Elements of Set Theory

Péter Mekis
Department of Logic, ELTE Budapest

December 11, 2017

Preface

This is an as yet incomplete draft of what is going to be the lecture notes for a course under the same title for students of the *Logic and the Theory of Science* MA program of ELTE Budapest. The text is written for students that are logically and philosophically minded, but do not necessarily have a mathematical background. However, it requires familiarity with the basics of first-order logic. A reader who is skilled in reading mathematical texts will find most of the proofs too detailed. Note that any proof can be skipped and taken as an exercise to avoid boredom.

Instead of the more well-known system of Zermelo–Fraenkel set theory, these notes use the framework of Gödel–Bernays set theory (GB). Beyond the personal taste of the author, the reason for this choice is that there is a follow-up of this course under the title *Alternative set theories*, the basic task of which is to investigate the philosophical motivations and implications of various versions of the set/class distinction, which is at the heart of any axiomatic set theory, and which is more convenient to discuss in a language where one can quantify over classes.

A mathematically minded reader may be surprised by the length of discussing certain basic topics, even without any apparent practical profit, and even at the cost of giving a rather sketchy presentation of more advanced results. An example is the detailed comparison of three different ways to represent natural numbers in GB. The reason is that the discussion aims at foundational questions, and how to represent numbers by sets adequately is one of the fundamental questions of the foundational studies. Also, note that different representations of numbers fit different set theoretical frameworks, so numbers are nice and relatively simple objects for a comparative study of alternative set theories. In other words: our focus is on establishing theories, not establishing results within a theory.

Although no mathematical background is assumed, the notes are written for students who have already completed the two semesters of an introductory logic course of our Masters program, and thus they are familiar with the syntax and semantics of standard firt-order logic, have some skills in reading and writing complex formulas, are familiar with basic logical laws including the basic Boolean laws and laws of quantification, with the concept of a first-order theory, and also with the axioms of first-order Peano arithmetic. These students are also able to recall some basic metalogical results like compactness theorem, the Löwenheim–Skolem theorem, and Gödel's incompleteness theorem; however, we make only sporadic use of these, so a detailed knowledge of them is not necessary to understand these notes.

The author apologizes for all mistakes due to his poor English, and also for the innumerable number of typos due to the work being under preparation. He hopes that the number of major mistakes is decreasing in the process of revisions. Questions, corrections and suggestions are welcome to `mekis.peter@gmail.com`.

1 Introduction: From Hilbert's Hotel to Cantor's Paradise

As William Weiss put it, "set theory is the true study of infinity." Introducing a technical concept of infinity, free from any theological or metaphysical burden, is a major achievement of human culture, made possible by a systematic study of sets (classes, collections). In this introductory section we consider some basic arguments in this area in an intuitive manner. In the subsequent sections we will develop a technical framework in which these intuitive results can be reconstructed with more precision.

A first and seemingly paradoxical property of an infinite collection is that its size is in some sense resistant to adding or withdrawing elements. This was already seen by Galileo, and systematically discussed in Bernard Bolzano's *Paradoxes of Infinity.* We present it in its popular form, as a thought experiment about an infinite hotel, the hotel's many guests, and its smart manager.

In Hilbert's Hotel, named after the German mathematician David Hilbert, the rooms are numbered in the usual way, except that there is no largest room number. Right after room number 100, we find room number 101; right after room 1000, there comes room 1001; and so on, *ad infinitum.* Each one is a single room, suitable to accommodate one guest only. At the beginning of our story, the hotel is full; each room is taken by one guest.

On the first day, a new guest arrives, and asks for a room at the reception desk. There is no free room, but the receptionist solves the problem quickly. She asks each guest to move to the subsequent room; from room 1 to room 2, from room 2 to room 3, and so on, from room n to room $n + 1$. The new guest takes room 1. Noone had to move out, one person moved in, and there is still one and only one person in each room.

On the second day, an infinite bus of new guests arrives. It is similar to the hotel in that it has no largest seat number, and all the seats are taken. Can the new guests fit in the hotel? Yes; the receptionist asks each guest of the hotel to move to the room the number of which is twice the number of her original room. Thus the guest in room 1 moves to room 2, the guest in room 2 moves to room 4, the guest in room 3 moves to room 6, and so on, the guest in room n moves to room $2n$. Now all the even numbered rooms are taken, and all the odd numbered rooms are free, so the new guests can move in them; from seat 1 to room 1, from seat 2 to room 3, from seat 3 to room 5, and so on, from seat n to room $2n - 1$.

On the third day, an infinite sequence of infinite buses arrive to the hotel; there is a bus number one, a bus number two, and so on; there's no bus with a largest number. Can they all fit in the hotel? After a bit of thinking the receptionist creates a diagonal ordering of all the guests as shown in the following tabular. (The arrows in the lower index point at the number of the destination room.)

hotel	$1_{\to 1}$	$2_{\to 3}$	$3_{\to 6}$	$4_{\to 10}$	$5_{\to 15}$	$6_{\to 21}$	$7_{\to 28}$	$\dots$
bus 1	$1_{\to 2}$	$2_{\to 5}$	$3_{\to 9}$	$4_{\to 14}$	$5_{\to 20}$	$6_{\to 27}$	$7_{\to 35}$	$\dots$
bus 2	$1_{\to 4}$	$2_{\to 8}$	$3_{\to 13}$	$4_{\to 19}$	$5_{\to 26}$	$6_{\to 34}$	$7_{\to 43}$	$\dots$
bus 3	$1_{\to 7}$	$2_{\to 12}$	$3_{\to 18}$	$4_{\to 25}$	$5_{\to 33}$	$6_{\to 42}$	$7_{\to 52}$	$\dots$
bus 4	$1_{\to 11}$	$2_{\to 17}$	$3_{\to 24}$	$4_{\to 32}$	$5_{\to 41}$	$6_{\to 51}$	$7_{\to 62}$	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

The above thought experiment suggests a conclusion that in the case of an infinite collection of items (guests, rooms, seats, etc.) expansion (e. g. by adding new guests) or reduction (e. g. by emptying even numbered rooms) does not affect the collection's size; any infinite collection has the same number of elements as any other. We will see shortly strong arguments against this position; but before that, let us distill preliminary definitions of some basic concepts from the above discussion.

The thought experiment suggests two different definitions of infinity. In one sense, the elements of infinite sets can only be enumerated in such a way that there is no natural number that would be the largest element number; or, in other words, *a set is infinite if* *and only if it has at least as many elements as the set of the natural numbers.* This is behind the idea that there is no largest room number in the hotel. If there was a largest room number, it would also be the number of the rooms, thus making the hotel finite. Since there is no largest room number, all the natural numbers are used in enumerating them; thus there are just as many rooms in the hotel as natural numbers. The other suggested definition exploits the idea that the hotel manager used in accommodating new guests in a full hotel; *a set is infinite if and only if it is possible to extract elements from it so that there are as many elements remaining as there were originally.*

In set theory, the first version became the standard definition of infinity; but the latter, labelled as Dedekind infinity (after the German mathematician Richard Dedekind) is also common. We are not yet in the position to tell whether they introduce the same concept or different ones; or, to phrase the question a bit more precisely, under what conditions do the two concepts fall together. (Later we will see that in the standard axiomatic framework of set theory this depends on whether we assume a weak version of the axiom of choice, called the principle of countable choice.) But we can observe that both definitions rely on the relational concept *as many as*; a proper definition of which will reflect the very essence of the thought experiment.

Each trick of the hotel manager established one-one correspondences between sets. Based on this strategy, we can say that *a set has as many elements as another one if and only if there is a one-one correspondence between them.* This idea is sometimes called Hume's principle, and although it is not entirely clear whether Hume actually held it regarding sets (he applied it somewhat confusingly to numbers), it was present in Gottlob Frege's *The Foundations of Arithmetic*, and several other early investigations on the numerousity of sets. When a set has as many members as another one, we call them equinumerous. Intuitively, equinumerousity is an equivalence relation; it is reflexive, symmetrical and transitive. This will be the case in the standard axiomatization of set theory, too, as well as in many alternative axiomatic frameworks.

From a set-theoretical point of view, a one-one correspondence from a set H to the set G may be regarded as a set, too; that is, a set f of ordered pairs that satisfy the following conditions:

1. the first members of each pair in f are from H;
2. each element of H occurs as a first member in exactly one pair in f;
3. the second members of each pair in f are from G;
4. each element of G occurs as a second member in exactly one pair in f.

As we have seen, the Hilbert's Hotel thought experiment suggests that any infinite set has as many members as any other. Let us now consider a counterargument against this, in the form of an additional episode to the Hotel saga. On the fourth day, after all the guests from the previous day left the hotel, a special infinite bus arrives, full of passengers. The seat numbers in this bus are real numbers between 0 and 1 (the latter reserved for the driver). There is a seat numbered as $2/3$ and one numbered as $4/5$; and in between them, there are seats number $\sqrt{2}/2$, $3/4$, $\pi/4$, etc. Some of these numbers, like $3/4$, are fractions of two integers; they are called rational numbers. Others, like $\sqrt{2}/2$, are roots of polynomial equations with integer coefficients (thus $\sqrt{2}/2$ solves the equation $2x^2 - 1 = 0$); these are the algebraic numbers, including the rationals. Yet others, like $\pi/4$, lack this connection with polynomials $\pi/4$ (as it was proved in the case of $\pi/4$ by Ferdinand von Lindemann); they are called transcendental.

We take it for granted that all of these seat numbers are expressible by infinite sequences of digits; $3/4$ as $0.75000\ldots$, $\sqrt{2}/2$ as $0.70710\ldots$, and $\pi/4$ as $0.78539\ldots$ (in the first case the three dots represent an infinite sequence of zeros; in the other two cases, various other digits without periodicity). We will identify seat numbers with their decimal expansions. To avoid notational ambiguity, we don't allow for infinite sequences of nines in the expansions, since, for example, 0.74999... (where the dots stand for further nines) denotes the very same number as 0.75000... above.

