An Operational Perspective on Sheaves

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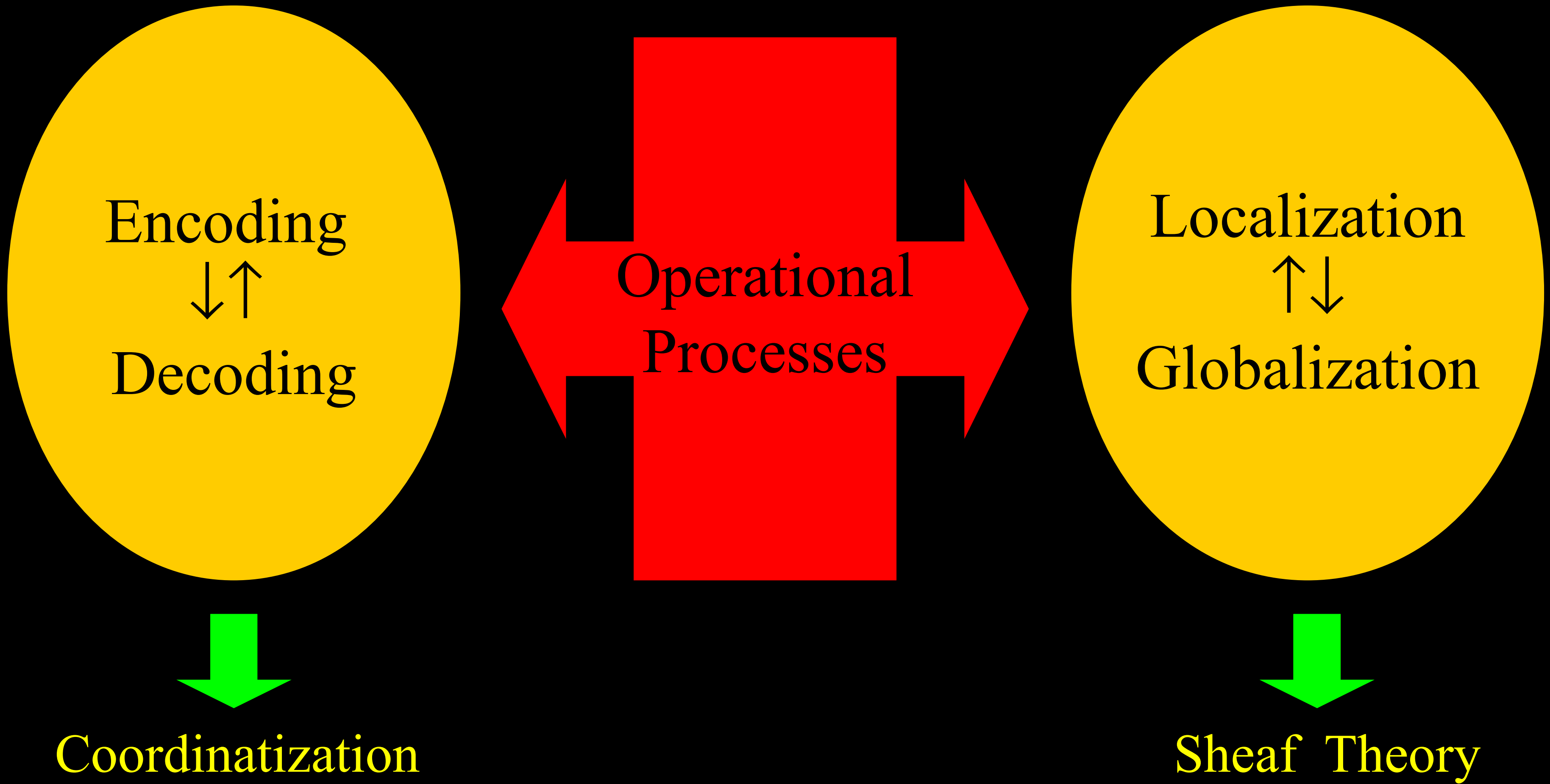
CONCEPTUAL SETTING

- Fundamental Operational Processes associated with modelling of physical systems via measurement:



Coordinatization

- Assignment of an observation frame, consisting of a set of "objectively individualized" coordinates, quantities in the most general sense, which are put into 1-1 correspondence with the homogeneous elements (points) of the physical geometry.

- The use of the term "quantity" by no means implies that the coordinates making up the frame must be ordinary numbers.

- The construction of these quantities and the study of their properties constitutes the domain of Algebra.

• The "number-like" quantities can be subjected to algebraic operations, so that they can form suitable algebraic structures closed under the action of the corresponding operations.

•Algebraic structures of "number-like" quantities of any particular operational form, can be thought of as solutions to corresponding physical measurement problems.

•Phases of coordinatization process:

1. Numerical Coordinatization
2. Functional Coordinatization
3. Sectional Coordinatization

Observables: Physical attributes which, in principle, can be measured. Collections of observables should have the closure property with respect to the operations of addition and multiplication, forming algebraic rings of scalar coordinates.

Localization-Globalization

•Bidirectional Compatibility of Observables under the Inverse Operations of:
[I]. Extension from the Local to the Global.
[II]. Restriction from the Global to the Local.

Localization: Extract information related with the local behaviour of a physical system, viz. discern observables locally.

Implementation:

Preparation of local reference frames for measurement of observables.

Globalization: Collate the local contextual pieces of observable information together globally, viz. make observables compatible under the extension from the local to the global.

