Codes: BMA-FILD-402.80, BMI-LOTD-351E.04, BBN-FIL-402.80, BMA-LOTD-351.04

Course: The logic of provability

Teacher: Attila Molnár

Location and time: I/-109, Thu, 14:15 – 15:45

Consultation: Thu, 16:00 – 17:00, molnar.h.attila@gmail.com, phil.elte.hu/attila

First occasion: 12/02/2015, 14:15 – 15:45

Courses required: Logic seminar and lecture

Requirements: Homeworks

Description: Mathematics has its own modality; the provability. According to the informal arithmetical provability interpretation, the modal sentence $\Box\varphi$ is true iff φ is provable in Peano arithmetic.

Formally, we say that a logic PRL is arithmetically sound and complete, iff

$$\mathbf{PRL} \vdash \varphi \iff \text{for any } *, \mathbf{PA} \vdash \mathrm{tr}_*(\varphi) \qquad (1)$$

where $\mathrm{tr}_*(\varphi)$, the *arithmetical interpretation* is

$$\begin{aligned} \mathrm{tr}_*(p) &\overset{\text{def}}{=} *(p), \text{ a sentence of } \mathbf{PA} \\ \mathrm{tr}_*(\neg\varphi) &\overset{\text{def}}{=} \mathrm{tr}_*(\varphi) \\ \mathrm{tr}_*(\varphi \wedge \psi) &\overset{\text{def}}{=} \mathrm{tr}_*(\varphi) \wedge \mathrm{tr}_*(\varphi) \\ \mathrm{tr}_*(\Box\varphi) &\overset{\text{def}}{=} \mathrm{Provable}(\ulcorner \mathrm{tr}_*(\varphi) \urcorner), \end{aligned}$$

where $\mathrm{Provable}$ is the provability predicate of PA and $\ulcorner - \urcorner$ stands for the Gödel-numbering.

Gödel showed in 1933 that no *normal alethic* logic, more precisely, no modal logic having the axiom and rule

$$\text{(T)} \quad \Box A \to A \qquad\qquad \text{(RN)} \quad \frac{A}{\Box A}$$

can be arithmetically sound. The obstacle here is *Gödel's second incompleteness theorem* itself:

$\vdash \Box(p \wedge \neg p) \to (p \wedge \neg p)$	instance of the axiom (T)
$\vdash \neg\Box(p \wedge \neg p)$	proof by contradiction
$\vdash \Box\neg\Box(p \wedge \neg p)$	(RN)

But the latter says that consistence is provable in the object language, contrary to Gödel's 2nd incompleteness theorem. Is there any modal logic $\mathbf{PRL}$ that can satisfy the equivalence (1)? *Solovay's first theorem* says that yes, there is, and it is the Gödel-Löb logic $\mathbf{GL}$:

$$\text{(K)} \quad \Box(A \to B) \to (\Box A \to \Box B),$$

$$\text{(RN)} \quad \frac{A}{\Box A},$$

$$\text{(GL)} \quad \Box(\Box A \to A) \to \Box A$$

Solovay's proof is as beautiful as Gödel's proof of his incompleteness theorems; it reveals the connection between possible world semantics and the provability predicate by embedding Kripke models into **PA**. If that would not be enough to convince the Reader to take that course, we show one last thing: the core idea of Solovay's proof is based on functions having the following ~~definition~~ property:

$$f(0) = 0$$

$$f(n+1) = \begin{cases} m, & \text{if } \ulcorner n \urcorner \text{ is a Gödel-number of a proof that shows that } m \text{ is not a limit of } f \text{ and } f(n) < m. \\ f(n) & \text{otherwise} \end{cases}$$

The course will contain

- a short but precise introduction to Peano arithmetics and modal logics,
- the construction of the provability predicate and its modal properties: the Hilbert-Bernays-Löb derivability conditions,
- the fixpoint theorems of PA *and* GL,
- the proof of Gödel's incompleteness theorems,
- the proof of Solovay's first theorem,
- investigations of some other provability logics of arithmetics.

Bibliography:

[1] George Boolos. *The Logic of Provability*. Cambridge: Cambridge University Press, 1993.