We will prove that the passengers of the bus cannot be accommodated in the hotel. We do this by means of an indirect argument; that is, we assume (towards a contradiction) that the whole busload of passengers has been successfully accommodated in the hotel. The room arrangement will be something like the following, with the room numbers on the left, and the seat numbers on the right:

$$\begin{array}{ll} 1 \longrightarrow & 0.\ 7\ 5\ 0\ 0\ 0\ \ldots \\ 2 \longrightarrow & 0.\ 7\ 0\ 7\ 1\ 0\ \ldots \\ 3 \longrightarrow & 0.\ 7\ 8\ 5\ 3\ 9\ \ldots \\ 4 \longrightarrow & 0.\ 3\ 9\ 2\ 2\ 8\ \ldots \\ 5 \longrightarrow & 0.\ 3\ 9\ 2\ 2\ 3\ \ldots \\ \vdots & \ \ \vdots \qquad \ddots \end{array}$$

By hypothesis all seat numbers occur in the list. We construct one that isn't there. The procedure runs as follows: take the nth digit (after the decimal point) of the nth member of the list, and if it is less then 5, substitute 5 for it; otherwise substitute 4 for it. This procedure gives us all the digits of a new number c as follows:

$$\begin{array}{ll} 1 \longrightarrow & 0.\ \mathbf{7}\ 5\ 0\ 0\ 0\ \ldots \\ 2 \longrightarrow & 0.\ 7\ \mathbf{0}\ 7\ 1\ 0\ \ldots \\ 3 \longrightarrow & 0.\ 7\ 8\ \mathbf{5}\ 3\ 9\ \ldots \\ 4 \longrightarrow & 0.\ 3\ 9\ 2\ \mathbf{2}\ 8\ \ldots \\ 5 \longrightarrow & 0.\ 3\ 9\ 2\ 2\ \mathbf{3}\ \ldots \\ \vdots & \ \ \vdots \qquad \ddots \\ c = & 0.\ 4\ 5\ 4\ 5\ 5\ \ldots \end{array}$$

By hypothesis c must be in the list; but it isn't, because it differs from each member on the list in at least one digit. We arrived at a contradiction, so our initial assumption must be rejected.

On the other hand, it is easy to see that the set of the passengers contains a part that is equinumerous with the set of the hotel rooms. For example, take the passengers with seat numbers of the form $1/n$ for positive integers n; moving the passenger on seat $1/n$ to room n amounts to a one-one correspondence between the two sets.

We call such parts of a set—which themselves are sets, too—subsets. We can phrase a definition of the term as follows: *a set G is a subset of a set H if and only if every element of G is also an element of H.* In this sense every set is a subset of itself; and the empty set is a subset of every set. (The latter is a very useful contruction included into the universe of sets in virtually all set theoretic frameworks; it is, e. g., the set of all natural numbers that are neither even nor odd.)

How many subsets does a set have? The question has a simple answer in the realm of finite sets. As an example, consider a set with three members, $\{a, b, c\}$. Its subsets include the singleton sets $\{a\}$, $\{b\}$, $\{c\}$, the two-membered sets $\{a, b\}$, $\{b, c\}$, and $\{a, c\}$, the set $\{a, b, c\}$ itself, and the empty set; eight subsets altogether. Or, counting more systematically, with each of the three members of $\{a, b, c\}$ there come two possibilities; it is either included or not included as an element of a particular subset. This amounts to $2 \times 2 \times 2 = 2^3 = 8$ possibilities altogether. Similarly, in the case of an n-membered set, there are 2^n subsets. Now for any positive number n, 2^n is larger than n; therefore a finite set has always more subsets than elements. This can be seen by simple arithmetic induction; $1 < 2^1$, and for every n, if $n < 2^n$, then

$$n + 1 \leq n + n < 2^n + 2^n = 2^{n+1}.$$

How about infinite sets? Do they have more subsets than elements? Let us call the set of all subsets of a set its power set. We will prove that (a) a set cannot be euinumerous with its power set; and (b) the power set of any set has a subset that is equinumerous with the original set. These two clauses are known as *Cantor's theorem*. Let's start with (b), and consider the set G of all singleton subsets of a set H. Obviously there's a one-one correspondence between the two sets; the one that assigns to each member a of H its singleton $\{a\}$.

As for (b), our proof will take the form of an indirect argument. Suppose (towards a contradiction) that for some set H and its power set G, there is a one-one correspondence between H and G. For each member a of H, its pair A is a subset of H. Thus for each a in H here are two possibilities: (1) a is in A; (2) a is not in A. By these possibilities we can divide H into two halves. Let C be the set of those elements a of H which are not elements of the corresponding subset A.

Now C is a subset of H, too, and thus it is an element of its power set G. Since by assumption we have a one-one correspondence between H and G, C must be assigned to some element c in H. Again, c either is or isn't an element of C; one of the above possibilities must hold. If (1) is the case, then by definition c is in C, so by the definition of C c is not in C, which is a contradiction. On the other hand, if (2) is the case, then c is not in C, which means that c satisfies the defining condition of C, therefore c is in C, which is, once again, a contradiction. There are no more possibilities; so the indirect assumption must be rejected.

Let us call a set G larger than a set H if the H is not equinumerous with G, but H is equinumerous with a subset of G. In these terms Cantor's theorem implies that there is no largest set; for every set H, finite or infinite, there is a set G that is larger than it. Can we conclude from this that in terms of largeness, there is an infinite hierarchy of infinite sets, the same way as there is an infinite hierarchy of finite sets? Not yet. First we have to prove that two sets cannot be mutually larger than each other. This result is widely known as the Schröder–Bernstein theorem. The proof we present was given by the Hungarian mathematician Gyula König.

Let H and G be arbitrary sets, so that there is a subset H' of H and a subset G' of G such that H is equinumerous with G' and G is equinumerous with H'. Based on these conditions we prove that H is equinumerous with G. Let f be a one-to-one correspondence between H and G'; and let g be a one-to-one correspondence between G and H'. Our strategy is to partition H and G into smaller sets between which either f or g work as one-one correspondences; and to construct a one-one correspondence e from these partial correspondences. The definition of the parts will be simpler if we introduce another concept: the image of a set H with regards to a correspondence f is the set of all elements that occur as the second member of a pair in f, the first member of which is from H. In other words, the image of H with regards to f is the set of the values assigned to elements of H by f. In this sense G' is the image of H with regards to f; and H' is the image of G with regards to g.

Consider now the following sequences of sets. Let H_1 be H, and let G_1 be G; and for every positive integer n, let H_{n+1} be the image of G_n with regards to g; and let G_{n+1} be the image of H_n with regards to f. This simultaneous definition makes, of course, H_2 to be the same as H', and G_2 to be the same as G'; and generates an infinite parallel sequence of sts in both sides in such a way that for every positive integer n, H_{n+1} is a subset of H_n, and G_{n+1} is a subset of G_n.

Now we can define the partitions of both sets as follows: for every positive integer n, let A_n be the set of those elements of H_n which are not in H_{n+1}; and let B_n be the set of those elements of G_n which are not in G_{n+1}. It may or may not be the case that every element of H is in A_n for some n, and every element of G is in B_n for some n. So let A^* be the (possibly empty) set of those elements of H that are not in A_n for any n; and let B^* be the (possibly empty) set of those elements of G that are not in B_n for any n.

Now we can start spotting equinumerous pairs in these sequences. By definition, an even-numbered member of the A-series is the image of the preceding member of the B-series with respect to g; in other words, the restriction of g to B_{2n-1} is a one-one correspondence between B_{2n-1} and A_{2n}. Te very same way, an even-numbered member of the B-series is the image of the preceding member of the A-series with respect to f; in other words, the restriction of f to A_{2n-1} is a one-one correspondence between A_{2n-1} and B_{2n}. And, also as a consequence of their definitions, the restriction of f to A^* is a one-one correspondence between A^* and B^* (and *vice versa*, the restriction of g to B^* is a one-one correspondence between B^* and A^*).

Based on the above considerations, we can define a one-one correspondence e between H and G as follows. If an element a of H is in A_{2n} for some n, then let e assign to a the element b of G to which g assigns a. If a is in A_{2n+1} for some n or a is in A^*, then let e assign to a the same element b that f assigns to it. thus defined, e is a one-one correspondence between H and G, demonstrating the equinumerousity of H and G, and thereby completing our proof of the Schröder–Bernstein theorem.

Now we can safely conclude from the above considerations that there is an infinite hierarchy of infinite sets. This hierarchy if often referred to as Cantor's paradise.

We have to take the results of this section with some care. In what sense have we proved the existence of a set larger than the set of natural numbers, or the existence of an infinite hierarchy of infinites? We certainly didn't, in the sense that anyone who reads these arguments should be convinced of the existence of this hierarchy. The arguments were based on intuitive assumptions, often left implicite. Let us enumerate some of the assumptions necessary to conclude that Cantor's theorem is true:

(1) there are sets of sets, not only sets of things that aren't sets;

(2) the subsets of any set form a set;

(3) you can specify a subset of a set by means of any well-defined property.

Of these assumptions, (1) is accepted in virtually all axiomatizations of set theory; assuming the iterability of the elementhood relation is part of the reasons why we call a theory a set theory. But, of course, it is by no means necessary to discuss sets in a set-theoretical framework. (One can choose, for example, monadic second-order logic as a framework to discuss sets, where the values of predicate variables are sets of individuals, but there's no way to talk about sets of sets.) As for (2), it is also assumed in most axiom sets of set theory, but not everywhere. (There are small theories with infinite sets that have no power sets.) Finally, while (3) is a core assumption in standard set theory and its akins, it's rejected in some alternative set theories like New Foundations and related systems, or the various versions of positive set theory favor of other principles.

Believing in the existence of Cantor's paradise, whatever the terms "believing" and "existence" may mean in this context, amounts to a metaphysical commitment. Technical considerations cannot solve metaphysical problems. The best that we can hope from a set-theoretical discussion of sets is making as clear as possible the consequences of such commitments, thus giving a technical basis for one's choice of metaphysical assumptions. As an extreme case, by showing that a certain set of assumptions leads to contradiction is in this sense a major achievement. This is what we will do in the next section regarding a very appealing system of set-theoretical axioms.

2 Naïve set theory

Let us try to extract the main presumptions behind the reasonings of section 1. We have already made such an attempt in the particular case of Cantor's proof, towards the end of the section; now we would like to

(1) There are things which are not sets. For example, rooms and people are not sets; and although we don't know much about the exact nature of numbers, they don't seem to be sets either.

(2) Besides sets of things which themselves are not sets, it makes also sense to talk about sets of sets. Moreover, every set can be used as an element of anoter set. Te universe cannot be devided into elements and sets.

(3) Sets have sharp boundaries. Something either is or isn't an element of a set; there is no third case.

(4) We can use well-defined properties in specifying a set; in general, the set of Ps is the collection of things that have the property P. P can be "room in Hilbert's hotel", "natural number", "subset of a set H", etc.

(5) It is the totality of its members, not its specifying property, that individuates a set. If a set H has exactly the same members as a set G, then H is exactly the same set as G. For example, the set of even numbers is the same set as the set of numbers that are predecessors of an odd number.

We are going to construct a formal system that captures the principles of the above list, except for (1). The reason behind dropping principle (1) is that we are not really interested in applications of set theory in other areas of knowledge; instead, what we have in mind is the possibility of using set theory as a foundational framework for mathematics; in particular, we want to reconstruct various mathematical concepts within the framework of set theory, thus defining numbers, arithmetic operations, functions, relations, etc. as sets. Therefore it is convenient to consider a universe in which everything is a set.