Epistemological Principle of Sheaf Theory:

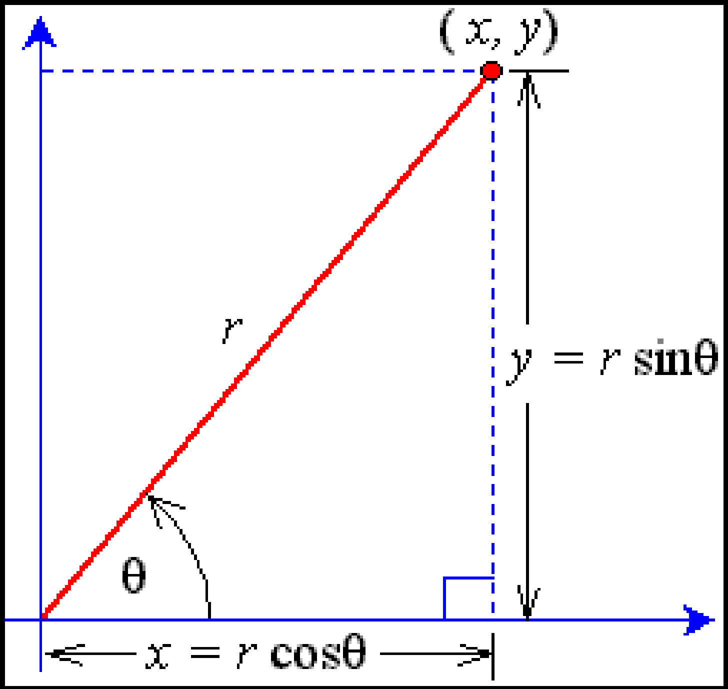
A global information structure can be synthesized by means of compatible interlocking families of simple, local or partial information carriers

Local or Partial Information Carrier: localization device or information filter or frame of measurement of observables.

Numerical Coordinatization

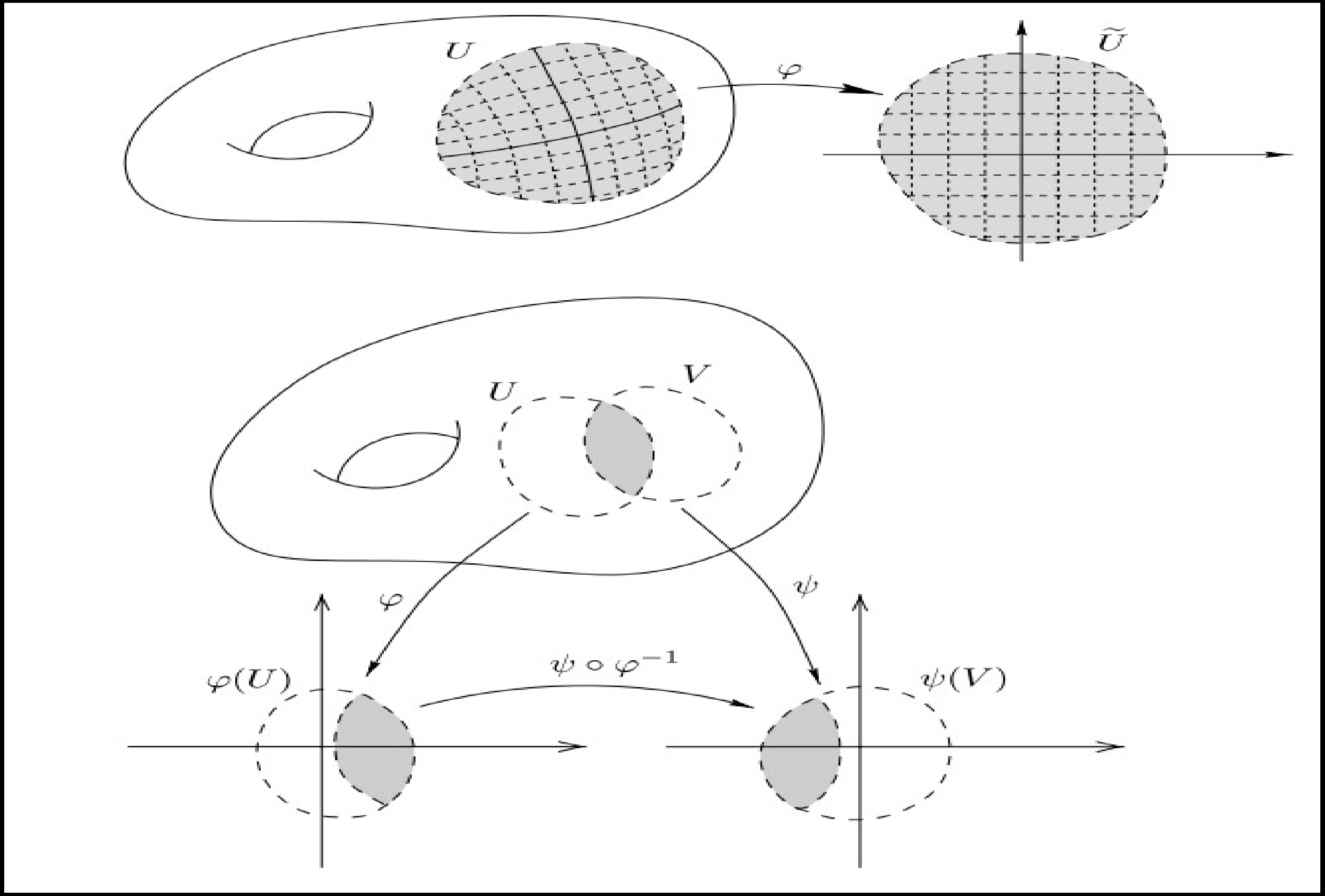
- Cartesian coordinates: Use three concurrent, pairwise perpendicular lines, each one endowed with an orientation and a unit length standard in order to produce an 1-1 mapping:

$$\text{Euclidian Space} \rightarrow \mathbb{R}^3, \quad P \longmapsto (x(P), y(P), z(P))$$



- The 1-1 mapping is called a Cartesian Coordinate System.
- There exist other Coordinate Systems (non-uniqueness of the mapping).
- Physical Geometry – Solution of Equations.

