TOPOLOGY WITH AND WITHOUT THE AXIOM OF CHOICE (COURSE SYLLABUS)

AMITAYU BANERJEE

- We will follow Chapter 2 from [Mun] and an initial part of Chapter 4 from [Mun] to cover the following.
  (1) **Basic topological definitions.**
    - Closed sets, limit points and topological closure.
    - Open sets, interior points, exterior points and boundary points.
    - Dense sets, nowhere dense sets, meager sets, perfect sets.
  (2) **Topological spaces.** Topological spaces and basis for a topology, Subspace topology, Product and Box topology, Quotient topology, Metrizable topological spaces, Completely metrizable topological spaces.
  (3) **Kolmogorovian classification of topological spaces.** Specifically Hausdorff space.
  (4) **Countability axioms and Compact spaces**. 1^{st}countable and 2^{nd} countable spaces, Seperable and Lindeloff spaces, Compact spaces.
- By a choice principle we mean a sentence ϕ in the language of set theory which is provable from ZFC but not provable from ZF (Zermelo–Fraenkel set theory without the axiom of choice). By a weak choice principle we mean a choice principle which is strictly weaker than the axiom of choice (AC). We will recall the following weak choice principles from [Her06].
  - *Dependent choice* (DC),
  - *Countable choice* (AC_ω),
  - *Boolean prime ideal theorem* ($B.P.I.T$) and
  - *Axiom of choice on reals* $AC(\mathbb{R})$.
- **Stone's representation theorem and** $B.P.I.T$**.** A fundamental theorem in Boolean algebra is *Stone's representation theorem*, which states that that every Boolean algebra is isomorphic to some set algebra. We will define certain topological spaces called *Stone's space* and observe the equivalence between Stone's representation theorem and $B.P.I.T$.
- **Variants of Tychonoff's theorem and Baire Category theorem without** AC**.** *Tychonoff's theorem* and *Baire Category theorem* provides the foundation of several important theorems in topology, functional analysis and other related areas of analysis. Tychonoff's theorem provides the foundation of *Ascoli theorem* and *Cech-Stone theorem* where as the *Baire Category theorem* implies some fundamental theorems in functional analysis like *Open Mapping theorem*, *Banach Isomorphism theorem*, *Closed Graph theorem* and *Uniform boundedness theorem*. We will focus on variants of Tychonoff's theorem and Baire Category theorem without AC. Specifically, we plan to cover the following.
  (1) **Compact spaces without** AC**.**
  (2) **Tychonoff's theorem with and without** AC**.**
    - Equivalence of *Tychonoff's theorem* with AC.

Key words and phrases. Axiom of choice, Topology.

    - Equivalence of *Tychonoff's theorem* for *compact Hausdorff spaces* and *finite discrete spaces* with $B.P.I.T$.
    - Relation of *Countable product of different topological spaces* with DC and AC_ω.
  (3) **Baire Category theorem with and without DC.**
    - Equivalence of *Baire Category theorem* for complete metric space and DC.
    - Relation of *Baire Category theorem* for compact Hausdorff spaces and DC.
    - Variants of *Baire Category theorem* without any weaker versions of AC.
- **Constructing a Lebesgue non-measurable set of reals.** We will see the following popular ways of constructing a *non-Lebesgue measurable* set of reals from $AC(\mathbb{R})$.
  - *Sierpiński's Construction.*
  - *Bernstein's Construction.*
  - *Vitali's construction* (with some details of *Hamel basis*).

Grades. Based on problem solving sessions or ability to present one of the topics (in details) that we will cover.

References

[Mun] James Munkres. *Topology*. 2nd edition.
[Her06] Horst Herrlich (2006). *Axiom of Choice*.

Department of Logic, Institute of Philosophy, Eötvös Loránd University, Budapest, Hungary

E-mail address: banerjee.amitayu@gmail.com