As a consequence of dropping (1), principle (2) amounts to the presumption that every set can be element of some sets. It also affects principle (5); in the light of principle (4) dropping (1) narrows down the range of definite properties that can be used in specifying sets. Only properties that can be expressed in terms of elementhood are good candidates for specifying sets.

We are going to use standard first-order logic with identity as a framework in developing an axiom system that captures principles (2)–(5) above. The choice of logical framework is far from obvious; we could use second-order logic, or some nonstandard version of first-order logic like first-order intuitionistic logic; and so on. We lack the space here to give a detailed motivation of our choice, so we leave it for the notes of other courses. We can't do betetr than refer, somewhat negatively, to a professional consensus: a vast majority of writings in set theory are set in a first-order framework. (We will briefly discuss set-theory in a second-order framework in an appendix to these notes.)

The decision that we made about principle (1) will be reflected in our choice of language. The only extralogical symbol in our language is the binary predicate letter $\in$ (*epsilon*). Since in standard first-order languages all variables belong to the same syntactic sort, for any variables x and y both $x \in y$ and $y \in x$ are well-formed atomic formulas. Accordingly, in the intended interpretation quantifiers will range over sets. These formal syntactic and informal semantic commitments also involve principle (2) above, that elementhood can be limitlessly iterated.

We will use throughall these notes a simple notational convention: the very same was as we write $x \neq y$ for x and y not being the same, we write $x \notin y$ for x not being an element of y:

$$x \notin y \overset{\text{def}}{\iff} \neg\, x \in y.$$

As another useful notational convention, we will use bounded quantification in formulas, introduced the usual way:

$$(\forall x \in Y)\, \varphi(x) \overset{\text{def}}{\iff} \forall x\, \big(x \in Y \to \varphi(x)\big);$$

$$(\exists x \in Y)\, \varphi(x) \overset{\text{def}}{\iff} \exists x\, \big(x \in Y \land \varphi(x)\big).$$

The role of these abbreviations is to make formulas more readable.

A set theory without elements that are not sets (with a term borrowed from German, *urelements*) is sometimes called *pure set theory*; and a language with the sole predicate $\in$ is called *purely set-theoretical.* We will use a purely set-theoretical language throughout these notes (although from the next section on we will distinguish between two kinds of collections, sets and *proper classes*, within the universe.)

Principle (3), which says that sets have sharp boundaries, is also accounted for by the logical framework we chose; in the standard semantics of logic, an atomic formula $x \in y$ is either true or false, there is no middle case.

What we have left as our pure set-theoretical principles are (4) and (5); and now we express them formally as our axioms. Principle (4), which says that a set is identified solely by its elements, will amount to the following formula:

Axiom 2.1 (Extensionality)**.** *Sets with the same elements are the same.*

$$\forall x\,\forall y\,(\forall z\,(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

Note that the converse claim, that if x is the same as y then they have the same elements, is logically valid.

Sincs in first-order logic we cannot quantify over properties or extensions of properties, we can only formulate principle (5) as a scheme; in very much the same way as we formulate the principle of arithmetic induction as an axiom scheme in first-order Peano Arithmetic.

Axiom 2.2 (Naïve comprehension)**.** *For every property defined in the language of set theory by means of an open formula with parameters, there is a set of those sets that have this property.*

$$\forall u_1 \ldots \forall u_n\,\exists y\,\forall x\,(x \in y \leftrightarrow \varphi(x, u_1, \ldots, u_n))$$

—where y has no free occurrence in $\varphi(x, u_1, \ldots, u_n)$.

The variables $u_1, \ldots, u_n$ are the parameters of the property expressed by φ. As a limiting case, the list may be empty. For easy readability, we will often compress such lists of parameters in a vector symbol $\vec{u}$:

$$\forall \vec{u}\,\exists y\,\forall x\,(x \in y \leftrightarrow \varphi(x, \vec{u})).$$

The above axioms, set in a standard first-order logical framework, form a theory that we will refer to as *naïve set theory.* This term has a twofold use the literature. Sometimes, like here, it refers to a particular system of axioms expressed in a formal language; and sometimes it refers to a vaguely defined discourse of sets, with arguments built on mainly implicite principles, like the one we put forward in section 1. We will refer to such a discourse as *intuitive set theory*, to distinguish it from the formal theory discussed in this section.

By the end of this section we will see that naïve set theory is inonsistent; that is, 2.2 has instances which are logical contradictions. However, it will be instructive to introduce some basic concepts and prove some basic results in this axiom system; among other things reconstructing some of the results of 1 in a formal context. (It is not true that an inconsistent theory is not worth discussing at all. If one aims at practical applications, one can of course hope for avoiding any encounter with an inconsistent sets of premises; butif one aims at constructing theories, usually she will have to struggle through many inconsistent versions before reaching one that passes the test phase.)

We will use axiom 2.2 to establish the existence of a set (relative to the parameters $u_1, \ldots, u_n$); and the scheme 2.1 to establish its uniqueness. Thus these axioms together warrant that for every well-formed formula φ and for every assignment of the parameters $u_1, \ldots, u_n$, there exists exactly one set of sets that satisfy the formula with regard to the variable x. This justifies the following notation familiar from high school math:

$$\{x : \varphi(x, u_1, \ldots, u_n)\}.$$

The term on the right hand side is often called a set abstract, a very useful tool in defining sets and set operations. However, it's not part of the language, so it itself needs to be introduced by a contextual definition:

$$y = \{x : \varphi(x, u_1, \ldots, u_n)\} \overset{\text{def}}{\Longleftrightarrow} \forall x\,\big(x \in y \leftrightarrow \varphi(x, u_1, \ldots, u_n)\big).$$

Sometimes we use a slightly more sophisticated kind of set abstract, in which some of the defining conditions are condensed before the colon. If $T(x_1, \ldots, x_n)$ is an individual term already defined in the language,

$$\{T(x_1, \ldots, x_m) : \varphi(x_1, \ldots x_m, u_1, \ldots, u_n)\} \overset{\text{def}}{=}$$
$$\{x : \exists x_1 \ldots \exists x_n\, T(x_1, \ldots, x_m) = x \wedge \varphi(x_1 \ldots x_m, u_1, \ldots, u_n)\}.$$

A simple example of a set abstract of this kind is

$$\{\langle x_1, x_2\rangle : \varphi(x_1, x_2)\},$$

where the angle brackets will be defined later. The extra condition can be condensed in a formula, too:

$$\{\psi(x, v_1, \ldots, v_m) : \varphi(x, v_1, \ldots, v_m, u_1, \ldots, u_n)\} \overset{\text{def}}{=}$$
$$\{x : \exists v_1 \ldots \exists v_n\, (\psi(x, v_1, \ldots, v_m) \wedge \varphi(x, v_1, \ldots v_m, u_1, \ldots, u_n))\}.$$

A simple example of a set abstract of this kind is

$$\{x \in v : \varphi(x, v_1, \ldots, v_m, u_1, \ldots, u_n)\},$$

which will play an important role when we seek for a suitable substitute for the principle of naïve comprehension.

Let us now see some applications of the comprehension schema and the axiom of extensionality, expressed by definitions with set abstracts.

Empty set If we choose $\varphi(x)$ to be a contradiction, it defines the empty set:

$$\emptyset \overset{\text{def}}{=} \{x : x \neq x\}.$$

Universal set If $\varphi(x)$ is a tautology, it defines the universal set:

$$\mathsf{V} \overset{\text{def}}{=} \{x : x = x\}.$$

Since the universal set contains every set as an element, it is also element of itself: $\mathsf{V} \in \mathsf{V}$. Sets that are elements of themselves are often treated with suspicion; they involve a kind of circularity that Bertrand Russell called vicious. The universal set is also often blamed as the source of the contradictions derivable in naïve set theory. We will see at the end of this section that this is not the case. We will return the case of sets that are elements of themselves in section 4, discussing the *axiom of regularity.*

Let us define now the subset relation. A set x is a subset of a set y (in other words, y includes x) iff very element of x is also an element of y:

$$x \subseteq y \overset{\text{def}}{\Longleftrightarrow} \forall z\,(z \in x \to z \in y).$$

We will use quantification bounded with the subset relation the same way we use quantification bounded with elementhood:

$$(\forall X \subseteq Y)\,\varphi(X) \overset{\text{def}}{\Longleftrightarrow} \forall X\,\big(X \subseteq Y \to \varphi(X)\big);$$

$$(\exists X \subseteq Y)\,\varphi(X) \overset{\text{def}}{\Longleftrightarrow} \exists X\,\big(X \subseteq Y \wedge \varphi(X)\big).$$

We can now formulate a simple theorem with our new concepts:

Theorem 2.1. *The empty set is part of every set, and every set is part of the universal set.*

$$\forall x\, \emptyset \subseteq x \wedge \forall x\, x \subseteq \mathsf{V}.$$

Proof: If the first conjunct was false, then the empty set would have an element, and if the second was false, there would be a set which is not an element of the universal set. □

Continuing the list of set abstractions, let us now introduce some basic set operations:

Complement The complement of a set u is the set of those sets that are not in u:

$$\overline{u} \stackrel{\text{def}}{=} \{x : x \notin u\}.$$

Boolean union The union of the sets u and v is the set of those sets that are either in u or in v:

$$u \cup v \stackrel{\text{def}}{=} \{x : x \in u \vee x \in v\}.$$

Boolean intersection The intersection of the sets u and v is the set of those sets that are both in u and in v:

$$u \cap v \stackrel{\text{def}}{=} \{x : x \in u \wedge x \in v\}.$$

Theorem 2.2. *The naïve set-theoretical universe with its distinguished elements $\emptyset$ and V, and its basic operations $^-$, $\cup$ and $\cap$ form a Boolean algebra; that is, the following are true:*

unit elements $\forall x\, x \cup \emptyset = x$, $\forall x\, x \cup \mathsf{V} = \mathsf{V}$, $\forall x\, x \cap \emptyset = \emptyset$, $\forall x\, x \cap \mathsf{V} = \emptyset$;

idempotence $\forall x\, x \cup x = x$, $\forall x\, x \cap x = x$;

associativity $\forall x \forall y \forall z\, x \cup (y \cup z) = (x \cup y) \cup z$, $\forall x \forall y \forall z\, x \cap (y \cap z) = (x \cup y) \cup z$;

commutativity $\forall x \forall y\, x \cup y = y \cup x$, $\forall x \forall y\, x \cup y = y \cup x$;

absorption $\forall x \forall y\, (x \cup y) \cap x) = x$, $\forall x \forall y\, (x \cap y) \cup x) = x$;

distributivity $\forall x \forall y \forall z\, (x \cup y) \cap z = (x \cap z) \cup (y \cap z)$, $\forall x \forall y \forall z\, (x \cap y) \cup z = (x \cup z) \cap (y \cup z)$;

rules of complementation $\overline{\emptyset} = \mathsf{V}$, $\overline{\mathsf{V}} = \emptyset$, $\forall x \overline{\overline{x}} = x$, $\forall x\, x \cup \overline{x} = \mathsf{V}$, $\forall x\, x \cap \overline{x} = \emptyset$;

De Morgan laws $\forall x \forall y\, (x \cup y) = \overline{(\overline{x} \cap \overline{y})}$, $\forall x \forall y\, (x \cap y) = \overline{(\overline{x} \cup \overline{y})}$.