•Smooth Manifolds: In the coordinate approach, a smooth manifold structure on a set is defined by a family of compatible charts constituting a smooth atlas. A chart (U, φ) on the set M is a bijective map φ: U → R^n of a subset U ⊂ M onto an open set φ(U) of the Euclidean space R^n. Two charts (U, φ), (V, ψ) are called compatible if the transition map is a diffeomorphism of open subsets of R^n.



Functional Coordinatization

• Let M be a smooth n-dimensional manifold, and A the linear space (over the reals) of all smooth real-valued functions on it. Then A is a commutative and associative R-algebra, that is A (as a commutative ring of functions) is endowed with two operations, + and · , satisfying the natural axioms of arithmetics.

A: R-algebra of smooth observables on M

•A smooth structure on a manifold M is a way of distinguishing in the algebra of all functions on this set the subalgebra of those functions that it is proper to call smooth.

• If we forget about charts and coordinate transformations and work only with A, what can we say about M? It turns out that A entirely determines M! Thus, the geometric object M is functionally coordinatized by the R-algebra A!

- Any smooth manifold M is determined by the R-algebra A of smooth observables, each point x on M being the R-algebra homomorphism $x: A \to R$ which assigns to every function f in A its value f(x) at the point x.

- A smooth manifold M is the R-spectrum of the R-algebra A of smooth observables, that is the set that it is visible by means of the functional coordinatizing frame A!

- Magic Inversion Formula: $x(f) = f(x)$

- Defines the homomorphism $x: A \to R$ when the functions f in A are given.

- Defines the functions $f : M \to R$, when the homomorphisms x are known.

•Didactic of Functional Coordinatization: Smooth R-algebra frames A allow us to observe smooth geometric forms (manifolds) M, where M is the R-spectrum realization of the observables in A (evaluation or measurement of observables).

•Smooth Functional Coordinatizing Frame ↔ Smooth commutative R-algebra
•Smooth Functional Coordinatizing Frame ↔ Physical Laboratory

•Observables of the Frame ↔ Smooth functions
•Observables of the Frame ↔ Measuring devices

•Smooth geometric form observed via the Frame ↔ Smooth Manifold M
•R-Spectrum Realization of the Frame ↔ Smooth Manifold M

•State ↔ R-algebra homomorphism $x: A \to R$
•State ↔ point of the R-Spectrum of the Frame ↔ point of Smooth Manifold M

• Natural Philosophy of Functional Coordinatization can be generalized for any type of functional frame preserving the ring-theoretic specification of observables.

Operational Role of Evaluating a Functional Frame: Translates geometrically the information collected by evaluations of A at rings of measurement scales.

Notion Of State

A state of a ring of observables A over a ring of measurement scales B, called a B-state of A, encodes geometrically the information acquired by evaluating the observables in A, at the measurement scales in B.

Notion Of Spectrum

The B-spectrum of a ring of observables A is the set of all B-states of A, where B is called the evaluation frame of those states. It is understood as a geometric space whose B-coordinatized points (B-states) denote elements, which can be B-observed by means of corresponding measurement procedures on A.

•Can we turn the R-Spectrum of A into a Topological Space using only information from the Frame A?

•Ideal of Frame A: A subset I of A is called an ideal of A if I is an additive subgroup of A and is absorbing every element of A under the action of the multiplication operation.

•Ideal of Frame A $\leftrightarrow$ Kernel of ring homomorphism from frame A

•Maximal Ideal of Frame A $\leftrightarrow$ [Kernel of $x: A \to R$] $\leftrightarrow$ point of R-Spectrum

•With every ideal I of the frame A we associate its zero locus (common set on which all the functions in I vanish):

$$V(I)=\{x \in M : \forall\ f \in I \quad f(x) = 0\}$$

•Declare all sets of the form V(I) to be the closed sets of a topology on M.

•Inversely, If $Z \subseteq M$ is a closed subset, then

$$I(Z) = \{ f \in A: f|Z = 0 \}$$

is an ideal of frame A.

<u>Generalized Concept of Functional Coordinatization</u>:

1. Every geometric (topological) state space can be coordinatized by some ring of observables (functions) on it, and

2. Any ring of observables coordinatizes some geometric state space identified as the observed spectrum of the ring.

Further Remarks:

1. Maximal Ideal of frame A: $I_x = \{ f \in A: f(x) = 0 \}$ at every $x \in M$
2. Decomposition of frame A: $A = R \oplus I_x$ at every $x \in M$
3. Frame of a point $x \in M$: $A_x = A / I_x = R$ always
4. Frame A is built up as the variable frame R over the set of points $x \in M$.

Questions:

1. What if the frame corresponding to a point x is variable with the point x?
2. What if topology is considered explicitly for specifying A_x?
3. Can we develop a theory of continuously variable frames over any topological space X?

