question Q1. What I am actually doing here is nothing but an application of the verifiability theory of meaning, in the following very weak sense: in order to determine the meaning of a scientific (mathematical) statement we follow up how the statement in question is confirmed or discomfirmed. So the question is, how do we, finally, know that a mathematical proposition is true? If a mathematical proposition is an assertion about the physical world, then its truth-condition would be the correspondence with the physical facts. That is to say, to decide whether a mathematical proposition is true or not, we would have to investigate the state of affairs in the physical world. In this case, at a certain point, just like the physicist, the mathematician would have to throw down his/her pen and go to the laboratory. In Gödel's words—if it is not an impudence to quote him in this context:

> If mathematics describes an objective world just like physics, there is no reason why inductive methods should not be applied in mathematics just the same as in physics.[1]

But, have you ever seen a mathematician in the laboratory? Isn't it true that Gauss' proposal to *measure* (by optical instruments) the sum of the angles in a triangle is regarded as illegal from the point of view of contemporary mathematics?

[1] Gödel 1951, p. 313.

It is illegal because the result of the measurement is irrelevant from the point of view of mathematics. Whether or not the distances in physical space, as physical quantities, can be described in terms of Euclidean geometry is indeed an empirical question. But this question has nothing to do with the truth of a mathematical statement like $a^2 + b^2 = c^2$. The truth of such a statement means only that $a^2 + b^2 = c^2$ follows from the axioms, according to the derivation rules of that very formal system called Euclidean geometry. Moreover, it would be difficult to imagine what kind of experiment should be performed in the laboratory in order to decide whether the group-theoretical statement $e(ee) = e$ (e is the identity element) is true or not. We do not refer to any experiment to decide this question. It is a true mathematical proposition in the sense that it is derived from the axioms of group theory.

Thus, even if someone associates "meaning" to a mathematical proposition, this meaning is outside of the scope of the decisive mathematical considerations and is irrelevant from the point of view of the truth of the mathematical proposition in question.

10 . According to the weaker version of physical realism, not all mathematical propositions have (empirical) meanings, only the most elementary ones which express certain elementary facts about the physical world that are evident to everyone without laboratory experiments.[2] Usually, these elementary propositions constitute the system of axioms for the mathematical theory in question. The meanings and the semantical truths of the rest of mathematical propositions are derived from the meanings and semantical truths of the axioms. In this way, the axioms' reference to the real world is transmitted to the more complex mathematical propositions. However, as the following reflections show, this weaker understanding

[2]Maddy presents a similar approach in her naturalised/physicalistic platonism (Maddy 1990).

of physical realism suffers from the same difficulties as the stronger version.

11 . It is commonly accepted that practically all mathematics can be "reconstructed" on the basis of first order set theory (and the first order predicate calculus with identity). Thus, the advocate of weak physical realism could claim that the axioms of set theory are good examples of those basic mathematical propositions which express evident truths about the physical world; the axioms of set theory express some elementary features of the "real sets" consisting of real objects of the world. But, what about the axiom of choice? It seems to express an elementary feature of the "real sets", we are yet baffled about whether it should be added to the list of axioms or not, depending on some delicate mathematical considerations—not to mention its counterintuitive consequences like the Banach–Tarski theorem. On the other hand, it would be difficult to tell what feature of "real sets" is reflected in the continuum hypothesis. Moreover, such an unquestioned axiom as the axiom of infinity does not reflect any feature of "real sets" in the physical world. It rather reflects the wish of the set-theorist to have infinite sets (without which set theory would be a boring subject). So, it seems that the axioms of the most fundamental mathematical structures are chosen on the basis of inherent mathematical reasonings, rather than on the basis of physical facts.

12 . Even if we assumed, for the sake of the argument, that a kind of semantical truth can be assigned to the axioms, it would not follow that this truth can automatically be transmitted to the more complex mathematical propositions derived from the axioms. For in this case, just like in a physical theory (see point **16**), there would be two different kinds of truth in mathematics: Truth_1, that is, that something is a theorem, and a Truth_2, which means that the proposition in question reflects an empirical fact of the world. The fact that the axioms are true$_2$ sentences plays, however, a marginal role in mathematics, except if Truth$_1$ and Truth$_2$ coincided, in the sense that if a proposition A is derived from a set of true$_2$ sentences (from some axioms), then A is true$_2$. But, whether or not such an A is true$_2$ is again an empirical question. It is because of the following two reasons: 1) It is, finally, an empirical question whether the logical axioms and the derivation rules we applied in the derivation of A preserve Truth$_2$ or not.[3] 2) In accordance with the general epistemic status of physical (scientific) theories (and now, from the point of view of Truth$_2$, mathematics would be a physical theory), it is entirely possible that a logical consequence of some true$_2$ propositions will not correspond to the observed empirical facts. In a physical theory, this is an indication that we need to change the axioms. This is in keeping with the wildly accepted hypothetico-deductive methodology of science.

Thus, even if we admitted that the axioms are true$_2$ sentences about the physical world, the Truth$_2$ of the mathematical propositions derived from the axioms would be an empirical question in all the rest of the mathematical theory in question. So we encounter the same difficulties as in the case of the strong version of physical realism.

13 . Finally, it is worth while mentioning that the nonchalant allusion of the physical realist to the "evident" truth of the axioms is, of course, completely untenable from philosophical point of view, if this truth is regarded as an *a priori* truth. The term "evident" should be understood as what is "known from our everyday experiences". But, as we can see in the history of modern sciences, sometimes the most "evident" concepts and laws of nature have to be revised under

[3]The explanation of the double-slit experiment via the violation of classical logic could be a good example for such an empirical disconfirmation of the laws of logic. See Reichenbach 1944 and Putnam 1979. (Although, the double-slit experiment itself does not lead us to such a radical conclusion. See Szabó 2001.)

the pressure of new empirical findings. (See point **41**.) So, if a mathematical theory is constructed—as the physical realists claim—to say something about the real world with the same rigor as sciences, then the mathematician should, at least, start by mentioning these "evident" empirical facts confirming the axioms. How are they precisely formulated? What kind of *objective* observations, experiments, or measurements lead us to these initial propositions about the physical world, etc.? However, no book, for example, on set theory ever starts with such an experimental introduction. The reason is very simple: because such empirical/experimental introductions are not necessary. This is because any claim about the "meaning" of a mathematical proposition, any reference to the physical world is irrelevant. This is merely "verbal decoration" in mathematics, which can be completely ignored.

14 . According to our stipulated empiricist/physicalist philosophical framework, the mental/psychological phenomena constitute a particular part of physical reality. In contrast, a platonic realm cannot be accommodated in this empiricist/physicalist ontology. Nevertheless, I want to note that everything I told about physical realism, *mutatis mutandis,* can be applied to intuitionism and platonic realism. This entails replacing "physical world" with the "world of mental/psychological phenomena" or with the "platonic world", depending on the respective realm. Likewise, the "laboratory experiment" of physical realism is replaced by some other means by which knowledge can be obtained about these other realms. In the case of intuitionism, knowledge could be obtained by psychological experiments and observations, sociological surveys, or anything else by which we can discover the universal laws of human thinking, beliefs, and intuitions. But we never see any reference in mathematics to the things like these. Similarly, no matter how platonists account for the epistemic access to the platonic realm, through *anamne-*

sis, apprehension, meditation, rational analysis of concepts, etc., we never see any reference to these epistemic faculties in mathematics.[4] Again, it is because they are irrelevant from the point of view of the truths with which mathematics is concerned. In Dummett's words:

> Like the empiricist view, the platonist one fails to do justice to the role of proof in mathematics. For, presumably, the supra-sensible realm is as much God's creature as is the sensible one; if so, conditions in it must be as contingent as in the latter. [...] There are indeed hypotheses and conjectures in mathematics, as there are in astronomy; but, while both kinds may be refuted by deducing consequences and proving them to be false, the mathematical ones cannot be established simply by showing their consequences to be true. In particular, we cannot argue that the truth of a hypothesis is the only thing that would explain that of one of its verified consequences; there is nothing in mathematics that could be described as inference to the best explanation. Above all, we do not seek, in order to refute or confirm a hypothesis, a means of refining our intuitive faculties, as astronomers seek to improve their instruments. Rather, if we suppose the hypothesis true, we seek for a *proof* of it, and it remains a mere hypothesis, whose assertion would therefore be unwarranted, until we find one.[5]

[4]Except if rational insight in mathematics means formal derivation. But in this case the platonist Truth_2 is nothing more but Truth_1, the formalist's only truth in mathematics, and the whole platonistic semantics becomes negligible in mathematics.

[5]Dummett 1994, p. 13.