Proof: Note that many items in the list are redundant; for example, $\overline{\mathsf{V}} = \emptyset$ is an obvious consequence of $\overline{\emptyset} = \mathsf{V}$ and $\forall x \overline{\overline{x}} = x$. There are many ways to reduce the list. The proofs of the list items are obvious consequences of the corresponding Boolean laws of the logical constants $\perp$ and $\top$, and the logical connectives $\neg$, $\wedge$, and $\vee$. As an example, we give an indirect proof of one of the De Morgan laws. Assume that for some sets x and y,

$$(x \cap y) \neq \overline{(\overline{x} \cup \overline{y})}.$$

By the axiom f extensionality, this means that either (i) there is a z such that $z \in x \cap y$, but $z \notin \overline{(\overline{x} \cup \overline{y})}$, or there is a z such that $z \in x \cap y$, but $z \notin \overline{(\overline{x} \cup \overline{y})}$, or (ii) there is a z such that $z \notin x \cap y$, but $z \in \overline{(\overline{x} \cup \overline{y})}$. In case (i) we get that the formula

$$z \in x \wedge z \in y$$

is true, while

$$\neg(\neg\, z \in x \vee \neg\, z \in y)$$

is false; while in case (ii) we get that the first of these formulas is false, while the second is true. But these formulas are equivalent in standard first-order logic. The rest of the proofs follow the same pattern. □

Instead of describing the naïve set-theoretical universe as a Boolean algebra, we can also describe it as a Boolean lattice. For this, besides the above results, it suffices to prove that the subset relation is a partial order on the universe. The formulation and the proof of this claim are routine.

To reconstruct the results of section 1, we need to introduce the concept of a one-one correspondence. We will represent a one-one correspondence as a set of ordered pairs; so first we need to definee this concept. There are many ways to do this; let us see one that is standard in the literature, due to Kazimierz Kuratowski (1922). It is based on the concepts of a singleton set and of a pair set; so let's start with these.

Singleton set To define a set that has the sets exclusively u as its elements, $\varphi(x)$ has to express the property of being identical with u:

$$\{u\} \stackrel{\text{def}}{=} \{x : x = u\}.$$

Pair set Similarly, to define a set that has the sets exclusively u and v as its elements, $\varphi(x)$ has to express the property of being identical with either u or v:

$$\{u, v\} \stackrel{\text{def}}{=} \{x : x = u \vee x = v\}.$$

We can iterate pairing to get many membered sets the following way:

$$\{x_1, \ldots, x_n, x_{n+1}\} \stackrel{\text{def}}{=} \{x_1, \ldots, x_n\} \cup \{x_{n+1}\}.$$

Ordered pair The ordered pair of u and v is the pair set of the singleton set $\{u\}$ and the pair set $\{u, v\}$:

$$\langle u, v\rangle \stackrel{\text{def}}{=} \{x : \forall y\,(y \in x \leftrightarrow y = u) \vee \forall y\,(y \in x \leftrightarrow y = u \vee y = v)\}$$

—or more simply

$$\langle u, v\rangle \stackrel{\text{def}}{=} \{\{u\}, \{u, v\}\}.$$

This is the first case that we see a construction with iterated use of elementhood: $u \in \{u\}$ and $\{u\} \in \langle u, v\rangle$. There will be many of these later on; in the most interesting cases the iteration will be unlimited. The simplest case of such an unlimited iteration is the case of n-tuples. 2-tuples are ordered pairs; and once we have n-tuples, we can define ordered $n + 1$-tuples as

$$\langle u_1 \ldots, u_n, u_{n+1}\rangle \stackrel{\text{def}}{\iff} \langle\langle u_1 \ldots, u_n\rangle, u_{n+1}\rangle.$$

We prove a simple theorem about the adequacy of Kuratowski's definition of an ordered pair:

Theorem 2.3. *Ordered pairs are identical iff both their first members and their second members are identical.*

$$\forall x\,\forall y\,\forall x'\,\forall y'\,\big(\langle x, y\rangle = \langle x', y'\rangle \leftrightarrow (x = x' \wedge y = y')\big).$$

Proof: One direction is obvious. As for the other direction, assume that for some x, y, x', and y', $\big(\langle x, y\rangle = \langle x', y'\rangle$; that is, $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$. By the axiom of extensionality there are two options: (a) $\{x\} = \{x'\}$ and $\{x, y\} = \{x', y'\}$; (b) $\{x\} = \{x', y'\}$ and $\{x, y\} = \{x'\}$. In case (a), $x = x'$ by extensionality, and by another application of extensionality, either $y = y'$, or $x = y'$ and $y = x'$, which also implies $y = y'$. In case (b), by extensionality we have $x = x'$, $x = y'$, and $y = x'$, which imply $y = y'$. □

Now we are in the position to talk about sets of ordered pairs. In general, such a set is called a relation. A one-one correspondence between sets is a special kind of relation, and henceforth we will call it a *bijection* from one set into the other. For now, we give a direct definition of a bijection; in section **??** it will be part of a more elaborated system of related concepts.

Bijection A set x is a bijection from a set y to a set z if x if and only if

(i) every element of x is an ordered pair of an element of y and an element of z;

(ii) every element of y occurs exactly once as a first member in such a pair;

(iii) every element of z occurs exactly once as a second member in such a pair.

$$\mathsf{Bij}(x,y,z) \stackrel{\text{def}}{\Longleftrightarrow} (\forall u \in x)\,(\exists v \in y)\,(\exists w \in z)\, u = \langle v, w\rangle \wedge$$
$$(\forall v \in y)\,(\exists! w \in z)\,\langle v, w\rangle \in x \wedge (\forall w \in z)\,(\exists! v \in y)\,\langle v, w\rangle \in x.$$

Now we can talk about equinumerosity in terms of bijection:

Equinumerosity A set y is equinumeros with a set z—in symbols: $y \approx z$—iff there is a bijection from y to z:

$$y \approx z \stackrel{\text{def}}{\Longleftrightarrow} \exists x\, \mathsf{Bij}(x,y,z)$$

We call a set y larger than or equinumerous with a set x—in symbols: $x \lesssim y$—iff x is equinumerous with a subset of y:

$$x \lesssim y (\exists y' \in y)\, x \approx y'.$$

Accordingly, y is strictly larger than x iff $x \lesssim y$, but $x \not\approx y$.

Let us state a simple fact regarding equinumerosity:

Theorem 2.4. *Equinumerosity is an equivalence relation; it is*

(i) *reflexive:* $\forall x\, x \approx x$,

(ii) *symmetrical:* $\forall x\, \forall y\, (x \approx y \rightarrow y \approx x)$*, and*

(iii) *transitive:* $\forall x\, \forall y\, \forall z\, ((x \approx y \wedge y \approx z) \rightarrow x \approx z)$.

Proof: For (i), let x be an arbitrary set, and consider $e = \{\langle v, v\rangle : v \in x\}$. For (ii), let x, y, and a bijection e from x to y be given; and consider $f = \langle v, w\rangle : \langle w, v\rangle \in x$. For (iii), let x, y, and z be given, along with the bijection e from x to y, and the bijection f from y to z. Consider $g = \{\langle v, w\rangle : \exists u\, (\langle v, u\rangle \in e \wedge \langle u, w\rangle \in f)\}$. □

We are not yet able to prove the corresponding claim about $\lesssim$ being a partial ordering. Reflexivity and transitivity are roitine to prove; but we can't prove antisymmetry until we complete the proof of the Schröder–Bernstein theorem, which relies on the concept of natural numbers. A formal reconstruction of other results of section 1 also demands an introduction of the concept of natural numbers; so let us turn to them now. There are many options to do that; including

(i) the Frege Numbers;

(ii) the Neumann numbers;

(iii) the Zermelo numbers.

In the subsequent chapters we will use Neumann numbers, since Zermelo' number construction has some technical disadvantages, and the Fregean construction simply doesn't go through. However, it will be instructive to see how and where Frege's number definition fails, so it is worth to discuss Frege numbers here.

The Fregean idea is to identify the cardinality of a set with the set of the sets equinumerous with it. This is an application of a general strategy to identify abstract objects with equivalence classes. We have just seen that equinumerosity is an equivalence relation, so it partitions the universe into equinumerosity classes. Thus the Fregean definition of cardinality is simple and straightforward:

$$|u|_{\mathsf{F}} \overset{\text{def}}{=} \{x : x \approx u\}.$$

We use the subscript F along with each notation in the Fregean construction, to distinguisb them from the analogous notations, in the Neumann construction. Let us see two examples. The Frege cardinality of the empty set is the set of all empty sets:

$$|\emptyset|_{\mathsf{F}} = \{\emptyset\};$$

while the cardinality of this set is the set of all singletons:

$$|\{\emptyset\}|_{\mathsf{F}} = \{x : \exists z\, x = \{z\}\}.$$

Note that the following is a bijection from the universal set to the set of singletons:

$$f = \big\{\langle x, x'\rangle \wedge x' = \{x\}\big\}.$$

Thus, while the cardinality of the empty set is singleton, the cardinality of the cardinality of $\emptyset$ is as large as the universe. And the same is true regarding all further cardinals. The set of cardinals can also be defined simply as

$$\mathsf{Card}_{\mathsf{F}} \overset{\text{def}}{=} \{x : \exists z\, |z|_{\mathsf{F}} = x\}.$$

Let us now turn to natural numbers. We define zero as the cardinality of the empty set:

$$0_{\mathsf{F}} \overset{\text{def}}{\iff} |\emptyset|_{\mathsf{F}}.$$

The next step is successorship, the operation that assigns to each number its successor. The number n is a collection of all n-membered sets; and its successor $n+1$ is the collection of all $n+1$-membered sets. So what the operation does is that it adds a new element to each member of n in every possible ways. Putting it formally:

$$\mathsf{s}_{\mathsf{F}} x = \{x : \exists z\, \exists v\, (z \in x \wedge v \notin z \wedge x = z \cup \{v\})\}.$$

Now we can define one as the successor of zero (this is the set of singletons discussed above), two as the successor of one (the set of doubletons or pair sets), and so on for all the standard natural numbers:

$$1_{\mathsf{F}} \overset{\text{def}}{=} s_{\mathsf{F}} 0$$
$$2_{\mathsf{F}} \overset{\text{def}}{=} s_{\mathsf{F}} 1$$
$$\dots$$

But the question is still open whether there exists a set of natural numbers. Our next job is to introduce this set by set abstraction.