Sectional Coordinatization : Sheaves

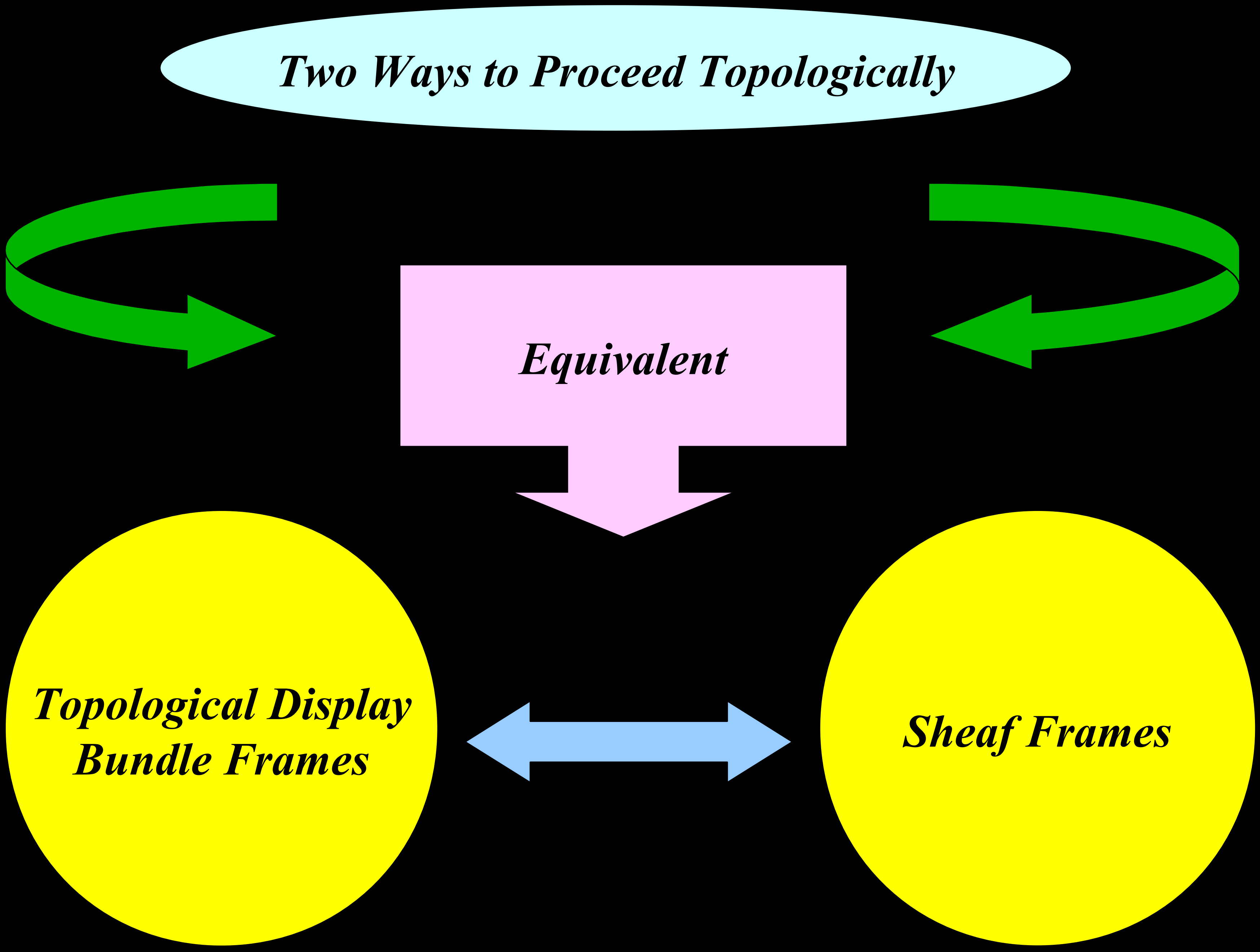
• Let M be a smooth n-dimensional manifold, and A the functional frame (R-algebra) of smooth real-valued observables (functions) on M. A function from M to R can also be thought of as an inverse to the bijective projection map from M × R to M. Such an inverse is called a global section.

• The projection from M × R to M has, for each point x of M, simply a copy of R as its inverse image. The value that we choose at the point is simply the value taken by the function. Thus, this kind of construction is appropriate for capturing point-properties of observables.

•But what about local properties of observables?

•Can we make sense of local sections and local frames?

•Think of M as a topological space. In a topological space it is only the open sets (or the closed sets) that matter and their partial order and not the points!


Two Ways to Proceed Topologically
Equivalent
Topological Display
Bundle Frames
Sheaf Frames

Sheaf Frames

•On each open set $U \subset M$, we have a frame of smooth functions. We denote this ring of functions by A(U). Given a smooth function on an open set, we can restrict it to a smaller open set, obtaining a smooth function there. In other words, if $V \subset U$ is an inclusion of open sets, we have a "restriction map" :

$$r(U \mid V): \mathbf{A}(U) \to \mathbf{A}(V)$$

•Take a smooth function on a big open set, and restrict it to a medium open set, and then restrict that to a small open set. The result is the same as if you restrict the smooth function from the big open set directly to the small open set. So restrictions commute and the identity inclusion gives rise to the identity restriction.

a. $r(U \mid U) =$ identity at $\mathbf{A}(U)$ for all open sets U of M and

b. $r(V \mid W) \circ r(U \mid V) = r(U \mid W)$ for all open sets $W \subseteq V \subseteq U$ of M .

Elements of $\mathbf{A}(U)$ are called local sections over U.

Notationally we agree to denote:

$$r(U \mid V)(s) := s \mid V$$

Abstracting from Smoothness: Presheaf Frame
Global → Local Compatibility

A presheaf frame **F** of rings of observables on a topological space X, consists of the following data:

1. For every open set U of X , a ring (of local observables) **F** (U), and
2. For every inclusion V ⊆ U of open sets of X , a restriction morphism of rings in the opposite direction:

$$r\,(U \mid V) : \mathbf{F}(U) \to \mathbf{F}(V)$$

   such that:

a. $r\,(U \mid U)$ = identity at **F** (U) for all open sets U of X and

b. $r\,(V \mid W) \circ r\,(U \mid V) = r\,(U \mid W)$ for all open sets W ⊆ V ⊆ U of X .