No doubt, Gödel is right in arguing that "we do have something like a perception also of the objects of set theory" and that there is no reason "why we should have less confidence in this kind of perception, i. e., in mathematical intuition, than in sense perception".[6] But, he seems to overlook the fact that both the sense perceptions of the external world and the internal perceptions of our intuitive ideas[7] are irrelevant from the point of view of the truths which mathematics is concerned with, because they both are perceptions of that part of reality which is external to mathematics.[8]

15 . The only "truth" mathematics is concerned with is Truth_1, the concept of that something is being proved. If we feel obliged to express the statements of mathematics in our everyday, no doubt referential, language, we should say that mathematics is a discipline which does not have such assertions as '$a^2+b^2=c^2$', or '$e(ee)=e$', or '$3+2=5$', but it has assertions like 'formula $a^2+b^2=c^2$ derives from the formulas called Euclidean axioms', 'formula $e(ee)=e$ derives from the formulas called the axioms of group', and 'formula $3+2=5$ derives from the formulas called the axioms of arithmetic', etc.— according to a given set of logical axioms and rule(s) of inference. That is what mathematics tells us about the world. (In point **23** we will see in what sense this statement is *about the world.*)

Many are unsatisfied with this simple "if-thenism". David Papineau writes:

> If if-thenism were true, then of course there would not be any gap between mathematical practice

[6]Gödel 1964, p. 484.

[7]Even if they are "naturalized" as in Maddy's approach (Maddy 1990, pp. 266-268).

[8]As we will see in points **19**–**24**, there are particular perceptions—internal to mathematics—which are relevant to mathematical truth, namely the perceptions of the formal system itself where the mathematical statement in question is formulated.

> and mathematical truth, for mathematical truth would not answer to anything more than logical facts about which theorems followed from which axioms. And so, if if-thenism were true, there would not be any difficulties about mathematical knowledge (or at least none which did not reduce to the more general topic of logical knowledge). However, despite these attractions, it is generally agreed that if-thenism is is false. The reason is that, although there are some branches of mathematics in which mathematicians are not committed to anything except exploring the consequences of postulates, there are other branches of mathematics where they are unquestionably committed to something more: for example, ... the number theorist who says there are infinitely many prime numbers is not just saying that this follows from Peano's postulates, but that there *are* all those numbers.[9]

Of course, very much depends on how we understand the practice of the mathematician. I think, that in the same way like ordinary people who dream about movie stars but live with their partner, mathematicians rave about various platonic objects, but if they are seriously asked what they are confident about, they reduce their claims to mere if-thenisms. All the rest is just "folklore". And this holds not only for the more complex branches of mathematics but also for arithmetic and set theory. When the number theorist says "there are infinitely many prime numbers", or simply '7 is a prime number lager than 5' then (s)he means that all these concepts as 'larger' and 'prime', etc., are defined with formal rigor, and that the statement in question is a *theorem* within the corresponding

[9]Papineau 1990, p. 167.

formal framework. This is true even if (s)he proves it by simple calculations, since it was previously *proved* that the employed algorithm is correct. (I will come back to this problem after some ontological reflections in point **25**.) Thus, concerning the rigorous, scientifically justified, non-folkloristic part of the claims of mathematicians, it is far from "unquestionable" that they are committed to something more than if-thenism.

3

Physical theories

16 . Objecting to the formalist approach, many ask "How is it possible then that mathematics is applicable to the real world?" As I tried to illustrate in the previous section, mathematics is not "applicable" to the real world. We have, however, *physical* theories that do refer to the elements of reality. I have no scruples in designating all theories describing the real world as "physical", but the reader may regard this as merely a terminological simplification. It has no significance from the point of view of the main objectives of this section as to illustrate the essential difference between mathematical truth and a semantical truth in a scientific theory describing something in the world.[1]

A physical theory P is a formal system L + a semantics S pointing to the empirical world. It is an interesting philosophical question, of course, how semantics S works, and I shall hark back to this problem in point **??**. In the construction of the formal system L one can *employ* previously prepared formal systems which come from mathematics and/or logic. That is, in general, L is a (first-order) language with some

[1]From this point of view, philosophy, being intent on telling something about the world, is much closer to the sciences than mathematics.

logical axioms and the derivation rules (usually the first-order predicate calculus with identity), the axioms of certain mathematical theories, and some physical axioms. A sentence A in physical theory P can be true in two different senses:

Truth$_1$: A is a theorem of L, that is, $L \vdash A$ (which is a mathematical truth within the formal system L, a fact[2] of the formal system L).

Truth$_2$: According to the semantics S, A refers to an empirical fact (about the physical system described by P).

For example, 'The electric field strength of a point charge is $\frac{kQ}{r^2}$' is a theorem of Maxwell's electrodynamics—one can derive the corresponding formal expression from the Maxwell equations. (This is a fact of the formal system L.) On the other hand, according to the semantics relating the symbols of the Maxwell theory to the empirical terms, this sentence corresponds to an empirical fact (about the point charges).

17 . In a physical theory, Truth$_1$ and Truth$_2$ are independent concepts, in the sense that one does not automatically imply the other. Of course, one of the aims of a physical theory is to keep Truth$_1$ and Truth$_2$ in synchrony throughout the region of validity of the theory in question. However, assume that Γ is a set of true$_2$ sentences in L, i.e., each sentence in Γ refers to an empirical fact, and also assume that $\Gamma \vdash A$ in L. As I already mentioned in point **12**, it does not automatically follow that A is true$_2$. Whether A is true$_2$ is again an empirical question. If so, then it is new empirically obtained information about the world, confirming the validity of the whole physical theory $P = L + S$. In Kevin J. Davey words:

[2] I use the word "fact" here, since I actually mean "empirical fact" about the formal system—as will be seen.

> The world is built in such a way that from certain bodies of knowledge, certain types of valid mathematical deductions take us from true claims about reality to other true claims about reality. But not all such otherwise valid mathematical deductions do this. This is a fact about the world that physicists must struggle to get their hands around – to learn which deductions are good in which contexts, and which are not.[3]

But if it turns out that A is not true$_2$, then this information disconfirms the physical theory, as a whole. That is to say, one has to think about revising one of the constituents of P, the physical axioms, the semantics S, the mathematical axioms, or the axioms of logic or the derivation rules we applied in the derivation of A—probably in this order. (Here we can see how Quine's semantic holism works. The unit of meaning is not the single sentence, but systems of sentences or even the whole of language). Usually, changing mathematical/logical axioms means that we change one entire mathematical/logical theory for some other. For example, when we learned new empirical facts about physical space(-time), we replaced the whole Euclidean geometry with another one. This is, however, an unimportant sociological fact about how the task is shared between the physicists and mathematicians. What is important is that the empirical disconfirmation of a physical theory, in which the Euclidean geometry is applied, can disconfirm the *applicability* of Euclidean geometry in the physical theory in question, but it leaves Euclidean geometry itself intact. In this way, one cannot disconfirm mathematical truths like 'formula $a^2 + b^2 = c^2$ derives from the formulas called Euclidean axioms'. (Here we can observe how Quine's confirmational holism fails. It is not the case that if an empirical finding disconfirms a physical theory which employs a

[3]Davey 2003, p. 100.

given mathematical theory then it also disconfirms the mathematical theory. It merely can disconfirm the applicability of the mathematical theory in question in the physical theory in question.[4])

18 . The fact that the truths of mathematics and the truths of physics are independent raises the question of how it is, then, possible that mathematical structures prove themselves to be so expressive in the physical applications. As Feynman put it: "I find it quite amazing that it is possible to predict what will happen by mathematics, which is simply following rules which really have nothing to do with the original thing."[5] This "miracle" is the main motivation of the indispensability arguments for realism:[6] if the statements of mathematics are indispensable for physics, they must be true in the sense that they correspond with the facts of the world. To me, however, there is nothing miraculous here.

- First of all, we give too much importance to the correspondence between mathematics and structures observed in the empirical world. One has to recognize that the "storehouse" of mathematics has a much larger stock of mathematical structures than we have ever applied in describing the real world.

- It is not mathematics alone by which the physicist can predict what will happen, but physical axioms and mathematics together. The physical axioms are determined by empirical facts. More exactly, the physicist, keeping, as long as possible, the logical and mathematical axioms fixed, *tunes* the physical axioms such that the theorems *derivable* from the unified system of logical, mathematical, and physical axioms be compatible

[4] Cf. Quine 1980, p. 41.
[5] Feynman 1967, p. 171.
[6] See Puttnam 1975, p. 73. See also point **26**.

with the empirical facts. Consequently, the employed logical and mathematical structures in themselves need not reflect any Truth_2 about the real world in order to be useful.[7]

- Due to the empirical underdetermination of scientific theories, it is often the case that more than one possible mathematical structure is applicable to the same empirical reality.

- Finally, there is no miraculous "preadaption" involved just because certain aspects of empirical reality "fit themselves into the forms provided by mathematics". This is simply a result of selections made by the physicist. Just as there is no preadaption at work when you successfully can install kitchen units obtained from a department store in your kitchen. The rules according to which the shelves, cupboards and doors can be combined show nothing about the actual geometry of your kitchen. But the final result is that the kitchen "fits itself" to the form of the whole set, as if through a kind of preadaption. Similarly, as G. Y. Nieuwland points out:

> From time immemorial, mankind has learnt to deploy mathematics in order to cope with the empirical world, finding helpful notions such as quantity, measure, pattern and functional dependence. For many and obvious reasons

[7] You can expirience a similar situation when you change the mouse driver on your computer (or just change the mouse settings): first you feel that the pointer movements ("derived theorems") generated by the new driver ("mathematics") according to your previously habituated hand movements ("physical axioms") do not faithfully reflect the geometry of your screen. Then, keeping the driver (and driver settings) fixed, you *tune* your hand movements – through typical "trial and error" leaning – such that the generated pointer movements fit to the arragment of your screen content.