Let us call a set that contains zero and is closed under successorship a Fregean inductive set:

$$\mathsf{Ind}_{\mathsf{F}}(x) \overset{\text{def}}{\iff} 0_{\mathsf{F}} \in x \wedge (\forall y \in x)\, \mathsf{s}_{\mathsf{F}} y \in x.$$

The set of natural numbers will be such a set. But it is certainly not the only one; the universal set V is Fregean inductive, too. But V has also elements that are not natural numbers; for example, V itself is in V, and it is not a natural number, since not all elements of V are equinumerous. So, what distinguishes the set of natural numbers is that it doesn't have redundant elements. This is actually a property that we can use defining it; the set of Fregean natural numbers is the set of sets which are elements of every Fregean inductive set:

$$\mathbb{N}_{\mathsf{F}} \overset{\text{def}}{=} \{x : \forall z\, (\mathsf{Ind}_{\mathsf{F}}(z) \rightarrow x \in z)\}.$$

The fact that there is at least one inductive set warrants that this set defines a nonempty set. Moreover:

Theorem 2.5. *The set of Fregean natural numbers is inductive.*

$$\mathsf{Ind}(\mathbb{N}_\mathsf{F}).$$

Proof: Zero is in every inductive set, so it is in $\mathbb{N}_\mathsf{F}$, too. Let x be in $\mathbb{N}_\mathsf{F}$. By the definition of $\mathbb{N}_\mathsf{F}$ x is in every inductive set, and by the definition of an inductive set $\mathsf{s}_\mathsf{F}x$ is in every inductive set, too. So, by definition of $\mathbb{N}_\mathsf{F}$ $\mathsf{s}_\mathsf{F}x$ is in $\mathbb{N}_\mathsf{F}$, too. □

It is time now to build arithmetic over $\mathbb{N}_\mathsf{F}$. First we define the basic binary operations of addition and multiplication. We model operations as sets of ordered triples, the first two members of which being the input, and the third being the output. We will denote these sets as Add_F and Mult_F. In their definitions, We use the same technique as we did in the case of the set of natural numbers; we introduce Add_F and Mult_F as the smallest sets of triples of natural numbers that obey the Peano axioms of these operations. In the case of addition, one of the axioms says that for all numbers n, $n + 0 = n$; in terms of triples it will amount to the condition that triples of the form $\langle x, 0, x\rangle$ belong to the set Add_F. The other axiom says that for all numbers n and m, $n + \mathsf{s}m = \mathsf{s}(n + m)$; in terms of triples it amounts to the condition that whenever $\langle x, x', x''\rangle$ is in Add_F, $\langle x, \mathsf{s}_\mathsf{F}x', \mathsf{s}_\mathsf{F}x''\rangle$ is also in Add_F. Let us call a set that satisfies these two conditions addition-inductive:

$$\mathsf{Ind\text{-}add}_\mathsf{F}(y) \stackrel{\text{def}}{\iff} \forall z\, \langle z, 0, z\rangle \in y \land \forall z\, \forall z'\, \forall z''\, (\langle z, z', z''\rangle \in y \to \langle z, \mathsf{s}_\mathsf{F}z', \mathsf{s}_\mathsf{F}z''\rangle \in y))$$

Now Add_F is the smallest induction-additve set; that is, it contains exactly those triples which are in every induction-additive set.

$$\mathsf{Add}_\mathsf{F} \stackrel{\text{def}}{=} \{\langle x, x', x''\rangle : \forall y(\mathsf{Ind\text{-}add}(y) \to \langle x, x', x''\rangle \in y)\}.$$

Whenever a triple $\langle x, x', x''\rangle$ is in Add_F, we say that x'' is the sum of x and x'. Since we have no warrant that there is always exactly one sum, both the definite article and the following familiar notation are as yet unjustified:

$$x +_\mathsf{F} x' = x'' \stackrel{\text{def}}{\iff} \langle x, x', x''\rangle \in \mathsf{Add}_\mathsf{F}$$

The definition of multiplication goes a very similar way, and it is left for the reader as an exercise.

Now we prove that the resulting system is a model of Peano arithmetic, and that addition and multiplication are operations, that is all pairs of natural numbers x, x' have exactly one sum and exactly one product:

Theorem 2.6. *The structure* $\langle \mathbb{N}_\mathsf{F}, 0_\mathsf{F}, \mathsf{Add}_\mathsf{F}, \mathsf{Mult}_\mathsf{F}\rangle$ *satisfies the Peano axioms:*

(i) $\neg(\exists x \in \mathbb{N}_\mathsf{F})\, \mathsf{s}_\mathsf{F}x = 0$;

(ii) $(\forall x \in \mathbb{N}_\mathsf{F})\, (\forall y \in \mathbb{N}_\mathsf{F})\, \mathsf{s}_\mathsf{F}x = \mathsf{s}_\mathsf{F}y \to x = y$;

(iii) $(\varphi(0_\mathsf{F}) \land (\forall x \in \mathbb{N}_\mathsf{F})\, (\varphi(x) \to \varphi(\mathsf{s}_\mathsf{F}x)) \to (\forall x \in \mathbb{N}_\mathsf{F})\, \varphi(x)$;

(iv) $(\forall x \in \mathbb{N}_\mathsf{F})\, \langle x, 0_\mathsf{F}, x\rangle \in \mathsf{Add}_\mathsf{F}$;

(v) $(\forall x \in \mathbb{N}_\mathsf{F})\, (\forall y \in \mathbb{N}_\mathsf{F})\, (\langle x, y, z\rangle \in \mathsf{Add}_\mathsf{F} \to \langle x,_\mathsf{F} y,_\mathsf{F} z\rangle \in \mathsf{Add}_\mathsf{F})$;

(vi) $(\forall x \in \mathbb{N}_\mathsf{F})\, (\forall y \in \mathbb{N}_\mathsf{F})\, (\exists! z \in \mathbb{N}_\mathsf{F})\, \langle x, y, z\rangle \in \mathsf{Add}_\mathsf{F}$;

(vii) $(\forall x \in \mathbb{N}_\mathsf{F})\, \langle x \times_\mathsf{F}, 0_\mathsf{F}, 0_\mathsf{F}\rangle \in \mathsf{Mult}_\mathsf{F}$;

(viii) $(\forall x \in \mathbb{N}_\mathsf{F})\, (\forall y \in \mathbb{N}_\mathsf{F})\, (\forall z \in \mathbb{N}_\mathsf{F})\, (\forall u \in \mathbb{N}_\mathsf{F})$
$\big((\langle x, \mathsf{s}_\mathsf{F}y, z\rangle \in \mathsf{Mult}_\mathsf{F} \land \langle x, y, u\rangle \in \mathsf{Mult}_\mathsf{F}) \to \langle u, x, z\rangle \in \mathsf{Add}_\mathsf{F}\big)$;

(ix) $(\forall x \in \mathbb{N}_\mathsf{F})\, (\forall y \in \mathbb{N}_\mathsf{F})\, (\exists! z \in \mathbb{N}_\mathsf{F})\, \langle x, y, z\rangle \in \mathsf{Mult}_\mathsf{F}$.

Proof: (i) If zero were a successor, by definition of successorship all its elements would have at least one element; but the sole element of zero is empty.

Now we turn to the induction scheme (iii), which we will use in the proof of (ii).

(iii) Let us consider the set $\{x : \varphi(x)\}$. This is by assumption an inductive set; so the set of Fregean natural numbers is a subset of it. Now we can use the induction scheme proving the rest of the clauses.

Let us now prove a simple lemma: *Every Fregean number is a Fregean cardinal.* This is obviously true of zero, and inherited from each number to its successor; so by (iii) it is true of every number. We use this fact in the proof of (ii).

(ii) Let x, y and z be given in such a way that $z = \mathsf{s_F}x = \mathsf{s_F}y$. By definition of successorship $\mathsf{s_F}x$ is not empty; so let w be in $\mathsf{s_F}x$. By assumption there are sets z and v such that z is in x, $v \notin z$, and $w = z \cup \{v\}$. The same way there are sets z' and v' such that z' is in y, $v' \notin z'$, and $w = z' \cup \{v'\}$. Now let's define a bijection f from z to z' as follows: for every $u \in z$, if $u \neq v'$, then $f(u) = u$; and $f(v') = v$. This bijection warrants that z is equinumerous with z'. Since both x and y are Fregean numbers, by the above lemma they are also Fregean cardinals; therefore x is the set of all sets equinumerous with z, and y is the set of all sets equinumerous with z'. Since z and z' are equinumerous, x and y are the same set.

(iv)–(v) These two clauses amount to the claim that the set $\mathsf{Add_F}$ above is addition-inductive. This can be proved the same way as the claim that $\mathbb{N}_\mathsf{F}$ is additive. Every addition-inductive set contains all triples of natural numbers of the form $\langle z, 0, z\rangle$, so their intersection also does so. Now let $\langle z, z', z''\rangle$ be in $\mathsf{Add_F}$. By assumption it is also in every addition-inductive set; so by definition of addition-inductivity $\langle z, \mathsf{s_F}z', \mathsf{s_F}z''\rangle$ is also in every addition-inductive set; but then it is in their intersection, too.

(vi) We will use (iii). Let $\varphi(y)$ be the claim that for all x in $\mathbb{N}_\mathsf{F}$, there's exactly one z in $\mathbb{N}_\mathsf{F}$ such that $\langle x, y, z\rangle$ is in $\mathbb{N}_\mathsf{F}$:

$$(\forall x \in \mathbb{N}_\mathsf{F})\,(\exists! z \in \mathbb{N}_\mathsf{F})\,\langle x, y, z\rangle \in \mathsf{Add_F}$$

(a) First we prove $\varphi(0_\mathsf{F})$. For this step itself we need another application of (iii); this tim ex is the distinguished variable, and we consider the following formula $\psi(x)$: @@@

(b) Now we prove that if for some y in $\mathbb{N}_\mathsf{F}$ $\varphi(y)$ holds, then $\varphi(\mathsf{s_F}y)$ also holds. This requires us to use (iii) a third time. @@@

Finally, by (iii), (a), and (b), (vi) is proved. (Applications of (iii) are called proofs by induction; and constructions like te present one are called embedded induction.)