Elements of **F** (U) are called local sections s of the frame **F** over U.

Notationally we agree again to denote:

$$r\,(U \mid V)\,(s) := s \mid V$$

Back to Smoothness.

1. Take two smooth functions f and g on a big open set V of M , and an open cover of V by some $\{V_a\}$, $a \in I$: index set. Suppose that f and g agree on each of these V_a.

Then they must have been the same function to begin with!

Thus, if $\{V_a\}$ is an open cover of V, and f, g in A(V), such that:

$f|V_a = g|V_a$ for all $a \in I$, then f=g.

Conclusion 1: We can identify smooth functions on an open set by looking at them on a covering by small open sets.

2.Then, take a smooth function f_a on each of these V_a and assume they agree on all their pairwise overlaps. Then they can be "glued together" to make one smooth function on all of V. Given $f_a \in \mathbf{A}(V_a)$, $a \in I$, such that:

$$f_a |V_a \cap V_e = f_e |V_a \cap V_e \qquad \text{(for all indices } a, e \in I)$$

then there exists a unique function $f \in \mathbf{F}(V)$, such that:

$$f|V_a = f_a \in \mathbf{F}(V_a) \text{ for all } a \in I.$$

Conclusion 2: We can glue together uniquely smooth functions which are compatible on overlapping open sets.

Abstracting from Smoothness: Sheaf Frame
Local ↔ Global Compatibility

A presheaf frame **F** of rings of observables on a topological space X, is defined to be a sheaf frame if it satisfies the following two conditions, for every family V_a , $a \in I$, of open subsets of V: open in X ($V = \mathbf{U_a} \, \mathbf{V_a}$, $a \in I$):

1. Given $s, t \in \mathbf{F}(V)$ with $s|V_a = t|V_a$ for all $a \in I$, then $s = t$.
 ---------------Identity Axiom of a Sheaf Frame-------------------------

2. Given $s_a \in \mathbf{F}(V_a)$, $a \in I$, such that:
 $$s_a | V_a \cap V_e = s_e | V_a \cap V_e$$
 for all indices $a, e \in I$,
 then there exists a section $s \in \mathbf{F}(V)$, such that:
 $$s | V_a = s_a \in \mathbf{F}(V_a) \text{ for all } a \in I.$$
 Such a section s is unique because of condition 1.
 ---------------Gluing Axiom of a Sheaf Frame--------------------------

Topological Display Bundle Frames

1. Take two smooth functions f and g defined in open neighbourhoods U and V of a point x of M. We say that f ≈ g at x, if there is an open neighbourhood W of x contained inside both U and V, such that f(y) = g(y) for every point y in W.

f(y)=g(y) for every point $y \in W \subseteq V \cap U$, or

$f|W = g|W$ where $W \subseteq V \cap U$

Conclusion: The relation ≈ is an equivalence relation on smooth functions. The equivalence classes under this equivalence relation are called germs of smooth functions at the given point x.

2. The set of all germs of smooth functions at a given point x of M, that is, the quotient space of all pairs (f, U), $x \in U$, under the equivalence relation ≈ is called the stalk A_x at the point x of M, and the disjoint union of all these stalks over every x, is termed the display space E of the frame **A**.

Questions: I. Is the stalk A_x for every x of M a frame itself, meaning is it a ring of observable germs at x?

II. Is the display space E a topological space?

I. The stalk $\mathcal{A}_x$ at every x of M is a frame itself, meaning it is a ring of observable germs at x. We call $\mathcal{A}_x$ the contextual frame at x.

$$j: A(U) \rightarrow \mathcal{A}_x$$

$$f \rightarrow j(f)=f_x=\text{germ}_x f$$

$\mathcal{A}_x$ inherits the ring (R-algebra) structure from the local sections.

J_x: unique maximal ideal of frame $\mathcal{A}_x$ (set of germs vanishing at x).

Quotient Frame: $\mathcal{A}_x / J_x = R$ (evaluation frame of germs at x).

Thus: the value of a germ at a point x is an element of the contextual frame at x modulo the unique maximal ideal of this frame at x!

Physical Idea: Two local observables have the same germ at a point x if and only if they induce the same contextual information at x, encoded as an element of the frame $\mathcal{A}_x$.

II. The disjoint union of all the stalks A_x for every point x forms the display space E of the frame A.

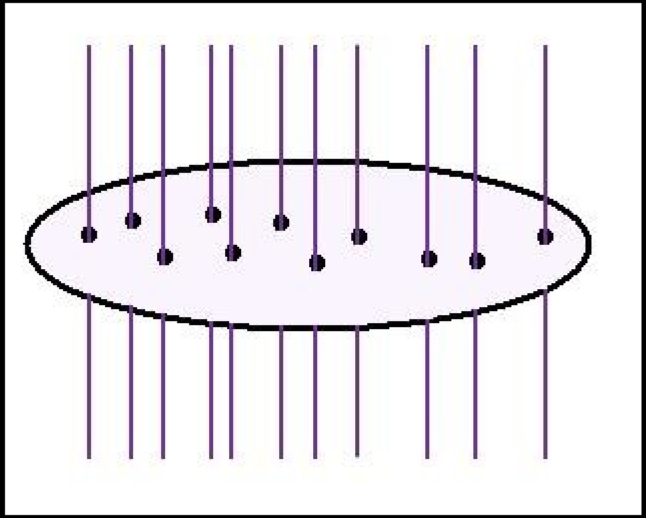
Think of the display space E as a plant (set of all germs) generated by the presheaf frame **A**!