> it proves expedient to order this experience into a system of knowledge, invoking notions of coherence, analogy, completion and logic. Doing so, it appears that the margins of interpretation, of whatever it is of our experience that can be organized into mathematical structure, are flexible. Such is the underdetermination of theory by the data.[8]

Of course, I do not want to understate the real epistemological problem here. Namely, to what extent a statement of a physical theory applying certain mathematical structures expresses an objective feature of the real object. This longstanding epistemological problem is, however, related to the Truth_2 of the statements of physical theories, which has nothing to do with what makes a mathematical proposition true, that is, with Truth_1.

[8] Nieuwland 2001

4

The physicalist ontology of formal systems

19 . It is a common belief that philosophy of mathematics must take account of our impression that mathematical truth is a reflection of fact. As Hardy expresses this constraint,

> [N]o philosophy can possibly be sympathetic to a mathematician which does not admit, in one manner or the other, the immutable and unconditional validity of mathematical truth. Mathematical theorem are true or false; their truth or falsity is absolute and independent of our knowledge of them. In *some* sense, mathematical truth is a part of objective reality.[1]

In this section, my aim is to determine what this objective fact actually is.

As we have seen in the previous sections, mathematical propositions, contrary to the propositions of physical theories, are not about anything outside of mathematics, neither in the

[1] Hardy 1929.

physical world nor a platonic world—even if the latter were not disqualified for various philosophical reasons. Therefore, this fact can only be a fact of inside the realm of mathematics. More exactly, taking into account that the only means of obtaining reliable knowledge about this fact is mathematical proof, it must be a fact of the realm inside of the scope of formal derivations. I shall argue that this fact is a fact of the formal system itself, that is, a fact about the physical signs and the mechanical rules according to which the signs can be combined.

In my—perhaps prejudiced[2]—reading, Hilbert's position was very close to this view:

> Kant already taught—and indeed it is part and parcel of his doctrine—that mathematics has at its disposal a content secured independently of logic and hence can never be provided with a foundation by means of logic alone; that is why the effortes of Frege and Dedekind were bound to fail. Rather, as a condition for the use of logical inferences and performance of logical operations, something must already be given to our faculty of representation [*in der Vorstellung*], certain extralogical concrete objects that are intuitively [*anschaulich*] present as immediate experience prior to all thought. If logical inference is to be reliable, it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ

[2]It is an arguable historical question how Hilbert actually considered the "concrete signs themselves" as intuitively present as immediate experience prior to all "logical inferences". In some readings, Hilbert's views are compatible with a kind of structural/conceptual realism/platonism. (Cf. Isaacson 1994.) Anyhow, in point **27** I shall formulate an argument against the view that mathematics has anything to do with some "abstract structure" over and above the physical signs and physical mechanisms constituting the formal system in question.

> from one another, and that they follow each other, or are concatenated, is immediately given intuitively, together with the objects, as something that neither can be reduced to anything else nor requires reduction. This is the basic philosophical position that I consider requisite for mathematics and, in general, for all scientific thinking, understanding, and communication. And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable.[3]

20 . Sometimes it is argued that symbolism is merely a "convenient shorthand writing" to register the results obtained by *thinking*. To be sure, according to the physicalist account of the mental, it is not important how much "thinking" is involved into the formal manipulations. It is worth while to mentioning, however, that thinking actually plays a marginal role in formal derivations—from the point of view of the truth conditions of a mathematical proposition. I am not concerned about the context of discovery but about the context of justification of a mathematical truth. The discovering of a new conjecture and the discovering of a proof of a conjecture or the definition of a new concept no doubt require the faculty of creative thinking. But, the justification of a mathematical truth, that is, to test that a given sequence of formulas constitutes a proof of a proposition, does not. The more strictly formalized the proof is, the less creative thinking involved. Consequently, the test of the truth conditions of a mathematical proposition is indifferent to complex creative thinking. For instance, when we perform a formal derivation on paper, since each step of manipulation is governed by strict rules, human beings could

[3]Hilbert 1967, p. 376.

be replaced by trained animals, robots, etc. Actually, the test whether a given sequence of formulas constitutes a proof of a proposition can be performed by a Turing machine.

Hilbert still emphasizes that the complete process of symbolic operations must be surveyable to us. This was a very common idea at his time. However, in the contemporary mathematics there are derivations which are not surveyable by the human mind—we cannot observe the whole derivation process, only the outcome of the process, the proved theorem. This is the case, for example, in the proof of the four-color theorem,[4] where certain steps of the proof are performed through very complex computer manipulations. Sometimes even the theorem obtained through the derivation process is not surveyable. It often happens, for example, that the result of a symbolic computer language manipulation is a formula printed on dozens of pages which is completely incomprehensible to the human mind.

21 . So far we have focused on the epistemological aspects of the problem of mathematical truth. The results of our investigation can be summarized in the following observations.

- Regarding the truth conditions of a mathematical proposition, contrary to the propositions of physical theories, we do not need to refer to the world outside of the formal system in which they are formulated.

- Testing whether a given string of signs is proof of a mathematical statement is completely determined by the formal system itself, no matter whether and to what extent human mental faculties are involved in the mechanical procedure of derivation.

- The process of derivation that leads us to the *knowledge* of the truth of a mathematical proposition is nothing

[4] See Tymoczko 1979.

> but the *truth-condition* of the mathematical proposition in question.

22 . The *ontology of formal systems* is crystal-clear: marks, say ink molecules diffused among paper molecules, more exactly, their interaction with the electromagnetic field illuminating the paper, or something like that. The rules according by which the marks are written on the paper form a strict mechanism which is, or easily can be, encoded in the regularities of real physical processes. From the point of view of the truth conditions of a mathematical proposition, human activity in this process plays a marginal role. Moreover, the marks and rules can be of an entirely different nature, like, for instance, the cybernetic states of a computer, supervening on the underlying physical processes.

Sometimes one executes simple formal derivations also in the head.[5] However, *from the point of view of the physicalist interpretation of mind* this case of formal manipulation does not principally differ from any other cases of derivation processes. If the signs and the rules of a formal system can be embodied in various physical states/processes, why not let them be embodied in the neuro-physiological, biochemical, biophysical brain configurations—more exactly, in the physical processes of the human brain? If this is the case, that one of the paths—as some rationalists believe, the only path—to trustworthy knowledge, the deductive/logical thinking, can be construed as a mere complex of physical (brain) phenomena, without any reference to the notions of "meaning" and "intentionality", or the concept of the acausal *global* supervenience on the physical,[6] then this is, actually, a very strong argument for physicalism.

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[5] Much more rarely than one would think. Even in the simplest cases, a proper formal derivation is much too complex to be executable in one's head.

[6] Cf. Chalmers 1996, pp. 33–34

That is to say, mathematical truths are revealed to us only through real physical processes. From this point of view we must agree with the quantum computer theorists David Deutsch, Artur Ekert, and Rossella Lupacchini:

> Numbers, sets, groups and algebras have an autonomous reality quite independent of what the laws of physics decree, and the properties of these mathematical structures can be just as objective as Plato believed they were (and as Roger Penrose now advocates). But they are revealed to us only through the physical world. It is only physical objects, such as computers or human brains, that ever give us glimpses of the abstract world of mathematics.[7]
>
> ⋮
>
> It seems that we have no choice but to recognize the dependence of our mathematical *knowledge* (though not, we stress, of mathematical truth itself) on physics, and that being so, it is time to abandon the classical view of computation as a purely logical notion independent of that of computation as a physical process.[8]

(Note that they still maintain a platonic concept of truth in logic and pure mathematics as independent of any contingent facts. The reason is the distinction they draw between knowledge and truth. They do not recognize what I emphasized above that the existence of a physical process of derivation that leads us to the *knowledge* of the truth of a mathematical proposition is nothing but the *truth-condition* of the mathematical proposition in question.)

[7] Deutsch *et al.* 2000, p. 265.
[8] Deutsch *et al.* 2000, p. 268.

23 . In order to determine what kind of objective fact is reflected in a mathematical truth, we apply our ontological test question Q2: In what respects is the world different if a given mathematical proposition is true or false? In other words, what kind of state of affairs in the world determine whether a mathematical proposition is true or false? Now we can answer this question. As the above epistemological analysis shows, the truth of a mathematical proposition is determined by a fact of the formal system in which the proposition is formulated. That is to say, it reflects a fact of the physical system consisting of the signs and the mechanism of derivation in question. In other words, the mathematical fact whether a formula of a formal system is a theorem or not locally supervens on the physical facts of the very piece of the (physical) universe occupied by the formal system as physical system. For example, imagine that the formal system in question is embodied in a notebook with a CD which contains the corresponding program causing the computer to list the theorems of the formal system in some order (Fig. 4.1). Whether or not a given formula is displayed by the notebook is entirely determined by the *physical* process going on within the region symbolized by the dotted line. In other word, assuming that the whole process is deterministic, it is predetermined by the physical laws and the initial state of the computer and the CD. (The "program" is nothing but a certain physical state of the surface of the CD.)