Clauses (vii)–(ix) are left for the reader as exercises. □

Finally we are in the position that we can formally prove the main intuitive results of the previous section. As for the Hilbert's Hotel arguments, the task is obvious; we can define various subsets of $\mathbb{N}_\mathsf{F}$, and prove the equinumerousities the arguments claims. One of these reconstructions is included in the exercises. Let us now consider the Schröder–Bernstein theorem:

Theorem 2.7. *(Schröder–Bernstein)*

$$\forall x\,\forall y\,\big((x \lesssim y \wedge y \lesssim x) \rightarrow x \approx y\big)$$

Proof: It goes the very same way as in the informal case. We use Fregean natural numbers in defining the four sequences of sets needed in the proof: H_n, G_n, A_n, and B_n; the numbers in the subscripts refer to one-one correspondences between themembers of the sequences and natural numbers. We make heavy use of set abstraction. □

To reconstruct Cantor's theorem, we need a formal definition of the concept of a power set:

Power set The power set of a set u is the set of all subsets of u:

$$\mathcal{P}(h) \stackrel{\text{def}}{=} \{x : x \subseteq h\}.$$

Now we can reformulate Cantor's theorem as follows:

Theorem 2.8. *(Cantor)*

(i) $\neg \exists x \, x \approx \mathcal{P}(x)$;

(ii) $\forall x \, (\exists y \subseteq \mathcal{P}) \, (x) x \approx \mathcal{P}(y)$.

Proof: The same way as we did in the last section. Once again, we make heavy use of set abstraction. □

After so many positive results, let us see the negative ones. We prove two of the many well-known paradoxes of naïve set theory.

Theorem 2.9. (i) *(Russell's Paradox) There is a set* R *such that* $\mathsf{R} \in \mathsf{R}$ *if and only if* $\mathsf{R} \notin \mathsf{R}$.

(ii) *(Cantor's paradox) The universal set is and is not both equinumerous with its power set.*

Proof: (i) Consider the set

$$\mathsf{R} \stackrel{\text{def}}{=} \{x : x \notin x\}.$$

By definition,

$$x \in \mathsf{R} \Leftrightarrow x \notin x.$$

As a special case,

$$\mathsf{R} \in \mathsf{R} \Leftrightarrow \mathsf{R} \notin \mathsf{R}$$

—which is a contradiction. (ii) On the one hand, every set is a subset of V, so $\mathsf{V} \subseteq \mathcal{P}(\mathsf{V})$. On the other hand, every subset of V is a set, so $\mathcal{P}(\mathsf{V}) \subseteq \mathsf{V}$. Thus, $\mathsf{V} = \mathcal{P}(\mathsf{V})$, and every set is equinumerous with itself. This contradicts Cantor's theorem, according to which V cannot be equinumerous with $\mathcal{P}(\mathsf{V})$. □

This completes our long discussion of an inconsistent theory. Now we have to construct a hopefully more secure one; but t do so, first we have to develop our strategy, based on a careful analysis of the paradoxes.

Exercises 1. Define the following sets and set operations with set abstraction:

(a) $\bigcup u$, the union set of u (the set of the elements of the elements of u);

(b) $u \times v$, the Cartesian product of u and v; (the set of all ordered pairs $\langle x, y \rangle$ such that x is in u and y is in v;

(c) $u \uplus v$, the disjoint Boolean union of u and v, where the elements of u and we are labeled in such a way that one can tell which element is from which set.

2. The first set-theoretical definition of ordered pair, put forward by the American mathematician Norbert wiener, is the following:

$$\langle u, v \rangle_{\mathsf{W}} = \big\{\{\{u\}, \emptyset\}, \{\{v\}\}\big\}$$

Prove theorem 2.3 regarding Wiener pairs.

3. Sometimes we find the following simplified version of Kuratowski pairs in discussions of set theory:

$$\langle u, v \rangle_{\mathsf{s}} = \{u, \{u, v\}\}$$

Why does this definition fail in a set theory where there's a universal set?

4. Define Fregean multiplication the same way we defined Fregean addition, and prove the basic facts above it, as we did with addition.

5. Define the set of Fregean even numbers, and show that it is equinumerous with the set of Fregean naturals.

3 Ways out of the paradoxes

If we compare the two paradoxes at the end of the previous chapter, we can observe that they are different in complexity. While Cantor's paradox depends on a number of set-theoretical assumtions enumerated at the end of section 1, all of which follow from different instances of the comprehension scheme, Russell's paradox comes down to a single instance of that scheme:

$$\exists y\, \forall x\, (x \in y \leftrightarrow x \notin x)$$

Note that this in itself is a contradiction. The naïve comprehension scheme is not only inconsistent in the sense that its various instances contradict each other; it has at least one contradictory instance.

As a consequence of the paradoxes, we have to get rid of naïve comprehension. This is a painful sacrifice, since this was our main resource in establishing the existences of set constructions that we needed in the proofs of our main results of the last section.

[. . .]

4 Gödel–Bernays set theory

Language We use standard first-order logic with identity. Like in naive set theory, the only extralogical constant is $\in$. Conventionally, individual variables are capitalized (smallcase variables will be defined soon). In the intended interpretation quantifiers range over classes, and "$X \in Y$" means that the class X is in the class Y.

Classes and sets Sets are classes that are elements of some classes:

$$M(X) \overset{\text{def}}{\iff} \exists Y\; X \in Y$$

Smallcase variables range over sets. They are introduced as shorthands in the contexts of formulas:

$$\forall x\, \varphi(x) \overset{\text{def}}{\iff} \forall X\, (M(X) \to \varphi(X))$$

$$\exists x\, \varphi(x) \overset{\text{def}}{\iff} \forall X\, (M(X) \land \varphi(X))$$

$$\varphi(x) \overset{\text{def}}{\iff} M(X) \land \varphi(X)$$

—where X has no free occurrences in $\varphi(x)$.

Remark In Gödel's original formulation of GB, sethood and classhood were primitive concepts. Instead of the above definition, special axioms guaranteed that every set is a class, and that elements of classes are sets.

Besides restricted variables, we also apply some usual notations ways to express restricted quantification:

$$(\forall x \in Y)\, \varphi(x) \overset{\text{def}}{\iff} \forall x\, \big(x \in Y \to \varphi(x)\big);$$

$$(\exists x \in Y)\, \varphi(x) \overset{\text{def}}{\iff} \exists x\, \big(x \in Y \land \varphi(x)\big);$$

$$(\forall X \subseteq Y)\, \varphi(X) \overset{\text{def}}{\iff} \forall X\, \big(X \subseteq Y \to \varphi(X)\big);$$

$$(\exists X \subseteq Y)\, \varphi(X) \overset{\text{def}}{\iff} \exists X\, \big(X \subseteq Y \land \varphi(X)\big).$$

The role of all these abbreviations is to make formulas more readable.

Now we can start formulating axioms for a more carefully built theory that we hope to be free from paradoxes. Of the two axioms of naïve set theory, extensionality should be true of classes in general.

Axiom 4.1 (Class extensionality: AxExt)**.** *Classes with the same elements are the same.*

$$\forall X\, \forall Y\, (\forall z\, (z \in X \leftrightarrow z \in Y) \rightarrow X = Y)$$

As for comprehension, it will introduce classes, not sets. Also, we apply a formal restriction on the formula $\varphi(x)$; quantification will be restricted to sets. The aim of this restriction is to keep the universe of classes moderate. We will see later that the resulting system is finitely axiomatizable.

Remark Without this restriction we do not have finite axiomatizability. the theory that results if we omit this restriction is called Morse–Kelley set theory, and abbreviated as MK (or KM). MK is much stronger than the present one, in the sense that a lot of formulas undecidable in GB are decidable in MK (including the arithmetic formula that expresses the consistency of GB).

Axiom 4.2 (Class comprehension: AxComp)**.** *For every property defined in the language of set theory with a formula in which quantification is restricted to sets, there is a class of those sets that have this property.*

$$\exists Y\, \forall x\, (x \in Y \leftrightarrow \varphi(x))$$

—where all bound variables in $\varphi(x)$ are set variables.

These two axioms provide us with a splitted universe of classes in which we can hopefully get rid of the classical set-theoretic paradoxes. Most of the definitions of naïve set theory make sense in the new framework, too, so we are going to use the symbols "$\emptyset$", "$\{x\}$", "$\{x, y\}$", "$\bigcup X$", "$\mathcal{P}X$", $X \subseteq Y$ etc. The only exceptions are disjoint union and Cartesian product, which require the sethood of Kuratowski-pairs.

Some of these definitions introduce proper classes, thus turning the classical paradoxes into harmless simple theorems. For example:

Theorem 4.1 (Russell)**.** *The Russell class is a proper class.*

$$\neg\, \mathsf{M}(\mathsf{R})$$

Proof. Suppose that R is a set. Then $\mathsf{R} \in \mathsf{R}$ if and only if $\mathsf{R} \notin \mathsf{R}$. □

Note that we did not show that there is no contradiction hidden in or in our refined comprehension principle. What this proof shows is merely that Russell's argument doesn't lead to contradiction in the way it did in naïve set theory.

But this solution of the paradoxes comes with a price. Nothing in the above two axioms guarantees the sethood of apparently non-paradoxical class constructions. It is even possible that the empty class is a proper class. In this case, the universe of sets is empty, and thus every instance of the class comprehension schema defines the empty class. Or, in other words, the structure $\mathcal{M} = \langle \{u\}, \emptyset \rangle$ is a model of the above axioms: $\mathcal{M} \models \{\mathrm{AxExt}, \mathrm{AxComp}\}$, where u is the individual that plays the role of the empty class, and the extension of the elementhood predicate is empty.

To avoid an empty universe of sets, one needs to introduce set existence axioms. The simplest such axiom is the axiom of the empty set:

Axiom 4.3 (Empty Set: AxEmpty)**.** *The empty class is a set.*

$$\mathsf{M}(\emptyset)$$

This axiom will later prove to be superfluous; however, it is good to include it here, since it has an important role in the heurictic process of constructing our axiom system. On the other hand, this axiom still does not provide us with a large set universe: $\mathcal{M} \not\models \{\mathrm{AxExt}, \mathrm{AxComp}, \mathrm{AxEmpty}\}$; but if we let , $\mathcal{M}' = \langle \{u_0, u_1\}, \{\langle u_0, u_1 \rangle\} \rangle$, we get a structure in which there are only two classes, of which u_0 plays the role of the empty set, and u_1 that of its singleton, which is a proper class, then $\mathcal{M}' \models \{\mathrm{AxExt}, \mathrm{AxComp}, \mathrm{AxEmpty}\}$.

