The Modal logic of Forcing

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In this reading course, we will attempt to cover the following reading materials (and will see some open problems in this area).

1. Diploma Thesis on The Modal Logic of Forcing, by Colin Jakob Rittberg; `https://www.uni-muenster.de/imperia/md/content/logik/c.rittberg_the_modal_logic_of_forcing.pdf`.

2. The Modal Logic of Forcing, by Joel David Hamkins, Benedikt Löwe; `https://arxiv.org/pdf/math/0509616.pdf`.

3. Structural connections between a forcing class and its modal logic, by Joel David Hamkins, G. Leibman, Benedikt Löwe; `https://arxiv.org/pdf/1207.5841.pdf`.

1 Aim

A set theoretical assertion ψ is *forceable or possible*, written $\Diamond\psi$, if ψ holds in some forcing extension, and *necessary*, written $\Box\psi$, if ψ holds in all forcing extensions. For example, $\Diamond\psi$ means that there is some forcing notion $\mathbb{P}$ and condition $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}} \psi$, and $\Box\psi$ means that for all forcing notions $\mathbb{P}$ and $p \in \mathbb{P}$, $p \Vdash_{\mathbb{P}} \psi$.

Definition 1.1. A modal assertion $\phi(q_0, ..., q_n)$ is a *valid principle of forcing* if for all sentences ψ_i in the language of set theory, $\phi(\psi_0, ..., \psi_n)$ holds under the forcing interpretation of $\Diamond$ and $\Box$. We say $\phi(q_0, ..., q_n)$ is a ZFC-provable principle of forcing if ZFC proves all such substitution instances $\phi(\psi_0, ..., \psi_n)$.

Example. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, $\Box\phi \to \phi$, $\neg\Diamond\phi \leftrightarrow \Box\neg\phi$, $\Box\phi \to \Box\Box\phi$, $\Diamond\Box\phi \to \Box\Diamond\phi$ are valid principles of forcing.

Definition 1.2. A forcing notion $\mathbb{P}$ satisfies the *κ-chain condition (κ-c.c.)* if every antichain in $\mathbb{P}$ has cardinality less then κ. The $\aleph_1$-c.c. is the c.c.c. (countable chain condition).

Definition 1.3. A forcing notion $\mathbb{P}$ is *κ-closed* if whenever $\gamma < \kappa$ and $\{p_\epsilon : \epsilon < \gamma\}$ is a descending sequence of elements of $\mathbb{P}$, then $\exists q \in \mathbb{P}, \forall \epsilon < \gamma, (q \leq p_\epsilon)$.

We recall Cohen forcing, Lévy collapse forcing, proper forcing, CH (Continuum Hypothesis), GCH (General Continuum Hypothesis) from forcing lecture notes. Recall the modal theories $\mathsf{S4.2}, \mathsf{S4.3}, \mathsf{S4}$ (mainly) from Modal logic notes. We shall study the following relevent results due to Hamkins, Löwe, Leibman, and Rittberg.

1. None of the modal theories most commonly considered beyond S4.2 (ex; S5, S4W5, S4.3, S4.2.1, S4.1, Dm.2, Dm, K4H, GL or Grz) are ZFC-provable principles of forcing.

2. The ZFC-provable principles of forcing are exactly those in the modal theory S4.2.

3. The ZFC-provable principles of ω_1-closed forcing are exactly those in the modal theory S4.2.

4. The ZFC-provable principles of c.c.c.- forcing contains the modal theory S4, but doesn't contain the modal theory S4.2.

5. The ZFC-provable principles of collapse forcing, Cohen forcing are in each case exactly those in S4.3.

6. The ZFC-provable principles of c.c.c.-forcing, proper forcing (over L (the constructible universe)) are each contained within $\mathsf{S4.3}$ and do not contain $\mathsf{S4.2}$.

7. The ZFC-provable principles of CH-preserving forcing, of GCH-preserving forcing, and of $\neg\mathsf{CH}$-preserving forcing are each exactly $\mathsf{S4.2}$.