Consequently, *a mathematical proposition is nothing but an ordinary scientific statement of the existence of a proof, that is, the existence of a particular physical process in the formal system in question.* If so, contrary to Gödel's claim[9] that "*some* implications of the form:

> If such and such axioms are assumed, then such and such a theorem holds,

[9] Gödel 1951, p. 305.

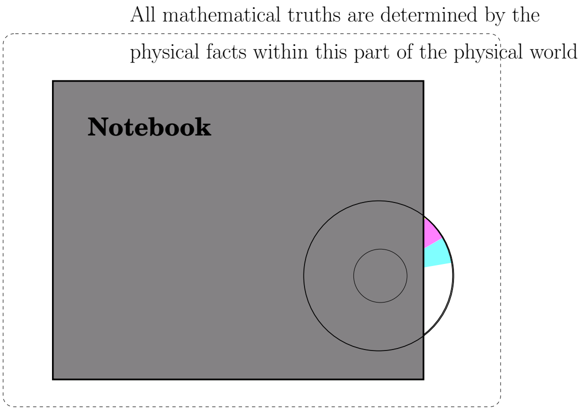

Figure 4.1: *The formal system is embodied in a notebook with a CD which contains the corresponding program making the computer to list the theorems of the formal system in some order*

must necessarily be true in an absolute sense", mathematics does not at all deliver to us absolute necessity. The mathematical proposition '$3+2=5$', which actually means—this is the usual formalist step, see point **15**—'formula $3+2=5$ derives from the formulas called the axioms of arithmetic', is nothing but—this is the physicalist step—the scientific assertion that there exists a proof-process in the formal system called arithmetic, the result of which is the formula $3+2=5$. This is an ordinary scientific assertion, just as the assertion of the chemist about the existence of the process $2H_2+O_2 \rightarrow 2H_2O$. In this way, mathematical truths express objective facts (of the physical world inside of the formal system as physical system). They are synthetic, *a posteriori*, not necessary, and not certain, they are fallible, but have contingent factual content, as any similar scientific assertion.

24 . According to this view, a mathematical proposition can be true before anybody can prove it. This simply refers to the normal situation in sciences, that things exist in the world that have not been discovered yet. It is true that process $2H_2 + O_2 \rightarrow 2H_2O$ exists in the world even if the chemist has not discovered the existence of this process yet. Actually, it would be better to give an example of a reaction in the chemistry of man-made materials such as plastic. The laws of nature predetermine whether a certain chemical process is possible or not, even if nobody has initiated such a process yet. But this kind of independence of the concept of objective truth of mathematical statement from the concept of 'having been proved' does not entitle us to claim—as the platonists do—that there is a "truth" in mathematics which is different from the concept of 'being a theorem in a formal system'. In this sense, the statement of Goldbach's conjecture is objectively true or false; it is an objective fact of the formal system, as a physical system, that there exists proof for such a statement or not. But this objective truth or falsity has nothing to do with such a platonic concept of truth and falsity as "either it is the case for all even numbers that it is the sum of two primes, or there exists an even number which is not". The reason is that the phrase that something "is the case for all natural numbers" is meaningless. Not because of the infinity involved in this phrase, but because there is no such a realm of "natural numbers" where the state of affairs may or may not correspond with such a statement.

25 . Not only is the ontological picture so obtained uniform but we have a uniform semantics for all discourse: mathematical and non-mathematical—just as the different sorsts of mathematical realism aim for. In order to achieve this semantical uniformity under the umbrella of the Tarskian theory, we only had to take a small step. We had to recognize that mathematical statements, if expressed in everyday language, have the form of 'Σ implies X' instead of 'X'. Our epistemological/methodological investigations concluded (point **15**) that only the 'Σ implies X'-sentences are scientifically justified *statements* in mathematics. X is not a statement. It is merely a formula of a formal system and it shouldn't be regarded as a linguistic object at all—in the ordinary sense of language. Consequently it has no meaning, it does not refer to anything, it is not a carrier of Tarskian truth. Just like we do not assign meaning and truth or falsity to a dishwasher or a brick, since they are not linguistic objects. One has to recognize that 'X'-sentences merely belong to the sloppy jargon of the mathematician and they are actually used as abbreviations for the corresponding 'Σ implies X'-sentences. If not, then they are negligible verbal decorations. 'Σ implies X'-sentences do have meanings and carry Tarskian truths about real physical entities of clear ontology.

Let me make this still clearer. Assume that Σ is a collection of formulas $Y_1, Y_2, \ldots Y_m$. Not only is X merely a formula of the formal system without meaning and Tarskian truth but the same holds for the formulas $Y_1, Y_2, \ldots Y_m$—even if they are axioms. Since $Y_1, Y_2, \ldots Y_m$ are not linguistic objects either.

26 . The 'Σ implies X'-sentences are the statements of mathematics that can be indispensable to physical theories—since they are no other scientific statements in mathematics. Let us accept, for the sake of the argument,[10] the Quine-Putnam indispensability argument:

[10] There have been many objections to both premises of the argument. See Field 1980, Maddy 1992; 1997, and Sober 1993. See also my objections in points **17**–**18**.

(1) We ought to have ontological commitment to all and only the entities that are indispensable to our best scientific theories.

(2) Mathematical entities are indispensable to our best scientific theories.

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(3) We ought to have ontological commitment to mathematical entities.

This means, we ought to have ontological commitment to the entities that mathematical *statements* are talking about. These statements are talking about formal systems, strings of signs, and about derivation processes relating certain strings (Σ) with some other string of signs (X), and so on. All these entities do exist, and to top it all they exist in the physical world.

This picture is entirely compatible with the other fact that (an originally non-linguistic) objects like X can also be indispensable for a physical theory (L, S). They can be indispensable as *formulas* in the formal system L, and they may carry physical Truths$_2$ according to the semantics S. For example, if $X \equiv \forall q \exists z\, q = z \cdot e$ is a formula expressing the (empirically confirmed) physical statement (according to S) that all electric charge is a multiple of the elementary charge, then—by virtue of the indispensability argument—we ought to have ontological commitment to the strings like X and the derivation processes in L, the electric charge, the elementary charge, and we may also have commitment to the reality of the property of electric charges expressed by the formula in question. Again, all these entities are accommodated in the physical world.

Since X is not a linguistic object in the ordinary sense of language, the quantifiers eventually used in X shouldn't be "interpreted" in the ususal way. For example, the arithmetical formula $\exists n\, n > 17$ should not interpreted as an ontological statement of the existence an entity which is natural number and larger than 17. Actually it shouldn't be interpreted at all, beacuse it has no meaning, since it is just a formula, a string of physical singns in a formal system.

5

Abstraction is a move from the concrete to the concrete

27. Many philosophers of mathematics, while admitting that formal systems are "represented" in the form of physical signs and mechanical rules, still assume that there is something behind this physical representation, an "abstract structure" that is "represented". Sometimes we find the same ambivalent views in the formalist school. Curry writes:

> ... although a formal system may be represented in various ways, yet the theorems derived according to the specifications of the primitive frame remain true without regard to changes in representation. There is, therefore, a sense in which the primitive frame defines a formal system as a unique object of thought. This does not mean that there is a hypostatized entity called a formal system which exists independently of any representation. On the contrary, in order to think of a formal system at all

> we must think of it as represented somehow. But when we think of it *as* formal system we abstract from all properties peculiar to the representation.[1]

What does such an "abstraction" actually mean? Let us first consider what we obtain if we abstract from some unimportant, peculiar properties of a physical system Z. In accordance with what we said about the physical theories, we obtain a theory $P = L + S$ about Z, that is, a formal system L with a semantics S relating the marks of the formal system to the (important) empirical facts of the physical system Z—where L is a formal system in the mind, or on paper, etc. Now, the same holds if the physical system is a formal system (a "representation of a formal system", in Curry's terminology) $Z = L_1$. Through the abstraction we obtain a theory $L_2 + S$ describing some important properties of the system L_1. That is, instead of an "abstract structure" we obtain another formal system L_2 "represented somehow"—in Curry's expression.

Thus, formal systems are always "flesh and blood" physical systems. These concrete physical systems should not be regarded as physical representations of some abstract formal systems. There are no such abstract things over and above the physically existing formal systems.