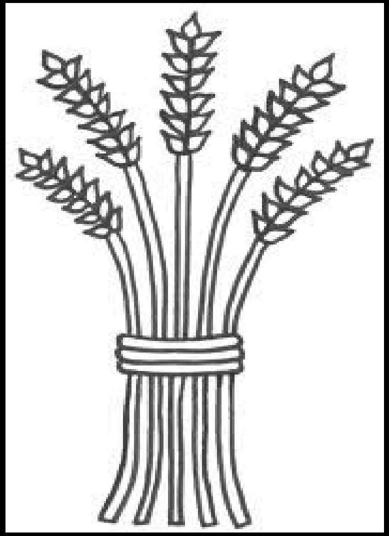
E=P**A** becomes a topological display space as follows:

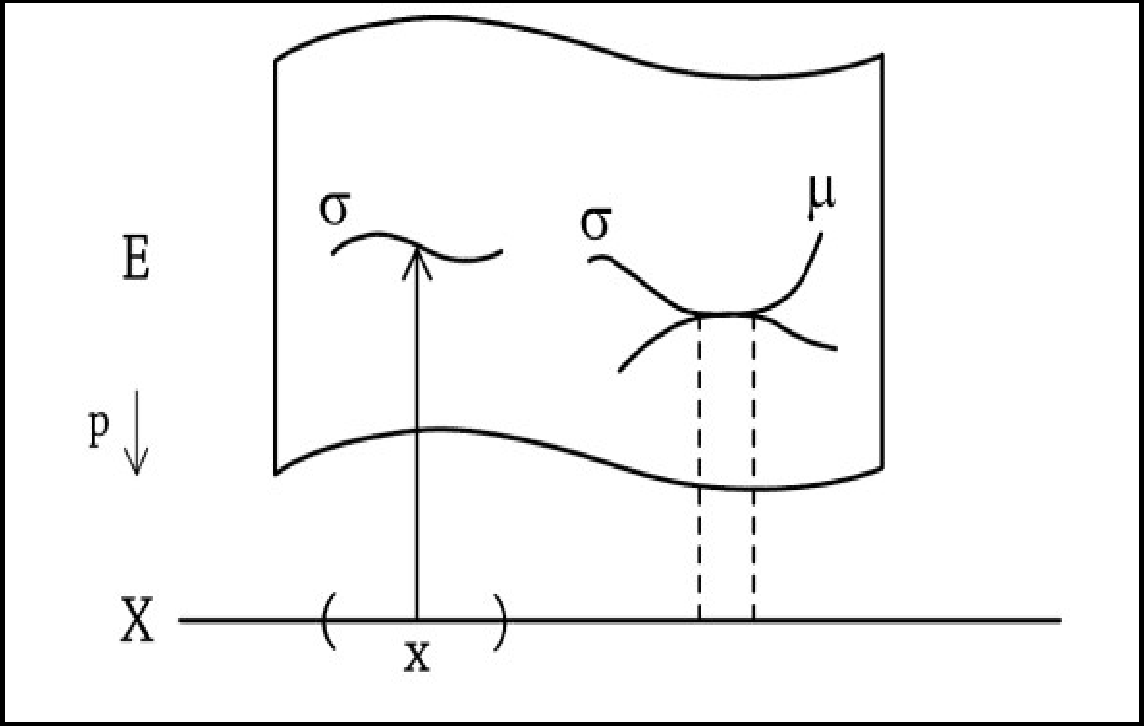
Fix a section f $\in$ **A** (U) and consider its germ at x: $f_x = germ_x f$
Declare $\{f_x = germ_x f : x \in U\} \subseteq$ P**A** to be open for all pairs (f, U), and let this be a basis for the topology on P**A**.

Thus: p: E $\longrightarrow$ M becomes a topological bundle. We call it the topological display bundle frame of A.

[i]. p: E $\longrightarrow$ M is a local homeomorphism of topological spaces.
[ii]. The contextual frame ring operations are continuous over M. Thus, the frame A becomes a continuous variable contextual frame A_x, $x \in M$.
[iii]. The stalks A_x are bound together by the topology of M.