To avoid a singleton universe, one might introduce an axiom of singletons:

Axiom 4.4 (Singletons: AxSing)**.** *Singleton classes are sets.*

$$\forall x\, \mathsf{M}(\{x\})$$

Clearly, $\mathcal{M}' \not\models \{\text{AxExt}, \text{AxComp}, \text{AxEmpty}, \text{AxSing}\}$. Moreover, in any model of the axioms intruduced sofar, there are infinitely many sets, since iterated application of AxExt shows that none of the sets in the series $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$ are identical.

However, the axiom of singletons still allows for a model $\mathcal{M}''$ which consists of infinitely many individuals $u_0, u_1, u_2, \ldots$ representing the members of the series $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$, and another infinite stack of individuals representing the definable collections of $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$ as proper classes. Elementhood is interpreted in an obvious way. Now, $\mathcal{M}'' \models \{\text{AxExt}, \text{AxComp}, \text{AxEmpty}, \text{AxSing}\}$; and we certainly need more than such a narrow universe.

To get sets with more than one element, one can introduce an axiom of pair sets:

Axiom 4.5 (Pairs: AxPair)**.** *Pair classes are sets.*

$$\forall x\, \forall y\, \mathsf{M}(\{x, y\})$$

Note that the AxSing is the special case of AxPair with $x = y$; thus AxPair $\Rightarrow$ AxSing.

With iterated application of of AxPair we can prove the existence and sethood of ordered Kuratowski-pairs (or simply ordered pairs):

Kuratowski-pair $\langle x, y\rangle =^{\text{def}} \{\{x\}, \{x, y\}\}$

The following theorem shows that this definition is correct:

Theorem 4.2. *Kuratowski-pairs are the same iff both heir first and second members are the same.*

$$\forall x\, \forall y\, \forall x'\, \forall y'\, \big(\langle x, y\rangle = \langle x', y'\rangle \leftrightarrow (x = x' \wedge y = y')\big)$$

Proof. One direction of the biconditional is obvious. As for the other direction, let $\langle x, y\rangle = \langle x', y'\rangle$; that is, $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$. By extensionality, either $\{x\} = \{x'\}$ or $\{x\} = \{x', y'\}$. If $\{x\} = \{x'\}$ is the case, then $\{x, y\} = \{x', y'\}$, which implies that $y = y'$. If $\{x\} = \{x', y'\}$, then by extensionality $x' = y'$, and thus $x = y$, which of course implies $x = x'$ and $y = y'$. □

Now we can use Kuratowski-pairs to define Cartesian product. Let X and Y be arbitrary classes. Then by the class comprehension schema, the class of ordered pairs of elements of X and Y exist, so the following definition is correct:

$$X \times Y \stackrel{\text{def}}{=} \{\langle x, y\rangle : x \in X \wedge y \in Y\}.$$

Cartesian power is a generalization of Cartesian product, defined recursively as

$$X^2 \stackrel{\text{def}}{=} X \times X;$$

$$X^{n+1} \stackrel{\text{def}}{=} X^n \times X.$$

Note that the natural numbers in this definition belong to the metalanguage; thus the second clause is a schema that abbreviates infinitely many defining clauses.

The concept of a Cartesian product will prove to be useful in the introduction of many basic concepts like that of a relation and a function. But to get those results, we need further axioms which guarantee that the Cartesian product of two sets is always a set. At the moment it is even undecidable whether the four-membered Cartesian product of two disjoint pair sets is a set or a proper class.

To get sets with more than two elements, we can use the well-known operation of set union. The union of a class H is the class of the elements of the elements of H; that is;

$$\bigcup H \stackrel{\text{def}}{=} \{x : \exists y\, y \in H\}.$$

The axiom of set union makes sure that whenever H is a set, $\bigcup H$ is a set, too.

Axiom 4.6 (Union: AxUnion)**.** *The union class of a set is a set.*

$$\forall x\, \mathsf{M}(\bigcup x)$$

Now we can construct finite sets as large as we want by iterated applications of AxPair and AxUnion on the empty set. For example,

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \bigcup \{\{\emptyset, \{\emptyset\}\}, \{\{\emptyset, \{\emptyset\}\}\}\}$$

—where every component of the expression on the right hand side is either the empty set or a singleton or a pair or a union set. Boolean union is a useful shorthand in expressing constructions like this, due to the fact that by definition of the class operations, $\bigcup$ and $\cup$,

$$\forall x\, \forall y\, x \cup y = \bigcup \{x, y\}.$$

Expressed in terms of Boolean union, the above construction becomes easier to read:

$$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\}$$

We can apply some other Boolean class operations like Boolean intersection, subtraction, or symmetric difference; but not complement. Nothing in the above axioms guarantee that the complement class of a set is a set. This is for a reason; we do not want very large classes like the universal class—the complement of the empty set—to be sets. Later on we introduce axioms that are incompatible with there being a universal set. With those axioms assumed, even the existence of an arbitrary set complement of which is a set results in contradiction.

We have seen that once we are equipped with the above axioms, we can handle finite set constructions. But there is still nothing in those axioms that would make the existence of an infinite set necessary. Remember from section 1 that we have two informal concepts of infinity; Dedekind's one and the one that is connected with the concept of natural numbers. Of these, the second one is stronger and we actually need this strength. So, we have to represent the series of natural numbers in our system in some way. The details of this are going to be developed in the next section; for now it will suffice to define some set-theoretic representations for zero and the successor operation. For zero, we choose the empty set, and we define the successor of a set h as $h \cup \{h\}$. These definitions originate in János (Johann von) Neumann's work in the 1920s, and we are going to investigate them in detail in section 5. For now it suffices to make it sure the class of what we are going to call Neumann numbers is a set. We call this class ω.

If we want to assume as an axiom that ω is a set, first we need to prove that it exists as a class; that is, we have to define it with a single formula. This is not as routine as it was in the case of the other set existence axioms; in fact, it is not even possible. The axioms that we have assumed sofar allow for a model $\mathcal{M}'''$ in which every Neumann number exists, but their class does not.

It is not hard to overcome this problem, but to understand the content of the axiom that comes up as a solution, it is worth to see first how the problem arises.

What we have for ω is a recursive definition, a kind of definition well-known from the theory of first-order languages, and one that appears in the theory of sets frequently, too. In the metatheory of logic, we generally use intuitive set theory and assume that this is a correct method of definition. In the case of an axiomatic system like GB one has to prove that it is. We return to this question in theorem 7.6 with regards to a generalization of recursive definitions called transfinite recursion.

1. $\emptyset$ is in ω;
2. if x is in ω, then so is $x \cup \{x\}$;
3. nothing else is in ω.

The closure clause—clause 3 above—of a recursive definition is usually omitted, and we are going to omit it later on; but in this particular case we want to emphasize that it is there.

To apply class comprehension, we have to transform this definition into a single formula that expresses a necessary and sufficient condition for any x to be in ω. We apply an idea here that will prove to be very useful later. Consider the closure clause, which says that ω has no elements that are not necessitated by the first two clauses. That is, any x in ω must be present in every superclass of ω. Even more precisely: x is in ω iff x is present in every class that satisfies conditions 1 and 2. This amounts to the following formula:

$$\forall x \left(x \in \omega \leftrightarrow \forall Y \left((\emptyset \in Y \wedge (\forall z \in Y)\, z \cup \{z\} \in Y) \rightarrow x \in Y\right)\right)$$

This construction is a nice example of a standard way to compress a recursive definition into a single formula, one that we will generalize in theorem 7.6.

On the other hand, the resulting formula is not an instance of the class comprehension schema. Observe that the formula that is substituted in the place of $\varphi(x)$ quantifies over classes without restriction. This is not allowed in class comprehension. For a satisfactory definition of ω, one has to substitute Y with a set variable. The resulting formula is already a formally correct one; so by class comprehension the following definition is correct, too:

$$\omega \stackrel{\text{def}}{=} \left\{x : \forall y \left((\emptyset \in y \wedge (\forall z \in y)\, z \cup \{z\} \in y) \rightarrow x \in y\right)\right\}$$

But we still cannot simply have as an axiom of infinity the assumption that ω thus defined is a set. Let us call a class Y inductive if Y contains the empty set and for every set z in Y, the set $z \cup \{z\}$. We know that there is at least one inductive class; the universal class V is an example. But we don't know whether there are inductive sets; and this question is actually undecidable based on the axioms we have accepted sofar.

From the negative result we can derive a positive idea: let us take the assumption that there is at least one inductive set as an axiom.

Axiom 4.7 (Infinity: AxInf)**.** *There is an inductive set.*

$$\exists y\, (\emptyset \in y \wedge (\forall z \in y)\, z \cup \{z\} \in y)$$

Three things deserve to be mentioned here. First, note that AxInf does not presuppose AxEmpty; instead, it implies AxEmpty. Second, AxInf does not imply the sethood of ω; all that we know is that ω is a subclass of a set. Third, we cannot prove the sethood of other infinite classes in general, except for obvious cases like $\omega \cup \{\{\emptyset\}\}$. For example, we cannot decide whether the Cartesian product $\omega \times \{\emptyset\}$, or $\omega \times \omega$ exists.

Let us start with the sethood of subclasses of sets. Since the known paradoxes of naïve set theory are all the result of unbounded set comprehension, it seems safe to assume the following axiom:

Axiom 4.8 (Separation: AxSep)**.** *A subclass of a set is always a set.*

$$\forall x\, \forall Y\, (Y \subseteq x \rightarrow \mathsf{M}(Y))$$

This is the GB version of Ernst Zermelo's famous *Axiom der Aussonderung*, also called the axiom of subsets. In ZF-style set theories it takes the form of a schema. We will see that AxSep is a consequence of some forthcoming axioms, too.

Now we can prove the sethood of ω:

Theorem 4.3. *ω is a set.*

$$\mathsf{M}(\omega)$$

Proof. By AxInf and the definition of ω, there is a set y such that ω is a subclass of y. Therefore ω is a set by AxSep. □

At this point we can make another important observation regarding the barrier between proper classes and sets. We have already mentioned it when we discussed Boolean set operations in finite set constructions.

Theorem 4.4. *There is no set such that the complement class of it is a set, too.*

$$\neg\,\exists x\,\mathsf{M}(\overline{x})$$

Proof. Towards a contradiction, suppose that for some h, $\overline{h}$ is a set. Thus by AxPair and AxUnion, $\mathsf{V} = h \cup \overline{h}$ is a set, so by separation every class is a set, including the Russell class. □

To establish the sethood of an infinite Cartesian product with the axiom of separation, we need a superset of that product. We are going use the power set of the power set of the Boolean union of h and g as a superset of $h \times g$, so we need an axiom of power sets. Of course, the power set axiom will prove useful in a number of other cases as well.