28 . By the same token, one cannot obtain an "abstract structure" as an "equivalence class of isomorphic formal systems" or something like that, since in order to think of such things as "isomorphism", "equivalence", "equivalence class" at all we must think of them as living in a formal system "represented somehow". For it is a categorical mistake to talk about "isomorphism" between two physical objects. To compare two formal systems L_1 and L_2 we have to use a meta-mathematical theory capable of describing both L_1 and L_2. That is to say we have to have a physical theory (M, S) where M is a third

[1] Curry 1951, p. 30.

formal system and the semantics S points partly to L_1, which is a part of the physical world, and partly to L_2, which is another part of the physical world (Fig. 5.1). Since "isomor-

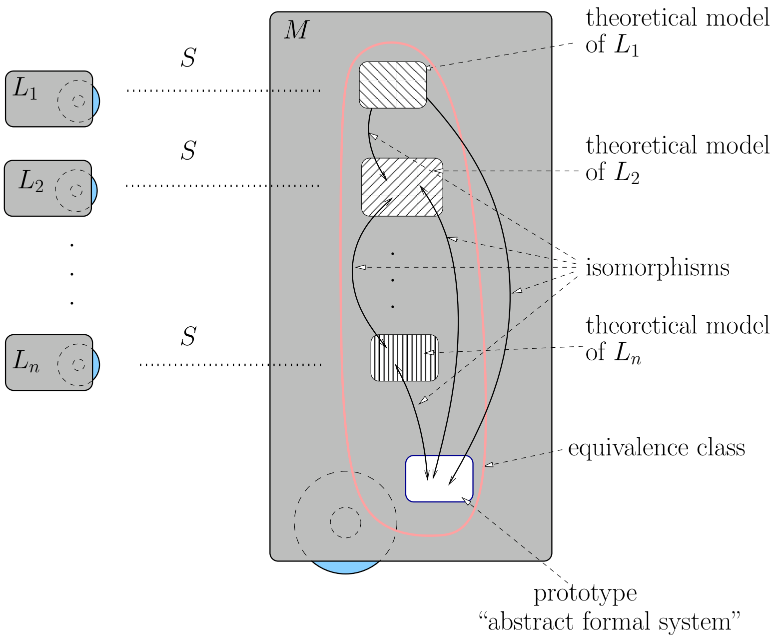

Figure 5.1: *It is a categorical mistake to talk about "isomorphism" between two formal systems. "Isomorphism" is a concept which is meaningful only in a formal system containing set theory. In order to say that formal systems are "isomorphic" we need a meta-mathematical theory (a third "flesh and blood" formal system together with a semantics) in which the object systems are simultaneously represented*

phism", "equivalence class", etc. are set-theoretic concepts, M must be a formal system containing set theory. Formal systems $L_1, L_2, \ldots L_n$ are simultaneously represented in the physical theory (M, S). Only in M it is meaningful to say that the theoretical models of $L_1, L_2, \ldots L_n$ are isomorphic and constitute an equivalence class. Only in M we can define the prototype of these structures, which can be regarded as an "abstract mathematical structure". And, more importantly, all these mathematical objects live in the formal system M, in a "flesh and blood" formal system existing in the physical world.

29 . This is neither a nominalistic view nor an attack on *scientific* realism. When a satisfactorily confirmed physical theory claims that a physical object has a certain property adequately described by means of a formal system, then this reflects—with or without the "foot-stamp" of the true realist—a real feature of physical reality. When many different physical objects display a similar property that is describable by means of the same formal system, then we may generalize and claim that these physical objects all possess the feature in question. This will be a true general feature of the group of objects in question, described by means of a formal system as a real physical system.

But, this realist commitment does not entitle us to claim that "abstract structures" exist over and above the real formal systems of physical existence. Again, according to the arguments in points **27**–**28**, the reason is that if we tried to consider such an "abstract structure" as a feature of the formal system itself, or as a general feature of many similar formal systems, then we would obtain another, "flesh and blood", formal system of physical existence.

30 . This observation, in conjunction with our previous observation (point **20**) that the role of human faculties in the formal machinery establishing the truth of a mathematical sentence is limited and unessential, is not a surprise from a physicalist point of view. For even if mathematical objects are "all in someone's mind", they would be nothing but physical processes going on in our brain and body. However, independently of the general physicalist account of the mental, the simple fact that abstraction is unable to go beyond the "physically represented" formal systems is a very strong argument against structuralism and concept platonism.

In brief: if, according to Frege's account,[2] the abstract things are items in the "third realm", which are non-mental and non-sensible, then mathematics has nothing to do with "abstract" things. This conclusion is in contradiction with the generally accepted view according to which mathematical objects constitute paradigm cases of abstract entities. The confusion is caused by the misunderstanding of the following facts:

(a) Mathematical truths are independent from the state of affairs in that part of the mathematician's external world, which is also external to the formal system in question. (Let us call this part of the world realm A.) That is, mathematical truths are independent of the realm traditionally described by physical theories. Therefore mathematical truths seem to be spaceless and timeless.

(b) Mathematical truths are independent of that part of the mathematician's internal world, which is external to the formal system in question. (Let us call this part of the world realm B.) Mathematical truths are intersubjective.

(c) Within the framework of a physical theory, a mathematical truth may correspond to a fact of realm A. This correspondence is depending on the concrete physical theory and its faithfulness is an empirical question. Similarly, a mathematical truth can correspond to an idea in realm B. This idea, however, is

[2] Frege 1968.

> subjective and may vary from person to person. As it follows from (a) and (b), mathematical truths are over and above the realms A and B.

Now, observation (c) is widely misinterpreted as saying that mathematical truths are truths about "abstract entities" existing over and above the "concrete representations" in both the internal and the external worlds. However, this claim obviously overlooks the fact that realm A is not identical with the external world and realm B is not identical with the internal world. What is missing from A and B is just what mathematical truths refer to, the formal systems themselves constituting a particular part of the real world, either in its external or in its internal parts—it does not make essential difference in our physicalist framework. And, as we have seen, abstraction does not lead outside this realm of concrete physical entities.

31 . To sum up, a formal system is a physical system, the marks of the formal system are embodied in different phenomena related to the system and the derivation rules are embodied in those regularities that govern the system's behavior. A mathematical derivation, making a mathematical proposition true, is nothing but a physical process going on in the formal system, and a theorem is the output of the process. To prove a theorem is nothing but to *observe* a derivation process in a formal system—that is, to observe a physical process in a physical system. That is all! In this physicalist ontological picture there are no "mathematical structures", as abstract thoughts, which are "represented" in the various formal systems.

Thus, physicalism—including the physicalist account of the mental—completes the formalist foundation of mathematics and removes the last residues of platonism. The physicalist ontology of mathematical truth makes it completely pointless in mathematics to introduce a concept of truth different from that of being proved. Mathematical proposition, as a formula in a formal system, does not carry meaning and semantic truth. At the same time, however, it corresponds to a physical fact. By this correspondence, a true_1 mathematical proposition reflects a truth in the usual sense of Tarski's semantical theory of truth, just like a true_2 sentence in a physical theory. Namely, it reflects a fact, a physical fact of the formal system itself. In this way, indeed, "mathematical truth is a part of objective reality".

This is the way I propose to "naturalize mathematics". In this way, mathematical knowledge is *not conventional*—except the choice of the topic itself, there is nothing conventional in the statement 'Σ implies X'. It is *not trivial*—sometimes it is highly non-trivial whether Σ implies X. It is *not perfect*, *not a priori*, and *not certain*. Just like non-mathematical sciences, mathematics delivers to us *knowledge of contingent facts* about a particular part of the physical world. Formal systems constitute this particular part of the physical world. This is what we can call "mathematical reality", and mathematicians rightly think themselves as scientists, exploring the intricacies of mathematical reality

32 . Since there are no "abstract formal systems" over and above the physically existing systems of signs and derivation processes, in order to simplify our further considerations, we make the following stipulation without the loss of generality:

Stipulation 1 *A formal system is a machine (like a computer) which has the following behavior: when it is started it prints out the list of axioms and derivation rules of the system in question and then it prints out a sequence of formulas constituting a proof of a theorem and stops.*

In order to remind the reader of this stipulation I shall sometimes call a formal system a formal machine. I do not want to specify what will happen if we start the machine again. In general it produces another sequence of theorems and stops.

An important consequence of this stipulation is that a statement is a “mathematical statement” only if it is a string printed out by the correponding formal machine. In other words, it is not a mathematical statement if its proof involves some faculties beyond the formal machine itself.

6

Physicalist account of semantics

7

Induction versus deduction

33 . It is a long tradition in the history of philosophy that—in Leibniz's words:

> There are ... two kinds of truths: those of reasoning and those of fact. The truths of reasoning are necessary and their opposite is impossible; the truths of fact are contingent and their opposites are possible.[1]

According to this tradition, one cannot justify a general statement about the world by *induction*. *Deduction*, contrary to induction, provides secure confidence because it is based on pure reasoning, without referring to empirical facts. According to the key idea of rationalism, cognition is an independent source of trustworthy knowledge. Moreover, it is the only secure source of knowledge, the rationalists say, because cognition is the only source of *necessary* truth, while experience cannot deliver to us necessary truths, i. e., truths completely demonstrated by reason.

[1] Rescher 1991, p. 21.

Let us leave aside the epistemological valuation of knowledge we obtain through inductive inference and consider in more detail the problem of deduction. As we pointed out in point **1**, the empiricist encounters the following difficulties: it must be either assumed that the truths of formal logic and mathematics are not necessary truths, in which case one must account for the universal conviction that they are; or one must say that they have no factual content, and then it must be explained how a proposition which is empty of all factual content can be true and useful and surprising.

