

# Foundations of Mathematics

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## ABSTRACT

In this course we will work in Zermelo Fraenkel Set Theory with Axiom of Choice (ZFC) and discuss the followings,

### 1. Sets are the foundation of all mathematical entities

- Motivation: Sets are the foundation of all abstract mathematical concepts (like functions, relations, relational structures) and all concrete mathematical objects (like 1, 2/3).

### 2. Construction of Number systems leading to different branches of mathematics

- Structures, Relational Structures, Algebraic Structures.
- Natural Numbers- Origin and structure of natural numbers, Countability.
- Rational Numbers- Construction of the structure of rationals from the structure of natural numbers and properties of rationals.
- Real Numbers- Completeness Axiom and Archimedean property, Construction of real numbers that leads to the study of real analysis. Continuum and properties of reals.
- Complex Numbers- Construction of complex numbers that leads to complex analysis.

### 3. Role of Axiom of Choice in the foundation of mathematics

- Axioms of ZF, Origin of Axiom of Choice(AC), Consistency and Independence of AC from other axioms, Equivalent versions of AC.
- Partially Ordered sets, Zorn's Lemma and their applications.
- Filters, Ultrafilters, Ideals and Prime Ideals, Ultrafilter theorem and its equivalents and applications.
- Dependent Choice and its equivalence and applications.

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4. Motivation to work in  $\mathbf{ZF+DC}$  or  $\mathbf{ZF+CC}$  or  $\mathbf{ZF+AD}$  where  $\mathbf{DC}$ ,  $\mathbf{CC}$ ,  $\mathbf{AD}$  represents **Dependent Choice**, **Countable Choice** and **Axiom of Determinacy** respectively