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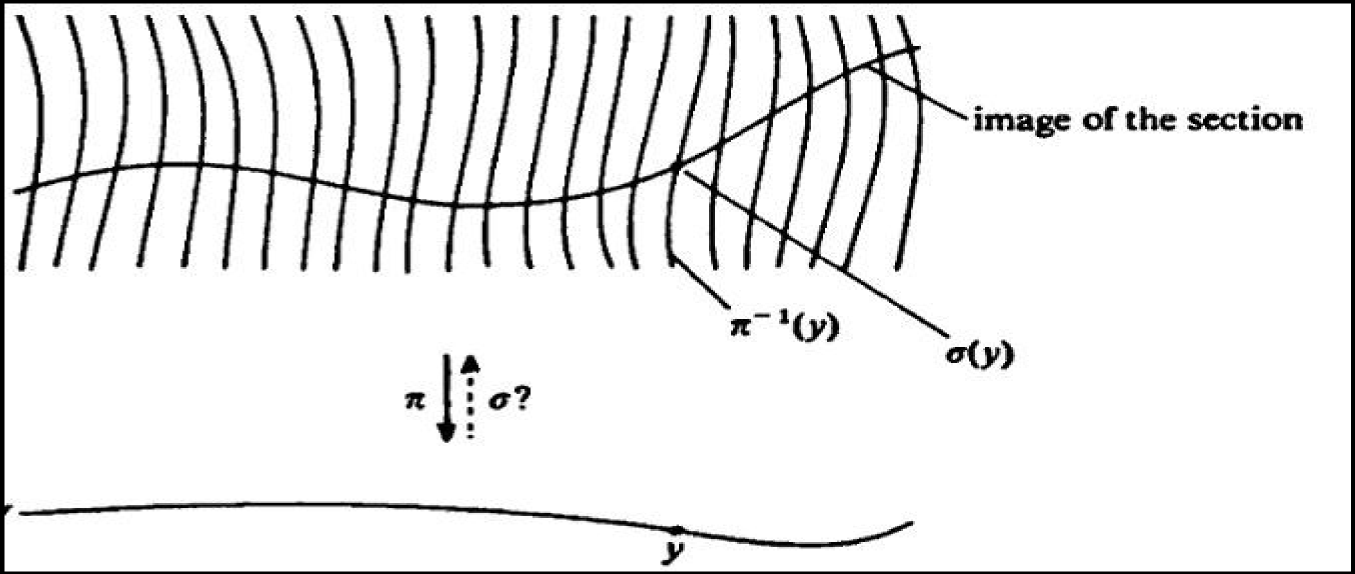
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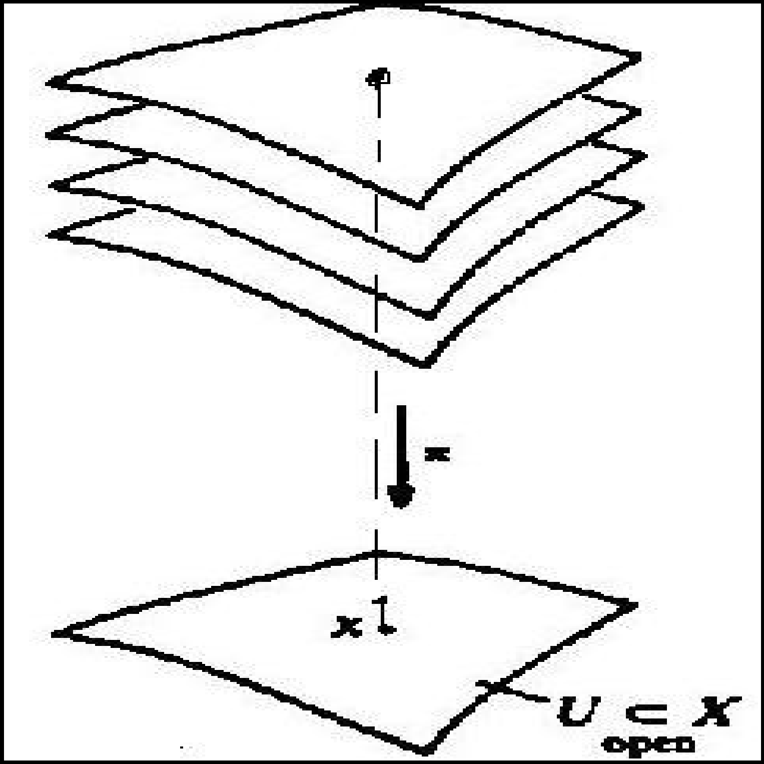
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Abstracting from Smoothness
Stalks and Germs of Presheaf Frame

We consider **F** to be a presheaf frame on a topological space X, and let $x \in X$ be a point of X. For any open neighbourhoods V, W of the point $x \in X$, and any sections $s \in$ **F** (V), $t \in$ **F** (W), we define:

$$s \approx t$$

if and only if , there exists an open subset R of $V \cap W$, such that:

$$x \in R, \text{ and } s|R = t|R$$

The relation $\approx$ is an equivalence relation on sections of **F.**
The stalk of the presheaf **F** at the point $x \in X$ is defined to be the set:
F_x = Disjoint Union of **[F (V)]** modulo the equivalence relation **$\approx$** for all open neighbourhoods V of the point $x \in X$.

The image s_x of a section $s \in$ **F** (V) at the stalk **F_x (**of the presheaf **F** at the point $x \in X$), is called the germ of s at x.

I. The stalk F_x at every x of X is a frame itself, meaning it is a ring of observable germs at x. We call F_x the contextual frame at x.