34 . According to the *physical realist*, mathematical and logical truths are not certain and not necessary, since they are nothing but generalizations of our fundamental experiences about the physical world, and, as such, they are, admittedly, fallible.

Logical empiricists, on the contrary, did not reject the necessity and certainty of mathematical and logical truths. According to their solution, analytical truths do not refer to the facts of reality. For we cannot obtain more information through deductive inference than that already contained in the premises. In other words, according to the logical empiricism, there are no synthetic *a priori* statements.

Popper's falsification principle also accepts the necessity and certainty of mathematical and logical truths. This is the basis of the principal distinction between induction and deduction. Similarly, this principal distinction between the "trustworthy deductive inference" and the "always uncertain inductive generalization" is the fundamental tenet upon which the widely accepted hypothetico-deductive and Bayesian theories of science are built up, seemingly eliminating the problem of induction.

35 . Now, from the standpoint of our *physicalist ontology of formal systems*, we have arrived at the conclusion that mathematical and logical truths are *not necessary and not certain,* *but they do have factual content referring to the real world.*

For "deduction" is a concept which is meaningful only in a given formal system. On the other hand, as we have seen, a formal system is nothing but a physical system, and derivation is a physical process. The knowledge of a mathematical truth is the knowledge of a property of the formal system in question—the knowledge of a fact about the physical world. *The formal system is that part of physical reality to which mathematical and logical truths refer.*

It must be emphasized that this reference to the physical world is of a nature completely different from that assumed, for example, by Mill in his physical realist philosophy of mathematics. In the terminology we introduced with respect to physical theories, the formal statements still do not have any reference to the real world in the sense of the truth-conditions of Truth$_2$, since mathematics does not provide us with a semantics directed from the formal system to the outside world. When we are talking about the empirical character of mathematical truths, we are still talking about Truth$_1$, namely we assert that even Truth$_1$ is of empirical nature, the factual content of which is rooted in our experiences with respect to the formal system itself. Mathematics is, in this sense, an empirical science.

The knowledge we obtain through a deductive inference is nothing but an empirical knowledge we obtain through the observation of the derivation process within the formal system in question. In other words, *deduction is a particular case of induction.* Consequently, the *certainty of mathematics, that is the degree of certainty with which one can know the result of a deductive inference, is the same as the degree of certainty of our knowledge about the outcomes of any other physical processes.*

For example, the reason why the truth of the height theorem is uncertain is not that our knowledge about the properties of "real triangles" is uncertain, as Mill takes it,[2] but rather that our knowledge about the deductive (physical) process, the outcome of which is the height theorem, is uncertain, no matter how many times we repeat the observation of this process.

36 . In order to explain the universal conviction that mathematical truths are necessary and certain, notice that there are many elements of our knowledge about the world which *seem* to be necessary and certain, albeit they have been obtained from inductive generalization. If we need a shorter stick, we break a long one. We are "sure" about the outcome of such an operation: the result is a shorter stick. This regularity of the physical world is known to us from experiences. The certainty of this knowledge is, however, no less than the certainty of the inference from the Euclidean axioms to the height theorem. Mathematical and logical truths are considered necessary and certain for the following two reasons: 1) Usually formal systems are simple and stable physical systems. 2) The knowledge of mathematical truths does not require observations of the world external to mathematics.

37 . Our physicalist approach resolves Ayer's problem raised in point **1** in the following way:

- Mathematical and logical truths express objective facts of a particular part of the physical world, namely, the facts of the formal systems themselves. They are synthetic, *a posteriori*, not necessary, and not certain, they are fallible, but have contingent factual content, as any similar scientific assertion.

- The fact that the formal systems usually are simple physical systems of stable behaviour and that the knowledge of mathematical and logical truths does not require

[2]The same kind of fallibility appears in Maddy's theory of naturarised mathematical intuition (Maddy 1990, p. 268).

> observations of the world external to the formal systems explains why mathematical and logical truths appear to everyone to be necessary, certain and *a priori*.

Empiricism is not challenged by the alleged necessary truths delivered by mathematical and logical reasoning. On the contrary, consequent physicalism can resolve the long-standing debate surrounding the truth-of-reasoning versus truth-of-facts dichotomy. Mathematical and logical truths are nothing but knowledge obtained through inductive generalization from experiences with respect to a particular physical system, the formal system itself. Since mathematical and logical derivations are reasonings *par excellence*, one must conclude that reasoning does not deliver to us necessary truths. Reasoning is, if you like, a physical experiment. Therefore, contrary to Leibniz's position, we must draw the following epistemological conclusion: *The certainty available in inductive generalization is the best of all possible certainties!*

8

Philosophically non-invariant parts of mathematics

38 . In my “philosophy first” strategy, it is necessary to pass all mathematics through the physicalist filter, and remove or reconsider everything that proves untenable from an empiricist/physicalist point of view. This would be, of course, a huge project, far beyond the scope of this work. Nevertheless I will try to use a few important examples for such a revision to illustrate my views.

Should the reader not be sympathetic to the physicalist philosophical framework, the examples given may be regarded as illustrations of philosophically non-invariant concepts in mathematics. I believe, that the critical approach that is applied to all scientific thinking should at least reveal that these concepts are metaphysically laden: that they are born of certain metaphysical claims, which are not necessarily acceptable or sometimes even meaningless from the point of view of other metaphysical positions. We have to point out the philosophical schizophrenia here: Why should the physicist who is per-

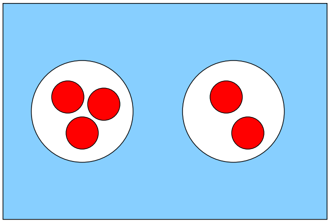

Figure 8.1: "The number of apples on plate I + the number of apples on plate II = the number of apples on the table" is a contingent physical fact

haps naturally committed to an empiricist/physicalist metaphysical position take at face value those claims of the mathematician that are simply meaningless outside of the scope of a bold platonism?

Nevertheless, I would like to emphasize that the physicalist approach is a minimalist one, in the sense that it removes many philosophically problematic concepts from the rigorous kernel of mathematics but without adding anything new that is not philosophically neutral.

39 . In the first place, one has to free mathematics from all kinds of "truth" that differ from Truth$_1$. It is completely meaningless to talk about "intuitive arithmetic", "naive set theory", "intended interpretation", and the like, or to differentiate "numbers" from "numerals" or to use the phrase "axiomatization of ...", etc. For instance, from the point of view of physicalism, there can be no arithmetic in mathematics before axiomatic arithmetic has been established. We do not "know" in advance that "$3+2=5$" until the corresponding axiomatic theory is constructed in which this formula exists, until this formula is derived from the arithmetical axioms. Counting using your fingers, or observing (Fig. 8.1) that

(A) When the apple-counter equipment interacting with

> the apples on plate I shows figure $\boxed{3}$ and the apple-counter interacting with the content of plate II shows figure $\boxed{2}$ then the apple-counter interacting with apples on the table shows figure $\boxed{5}$.

are nothing but observations about the physical world outside of the realm of mathematics. These physical observations have nothing to do with the mathematical truth of formula $3 + 2 = 5$. It does not follow from these observations that $3 + 2 = 5$ because this formula is a formula of arithmetic. And the only truth mathematics (arithmetic) is concerned with is Truth_1. We cannot infer the truth of formula $3 + 2 = 5$ from the physical observations of apples, plates and tables. This is not because "the general arithmetical truth $3+2=5$ is obtained by abstraction from many other similar observations". It is because we can infer its truth from a *completely different physical* observation: the observation of a sequence of formulas constituting a proof of formula $3+2=5$ within arithmetic. (Demonstration 1).

Demonstration 1

Sketch of the proof of "$3 + 2 = 5$" in arithmetic

`(PC1)` $(\phi \rightarrow (\psi \rightarrow \phi))$

`(PC2)` $((\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \chi))$

`(PC3)` $((\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi))$

`(PC4)` $(\forall x\,(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \forall x\psi))$ `given that` x `is not free in` ϕ.

`(PC5)` $(\forall x\phi \rightarrow \phi)$ `given that` x `is not free in` ϕ.

`(PC6)` $(\forall x\phi(x) \rightarrow \phi(y))$ `given that whenever a free occurance of` x `is replaced by` y, y `is free in` $\phi(y)$.

(MP) $\phi, (\phi \rightarrow \psi)$ implies ψ

(G) ϕ implies $\forall x \phi$

(E1) $E(x, x)$

(E2) $E(t, s) \rightarrow E\left(f^n\left(u_1, u_2, \ldots, t, \ldots u_n\right), f^n\left(u_1, u_2, \ldots, s, \ldots u_n\right)\right)$

(E3) $E(t, s) \rightarrow \left(\phi\left(u_1, u_2, \ldots, t, \ldots u_n\right) \rightarrow \phi\left(u_1, u_2, \ldots, s, \ldots u_n\right)\right)$

(A1) $\neg(0 = sx)$

(A2) $(sx = sy) \rightarrow (x = y)$

(A3) $x + 0 = x$

(A4) $x + sy = s(x + y)$

(A5) $x \cdot 0 = 0$

(A6) $x \cdot sy = (x \cdot y) + x$

(A7) $(P(0) \wedge \forall x\, (P(x) \rightarrow P(sx))) \rightarrow \forall x P(x)$

1 $sss0 + ss0 = s(sss0 + s0)$

2 $s(sss0 + s0) = ss(sss0 + 0)$

3 $sss0 + ss0 = ss(sss0 + 0)$

4 $sss0 + 0 = sss0$

5 $sss0 + ss0 = sssss0$

Of course, one can construct a *physical theory* describing a physical system consisting of apples, plates and a table. It would consist of a formal system L and a semantics S which relates some elements of the formal system L to observed phenomena. Of course, we observe the situation described in (A) many times and regard it as a law-like regularity:

(B) Whenever the apple-counter interacting with the apples on plate I shows figure $\boxed{3}$ and the apple-counter interacting with the apples on plate II shows figure $\boxed{2}$ then the apple-counter interacting with the apples on the table shows figure $\boxed{5}$.

This regularity can be expressed in the physical theory (L, S) in the following way: The formal system L contains the axioms of the first-order predicate calculus with identity, $PC(=)$, and the axioms of arithmetic. Semantics S relates the figures $\boxed{3}$, $\boxed{2}$ and $\boxed{5}$ shown by the counters to the symbols $3, 2$ and 5 (where 3 is an abbreviation for $sss0$, etc.) and the regularity (B) will be related with formula $3 + 2 = 5$.

When I say that the semantics "relates" some elements of the observed phenomena with some symbols in L, I do not mean a map, or function or relation of mathematical character. That would be a categorical mistake! Maps, functions and relations are mathematical objects, existing only in a formal system (incorporating set theory). Such a semantical relationship is always embodied in causal concatenations, relating the formal system with other physical systems. You see the figure $\boxed{3}$ on the instrument and put your finger on the symbol 3 on the paper. Simultaneously the neural configuration of your brain or—let me put it in a more Searlean way—the whole physical state of your brain and body is such that you have the impression in your mind that the two things are germane to each other.

From physicalist point of view, the involvement of human activity does not make an essential difference. In order to avoid any confusion about the role of human mind in this procedure, one can always imagine a robot doing the job. Thus, imagine a robot (a computer with all the required peripherals) designed (programmed) in the following way: he observes the figure shown by the apple-counter interacting with the apples on plate I. He identifies the observed figure with a definite symbol in L, namely, with one of the numbers ("numerals", if you like, I do not make a distinction), say 3. He observes the figure on the other counter and puts his finger on number 2. He also observes the figure shown by the apple-counter interacting with the apples on the whole table, and identifies this figure with, say, 5. Then the robot checks on whether $3 + 2 = 5$ is a theorem in L, that is, in arithmetic, or not. Since $\vdash_L 3 + 2 = 5$, he reports that the phenomenon he observed is compatible with the physical theory (L, S). And so on.

This example, I believe, gives an intuition about how arithmetic works within a physical theory that describes real phenomena. But, the success of such a physical theory—within its range of applicability, of course—does not entitle us to claim that "there are numbers" as entities—different from "numerals". Nor can we say that "$3 + 2 = 5$" is a truth about them, that we can know before proving the corresponding formula in ("axiomatic") arithmetic. $3 + 2 = 5$ is only a formula in arithmetic, and the only truth it holds is its Truth_1: that is it is a theorem of arithmetic.

Finally, it is worth while emphasizing that not only does the empirical fact (B) not imply that $3+2 = 5$ is a theorem of arithmetic, but, vice versa, the arithmetical theorem $3+2 = 5$ in itself does not imply the physical fact (B). Moreover, the arithmetical theorem $3 + 2 = 5$ in itself does not even imply the physical hypothesis, that (B) is true. Such a physical prediction only follows from the corresponding physical theory (L, S) incorporating arithmetic.

40 . *Mutatis mutandis*, the same holds for set theory. It is often said that we need "axiomatic set theory" to resolve the paradoxes produced by "naive set theory" and that naive set theory would be fine if it did not have such paradoxes. Although this explanation correctly reflects the history of set theory, it is misleading. We need "axiomatic" set theory in order to have set theory *at all*. For, according to the contemporary standards, there is not a branch of mathematics called "naive set theory". In fact, there is a lot of confusion about what "naive set theory" really is. Standard text-books,[1] tend to describe "naive set theory" as a kind of axiomatic theory presented with (limited) formal rigor: These books present basic assumptions (axioms) of "naive set theory" and the theorems derived from these basic assumptions. The presentations are often fairly informal, for didactical purposes. However, there are no descriptions of "naive set theory" in its pure form. Not as a science of truths about real sets nor as a collection of "pre-existing truths" which would be known independently of mathematics, supposedly "axiomatized" in set theory.

In her "naturalized platonism" Maddy is probably correct in describing how we gain intuitive beliefs which lead to the obviousness of the axioms of set theory (at least some of them). These intuitions are based on our experience of classes of discrete medium-size physical objects, or continuous phenomena, etc., in early life.[2] However, the fact that the set-theoretical axioms reflect our intuitive beliefs, does not imply neither that these beleifs correctly reflect the basic features of the physical world, nor that the set-theoretical statements are true or false independently of whether they are theorems of the axiomatic theory or not. We are not entitled to say that

$$\forall A \forall B \forall C \, [A \cap (B \cup C) = (A \cap B) \cup (A \cap C)] \tag{8.1}$$

"is true" and to ask "whether it is *also* provable in set theory". As was pointed out earlier, this view is contrary to the practice of contemporary mathematics. When the mathematician asserts (8.1), (s)he means that this formula is derivable from

[1] Let me refer to one of the best ones: Halmos 1960.

[2] Maddy 1990a Chapter 2; 1990b, pp. 266–268.

the axioms of set theory. (Note that it also is a *theorem* in "naive set theory", since it is derived from some more elementary assumptions.) For, if the mathematician is asked to confirm this assertion, (s)he presents a formal proof, but does not refer to the state of affairs neither in a *platonic* platonic world, nor in a *naturalized* platonic world, which is, in my opinion, nothing but the physical world. So, even if someone associates semantical truth to a mathematical proposition like (8.1), this is outside of the scope of the decisive mathematical considerations, because this truth is of type Truth_2, while the mathematical practice only is concerned with Truth_1.

41 . On the other hand, as I have already mentioned, the claim that sets and some elementary statements (including the axioms) of set theory reflect physical facts is not simply a harmless explanation of the intuitive origin of set-theoretical rules. If we take it seriously, it is a description of fundamental features of the physical world. That is to say, it is a physical theory. As such it must be clearly and objectively formulated in empirical terms, and it requires empirical confirmation. For example, whether the common view that the properties of physical objects can be identified with sets is correct or not, is an empirical question. This seems to be the case in classical physics, but it is far from obvious whether the same holds for quantum physics. If Jauch and Piron's description of the properties of physical objects[3] is correct, then the property lattice of a quantum system does not satisfy the distributive law (8.1). This means that the properties of a quantum mechanical object cannot be identified with sets. Thus, contrary to our intuitions about "properties", "classes", "having a property", "belonging to a class", etc., such a simple true_1 set-theoretic formula as (8.1) fails to be satisfied by the properties of a quantum mechanical object, that is to say, in this case it is not true_2.

[3] Jauch and Piron 1963.

Chapter 9

Epistemological status of meta-mathematical theories

42 . A meta-mathematical theory is a theory *describing* a formal system. This fact in itself provides compelling reason to follow Hilbert's careful distinction between mathematics (i.e., a system of meaningless signs) and meta-mathematics (meaningful statements *about* mathematics).[1] That is to say, a meta-mathematical theory is a theory describing a physical system. Consequently, meta-mathematical theory is *a physical theory.* All the truths that a meta-mathematical theory can tell us about its object are of the type Truth_2. And, since there is no *a priori* physical truth, there is no *a priori* meta-mathematical truth. This means that no feature of a formal system can be "proved" mathematically. (Not to mention that, according to the physicalist account of mathematical truth, there is no *a priori* mathematical truth either.) Genuine information about a formal system must be acquired

[1] See Nagel and Newman 1958, p. 28.

by *a posteriori* means, that is, by observation of the formal system and, as in physics in general, by inductive generalization. So, a properly construed meta-mathematical theory has the same structure as physical theories in general. Let L denote the object formal system described by the meta-theory in question. The meta-mathematical theory consists of two components, a formal system M and a semantics S. Now, in order to make any *prediction* using the meta-mathematical theory (M, S) about the properties of the formal system L, first one has to confirm that (M, S) is a faithful theory of L. And, as in the case of any other physical theory, there is no way to confirm this faithfulness other than to confirm it *empirically.*

Demonstration 2

Sentence calculus L

`Alphabet of symbols:`

$\sim, \supset, (,), p, q, r,$ `etc.`

`Well-formed formulas:`

1. $p, q, r,$ `etc. are wfs.`
2. `If` A, B `are wfs. then` $(\sim A)$, $(A \supset B)$, `are wfs.`
3. `All wfs. are generated by 1. and 2.`

`Axiom schemes:`

`(L1)` $\quad A \supset (B \supset A)$

`(L2)` $\quad (A \supset (B \supset C) \supset ((A \supset B) \supset (A \supset C)))$

`(L3)` $\quad (((\sim A) \supset (\sim B) \supset (B \supset A)))$

`Modus Ponens:`

`(MP)` A `and` $(A \supset B)$ `implies` B

43 . An important consequence of this fact is that one cannot "mathematically prove" a property of a formal system such as, for example, its consistency. Just as one cannot "mathematically prove" the conservation of energy in a certain physical processes. Let me illustrate this with a well known and simple example of the so called "absolute proof of the consistency of sentence calculus".[2]

The formal system in question is shown in Demonstration 1. The system L is called consistent if there is no formula X such that $\vdash_L X$ and $\vdash_L \sim X$. The standard "absolute proof" of the consistency of L is shown in Demonstration 2.