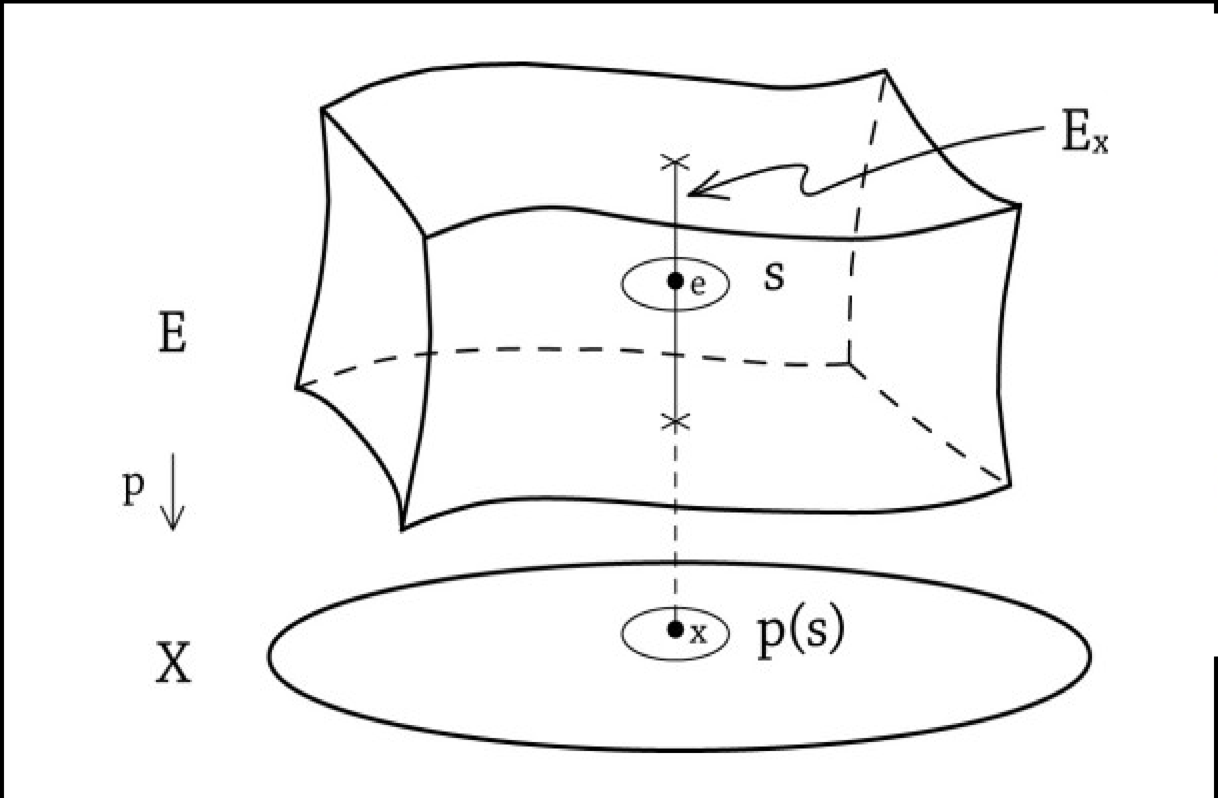
Physical Idea: Two local observables have the same germ at a point x if and only if they induce the same contextual information at x, encoded as an element (germ) of the frame F_x.

II. The disjoint union of all the stalks F_x for every point x forms the display space E of the presheaf frame **F**.

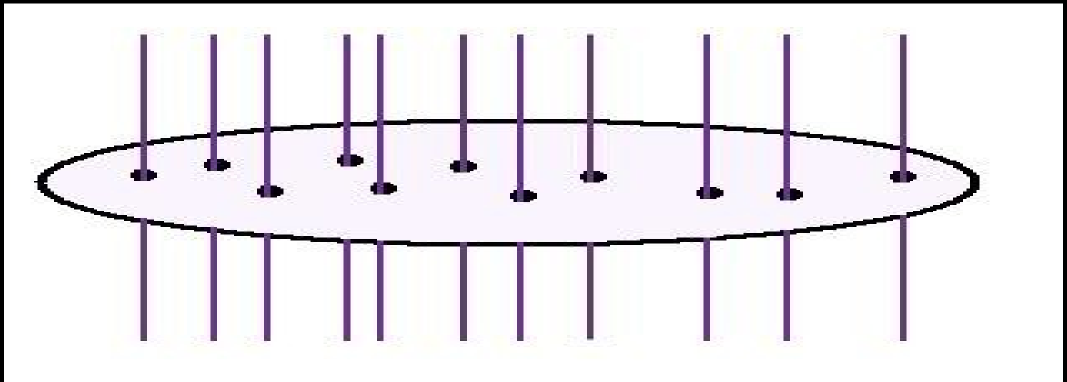
Think of the display space E as a plant (set of all germs) generated by the presheaf frame **F.**

p: E → X is a topological bundle. We call it the topological display bundle of the presheaf frame **F**.

[i]. p: E → X is a local homeomorphism of topological spaces.
[ii]. The presheaf frame **F** is a continuous variable contextual frame F_x.
[iii]. The stalks F_x are bound together by the topology of X.



Local Homeomorphism to the plane R×R : Through each point of the stalk there is a horizontal open disc on which the projection p is locally homeomorphic to a disc on the plane. The discs at different points of the stalk (pieces of lamb or onion, say) may come in very different sizes. All these servings over different points x of X are "glued together" by the topology of the display space E!



Taking Cross Sections of the topological display bundle $p: E \to X$ of the presheaf frame **F** : Look at small cross sections over small open sets U of X. A cross section is a continuous map $\sigma: U \to E$ such that: $p \circ \sigma = id(U)$

It selects one germ from each stalk over a point x in U!

Meaning of Cross Section: Each local observable s of the local presheaf frame **F**(U) determines a continuous function $\sigma: U \to E$ such that:

$$x \to germ_x s$$

for every point x in U. Thus, the correspondence $s \to \sigma$ transforms each local observable s of **F**(U) into a continuous function $\sigma: U \to E$.

Cross sections functionalize local observables!

Every local observable becomes a continuous function valued on E!

For each U open in X we may consider many different cross sections by selecting different germs from each stalk over points x in U.

So we define **E**(U) to be the set of all cross sections of E over U. The ring algebraic structure is inherited from the stalks and varies continuously. So **E**(U) is a local frame of cross sections over each U.

Since cross sections are continuous E-valued functions we may apply the operation of restriction, and thus obtain a new presheaf frame of cross sections **E.**

[1]. The Presheaf Frame of Cross Sections of E is a Sheaf Frame.

[2]. The stalks of E are identical with the stalks of F.

[3]. If F is a sheaf then it is a sheaf isomorphic with E.

Display Bundle Frames

Topological Bundle of Observable Germs Frame: Contextualize at spectrum points and then glue together all the contextual frames.

Sheaf of Local Observables Frame:
Make observables compatible under restriction (from global to the local frame) and under extension (from the local frames to the global).

Sheaf Frames