Demostration 3

Sketch of the standard "absolute proof" of consistency of L

`Definition:` `A` ***coloring*** `of` L `is a function` v `whose domain is the set of wfs. of` L `and whose range is the set` $\{red, blue\}$ `such that, for any wfs.` A, B `of` L`,`

`(i)` $v(A) \neq v(\sim A)$

`(ii)` $v(A \supset B) = blue$ `if and only if` $v(A) = red$ `and` $v(B) = blue$

`Definition:` `A wfs.` A `is` ***stably red*** `if for every coloring` v`,` $v(A) = red$.

`Proposition 1:` `For every formula` A`, if` A `is a theorem of` L `then` A `is stably red.`

[2] See Nagel and Newman 1958, pp. 45–56, or any mathematical logic text-book, for example, Hamilton 1988, Chapter 2.

Proof: Let A be a theorem. The proof is by induction on the number n of wfs. of L in a sequence of wfs. which constitutes a proof of A in L.

$n = 1$ A is an axiom. One can easily verify that every axiom of L is stably red.

$n > 1$ Induction hypothesis: all theorems of L which have proofs in fewer than n steps are stably red.
Assume that the proof of A contains n wfs. Now, either A is an axiom, in which case it is stably red, or A follows by (MP) from previous wfs. in the proof. These two wfs. must have the form B and $(B \supset A)$. But, since B and $(B \supset A)$ are stably red, it follows from (ii) that A is stably red.

Proposition 2: L is consistent.

Proof: As is known, one can easily proof that if both X and $\sim X$ are theorems in L then arbitrary formula is a theorem. Consequently, if there exists at least one formula in L which is not a theorem, then L is consistent. By virtue of Proposition 1 one has to show that there is a formula Y in L which is not stably red, that is, there is a coloring v such that $v(Y) = \mathit{blue}$. Let Y be $\sim p \supset q$. Taking into account (i) and (ii), $v(Y)$ is determined by $v(p)$ and $v(q)$. Since $v(Y) = \mathit{blue}$ whenever $v(p) = \mathit{blue}$ and $v(q) = \mathit{blue}$, Y cannot be a theorem of L.

Now, a properly formulated meta-mathematical theory of L, will be a formal system with a semantics, (M, S), where semantics S points to the empirical facts of the object formal system L. Like in other physical theories, the formal system M is generated from logical, mathematical and physical axioms. Assume that the meta-theory (M, S) is strong enough to accommodate something like the "absolute proof of consistency" of L. In the proof we talk about "*functions* over the formulas of L". 'Function' is a mathematical concept which is meaningful only in set theory. Also, we "prove *by induction*", which requires either set theory or arithmetic. Therefore, we assume that the formal system M contains set theory (say ZF), and, consequently, M also contains the first order predicate calculus with equality. One may think that there are some vicious circularities here, but this is not so. The object system L must be regarded as an entirely autonomous formal system, not as a particular part of the predicate calculus contained in M[3]. What concerns me is an entirely different problem.

The object formal system L stands as a physical system which has to be described by a physical theory (M, S). The elements of the alphabet, the complex strings, the derivation processes, etc., must be somehow represented in M. Then we introduce theoretical concepts like "stably red", which expresses a structural property of formulas of L. "To be a theorem of L" is another concept we define in M expressing another possible property of a formula of L. Then, we prove a theorem in M saying that if a formula of L is a theorem of L then it is stably red. What is important is that this claim is just a theoretical description of the system L, which *may be correct or incorrect.* There is no way to decide whether it is correct or not other than testing it empirically. Whether a formula is stably red or not is an empirical question. We

[3]I postpone the question whether we need to be sure about the consistency of the formal system M in order to use it in a physical (meta-mathematical) theory.

must observe the formula in question and analyze its structure. Whether a formula is a theorem or not is another empirical question. Consequently, whether such a statement of the theory (M, S) as 'Every theorem of L is a stably red formula.' is correct or not is an empirical question—in spite of the fact that a corresponding statement can be derived in M. (Truth$_2$ does not follow from Truth$_1$!) We *observe* that whenever a formula of L is a theorem of L, it is stably red. From these observations, through *inductive generalization*, we arrive at the conclusion that this law is empirically confirmed. And similar observations confirm the whole theory (M, S) describing L.

For example, let us consider such an ostensibly simple statement that a given formula of L, say X, is stably red. This is a meta-mathematical statement about the *structure* of X. In order to say anything about its structure, you have to observe and identify the different constituents of X. You have to ascertain the sequential order of its constituents. So you will count them (say, using your fingers): this is the first one, this is the second one, ... Then you will make a semantical, that is physical/causal connection between these constituents and some natural numbers (in set theoretic sense) in M, and also will connect them to different set-theoretic symbols in M. As a next step, you will classify the different constituents of X on the basis of some physical properties like their shape, etc. and you will define equivalence classes in M. Then you can introduce — in M — coloring functions defined over these equivalence classes. And so on and so forth.

To avoid any misunderstanding, I do not question the statement that sentence calculus is consistent. I believe in it as I believe in the conservation of energy. That is to say, what I claim is that the epistemological status of this statement is the epistemological status of a physical law, which is not the same as the epistemological status of a mathematical theorem. To be sure, both a mathematical theorem and a physical law are confirmed by physical observations. But, the mathematical theorem we derive in M is confirmed by the physical observation of a sequence of formulas in the formal system M, constituting a proof of the theorem of M, while the physical law in question, that is, the meta-mathematical statement about L is confirmed by observations of physical facts of the object formal system L. One ought to believe in the consistency of sentence calculus not because it is "proved" just like "$a^2 + b^2 = c^2$" is proved in Euclidean geometry, but because there have been no formula X observed, such that $\vdash_L X$ and $\vdash_L \sim X$.

44 . It is this physical-law-like nature of meta-mathematical laws what is misunderstood when mathematicians claim about unavoidable circularities in the formally rigorous foundations of mathematics or about infinite regress of meta-languages. Let me illustrate these problems with two quotations. Gödel writes:

> ...this theory, if it wants to prove the tautological character of the mathematical axioms, must first assume these axioms to be true. So while the original idea of this viewpoint was to make the truth of the mathematical axioms understandable by showing that they are tautologies, it ends up with just the opposite, that is, the truth of the axioms must *first* be assumed and *then* it can be shown that, in a suitably chosen language, they are tautologies. Moreover, a similar statement holds good for the mathematical concepts, that is, instead of being able to define their meaning by means of symbolic conventions, one must first know their meaning in order to understand the syntactical conventions in question or the proof that they imply the mathematical axioms but not

> their negations. Now, of course, it is clear that this elaboration of the nominalistic view does not satisfy the requirements set up [earlier], because not the syntactic rules alone, but all mathematics in addition is used in the derivations.[4]

The second quotation is from Elliott Mendelson's famous text book on mathematical logic:

> The word "proof" is used in two distinct senses. First, it has a precise meaning defined above as a certain kind of finite sequence of wfs of L. However, in another sense, it also designates certain sequences of sentences of the English language (supplemented by various technical terms) which are supposed to serve as an argument justifying some assertion about the language L (or other formal theories). In general, the language we are studying (in this case L) is called the *object language*, while the language in which we formulate and prove results about the object language is called the *metalanguage*. The metalanguage might also be formalized and made the subject of study, which we would carry out in a meta-metalanguage, etc. ... The distinction between "proof" and "metaproof" (i.e., a proof in the metalanguage) leads to a distinction between theorems of the object language and *metatheorems* of the metalanguage.[5]

As it follows from the physical-law-like nature of meta-mathematical statements, the important fact is that a "metaproof" in a meta-mathematical theory (M, S) does not *prove* a fact about the object formal system, even if—as it was

[4] Gödel 1951, p. 317.
[5] Mendelson 1964, p. 31–32.

assumed in point **42**—the formal system M is a properly formalized meta-language. Just as a formally proper derivation of the Coulomb law from Maxwell's equations only yields to a hypothesis, rather than it "proves" a contingent fact about point charges on the basis of other contingent facts. The formalization of the meta-language cannot change the original epistemological status of the whole meta-theory.

Chapter 10

The non-mathematical status of model theory

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