Axiom 4.9 (Power set: AxPow)**.** *The power class of a set is a set.*

$$\forall x\,\mathsf{M}(\mathcal{P}x)$$

And now we can prove a theorem asserting the sethood of Cartesian products:

Theorem 4.5. *The Cartesian product of two sets is always a set.*

$$\forall x\,\forall y\,\mathsf{M}(x \times y)$$

Proof. Every Kuratowski-pair is a pair of pair sets from the union of x and y. The class $x \cup y$ is a set by AxPair and AxUnion; by AxPow so is $\mathcal{P}(x \cup y)$ and $\mathcal{P}(\mathcal{P}(x \cup y))$. $x \times y \subseteq \mathcal{P}(\mathcal{P}(x \cup y))$, so by AxSep it is a set, too. □

Now we have settled the problem regarding the set status of the above mentioned classes $\omega \times \{\emptyset\}$ and $\omega \times \omega$. But for some purposes we will need more than this; we want every class that is equinumerous with ω to be a set. This is part of a very general requirement known as the principle of limitation of size. We would like to avoid the set-theoretic paradoxes by excluding very large classes like R and V from the universe of sets; that is, we distinguish between sets and proper classes in terms of size.

Remember from section 1 that we based the concept of equinumerousity on one-one correspondence. Now we will base our version of the limitation of size principle on the relation of being larger, which will in turn be based on many-to-one correspondence; better known as function. So, before introducing a new axiom that attempts to express the idea of limitation of size, we have to define function as a special kind of class. (In section 6 we are going to introduce the more general concept of a relation, of which function is a special case.)

We define a function F in terms of ordered pairs $\langle x, y\rangle$, of which x is the argument (or input), and y is the value (or output) of the function. A function is a class of ordered pairs such that it has a single value for every argument. The domain of a function is the class of sets x for which F has a value y; and the range of F is the class of those sets y which are the value of F for at least one argument x. The formal definitions are as follows:

$$\mathsf{fn}(H) \overset{\text{def}}{\iff} (\forall z \in H)\,\exists x\,\exists y\, z = \langle x, y\rangle \wedge \forall x\,\forall y\,\forall y'\,\big((\langle x, y\rangle \in H \wedge \langle x, y'\rangle \in H) \rightarrow y = y'\big).$$

$$\mathsf{dom}(H) \stackrel{\text{def}}{=} \{x : \exists y\, \langle x, y\rangle \in H\}.$$

$$\mathsf{rng}(H) \stackrel{\text{def}}{=} \{y : \exists x\, \langle x, y\rangle \in H\}.$$

Note that domain and range are defined for arbitrary sets, not only classes. The reason is that we would like to talk about domains ranges of relations, too. If F is a function with domain $\mathsf{dom}(F) = H$ and range $\mathsf{rng}(F) \subseteq G$, we may write $F : H \longrightarrow G$, but we will not use this arrow notation in formulas. What we will use is the notation $F(x) = y$ as a shorthand for $\langle x, y\rangle \in F$.

By definition, $F \subseteq \mathsf{dom}(F) \times \mathsf{rng}(F)$. Thus, if $\mathsf{dom}(F)$ and $\mathsf{rng}(F)$ are sets, then by theorem 4.5 F is a set, too. If $\mathsf{dom}(F)$ is a proper class, the principle of limitation of size suggests that F to is a proper class, too, since every element of $\mathsf{dom}(F)$ appears as the first member of at least one pair in F. But this follows from the axiom of separation, too. And what about the case of $\mathsf{dom}(F)$ being a set and $\mathsf{rng}(F)$ being a proper class? Limitation of size suggests that this cannot be the case; the size of the range of a function cannot exceed the size of its domain. We are going to grasp the idea of limitation of size in terms of this situation being impossible.

Now we are in a position to formulate the axiom in question.

Axiom 4.10 (Replacement: AxRepl)**.** *If the domain of a function is a set, then its range is a set, too.*

$$\forall X\, \big((\mathsf{fn}(X) \wedge \mathsf{M}(\mathsf{dom}(X))) \rightarrow \mathsf{M}(\mathsf{rng}(X))\big)$$

The axiom of replacement is sometimes called the axiom of projection. In the presence of it, we are able to prove the axiom of separation as a theorem:

Theorem 4.6. *The axiom of separation follows from class comprehension and replacement.*

$$\text{AxComp}, \text{AxRepl} \Rightarrow \text{AxSep}$$

Proof. Let x be a set and let Y be a subclass of x. We have to deal with two cases. 1. $Y = \emptyset$, which is a set by the axiom of infinity. 2. There is at least one element in Y. Let h be an arbitrary one. Now for every $z \in x$, let

$$F(z) \stackrel{\text{def}}{=} \begin{cases} x \text{ if } z \in Y; \\ h \text{ otherwise.} \end{cases}$$

By class comprehension, F exists. $x = \mathsf{dom}(F) = x$ is a set. By replacement, $Y = \mathsf{rng}(F)$ is a set, too. □

Note that we used only replacement and class comprehension in the proof. However, these are still rather strong assumptions. There are weaker versions of set theory without the axiom of replacement, and where the axiom of separation is indispensable. On the other hand, there are weaker versions of GB without full class comprehension. There, in order to make it possible to derive separation one needs to take a slightly stronger version of the axiom of replacement:

$$\forall X\, \big(\mathsf{fn}(X) \rightarrow (\forall y \subseteq \mathsf{dom}(X))\, \mathsf{M}(X[y])\big).$$

This version follows easily from AxRepl and AxCompr, and AxSep follows from it easily without AxCompr.

Before turning to the last axiom of GB, we would like to reformulate in an axiomatic context some informal results of section 1. With this in mind, we define bijection and equinumerosity as follows:

$$\mathsf{bij}(H) \stackrel{\text{def}}{\Longleftrightarrow} \mathsf{fn}(H) \wedge \forall x\, \forall x'\, \forall y\, \big((\langle x, y\rangle \in H \wedge \langle x', y\rangle \in H) \rightarrow x = x'.$$

$$F[H] \stackrel{\text{def}}{=} \{y : (\exists x \in H)\, \langle x, y\rangle \in F\}.$$

$$H \approx G \overset{\text{def}}{\Longleftrightarrow} \exists F \big(\mathsf{bij}(F) \wedge H = \mathsf{dom}(F) \wedge G = \mathsf{rng}(F)\big).$$

Now We can reformulate a number of results from the intuitive discussion of equinumerous sets, including Cantor's theorem and the Schröder–Bernstein theorem, and turn their intuitive proofs to axiomatic ones. We skip the details of this reconstructive process, but let us note two things. First, Cantor's theorem is only true of sets, but not of proper classes. Second, the Schröder–Bernstein theorem is true of any class. Finally, we prove the GB version of Cantor's paradox:

Theorem 4.7. *The universal class is not a set.*

$$\neg\, \mathsf{M}(\mathsf{V})$$

Proof. Towards a contradiction, suppose that V is a set. Then by AxPow $\mathcal{P}(\mathsf{V})$ is a set, too. Both $\mathsf{V} \subseteq \mathcal{P}(\mathsf{V})$ and $\mathcal{P}(\mathsf{V}) \subseteq \mathsf{V}$, so by the Schröder–Bernstein theorem, $\mathsf{V} \approx \mathcal{P}(\mathsf{V})$. But by Cantor's theorem, $\mathsf{V} \not\approx \mathcal{P}(\mathsf{V})$. □

Note that the proof could be reduced to the above obvious observation that the Russell class is a subclass of the universal class, so if the latter is a set and we accept separation, then the former is a set, too. We had this more complicated version to make it explicit how the argument of Cantor's paradox is domesticated in GB.

We introduce one more axiom which is far from necessary, and which we will not use until section **??**; but which is still part of the standard system of axioms. The role of this axiom is to rule out certain irregular, ungrounded set constractions, and thereby support the view that we will call the cumulative conception of set.

Axiom 4.11 (Regularity: AxReg)**.** *If a set x is nonempty, then x has at least one element y such that x and y are disjoint.*

$$\forall x\, (x \neq \emptyset \rightarrow (\exists y \in x)\, y \cap x = \emptyset)$$

It is an easy consequence of the axiom of regularity that classes are regular, too:

Theorem 4.8 (Regularity of classes)**.** *If a class X is nonempty, then X has at least one element y such that X and y are disjoint.*

Proof. Suppose that X is a nonempty class such that for every $y \in X$, $y \cap X$ is nonempty. Let $y_0 \in X$,, and define the following series of sets:

1. $z_0 = \{y_0\}$;
2. $z_{n+1} = z_n \cup \{y \cap X : y \in z_n\}$.

Now let $z = \bigcup_{n\in\omega} z_n$. z violates the axiom of regularity. (Replacement is needed both for the sethood of z_{n+1} and the sethood of z.) □

Now let us see what kinds of irregularities the axiom rules out exactly.

Theorem 4.9. *1. There are no $\in$-circles: there is no class $Y = \{x_0, x_1, \dots, x_n\}$ such that $x_0 \in x_1, \dots, x_n - 1 \in x_n$ and $x_n \in x_0$;*

2. there are no infinite descending $\in$-chains: there is no class $Y = \{x_0, x_1, \dots\}$ such that $x_0 \in x_1, \dots, x_n \in x_{n+1}, \dots$.

Proof. Such classes would contradict the theorem of class regularity. □

Two things deserve to be mentioned here. First, note that although the axiom of regularity rules out the existence of sets that are elements of themselves, it has nothing to do with Russell's paradox. In GB, Russell's paradox is avoided by a careful distinction between sets and proper classes. Second, note that we only disproved the existence of infinite $\in$-chains that are represented by classes. We will see in the discussion of the cumulative hierarchy that we cannot dispoove the possibility of infinite descending $\in$-chains that are not represented by classes.

5 Natural numbers

Athough the main concern of set theory is to investigate the properties of infinite sets and classes, it is also very important to describe the finite realm. To measure finite sets, we need a representation of natural numbers in GB. We already have the set ω, which is the standard set-theoretic representation of natural numbers; but before going with the standard definition, let us see some alternatives.

The first successful set-theoretic definition of natural numbers was developed in Gottlob Frege's concept theory (from 1879 to 1892), and adopted by Bertrand Russell and Alfred N. Whitehead in their type-theoretic system (1910). It is based on a very general idea that abstract concepts are to be introduced by equivalence classes, where the equivalence relation can be defined by less abstract concepts. In the case of natural numbers, it amounts to defining a natural number n as the equivalence class of sets with exactly n elements. We call this representation of natural numbers Frege numbers. Let us see a sketch of their definition in GB.

Frege numbers 1. Frege zero is the set of the empty sets; that is, the singleton of $\emptyset$:

$$0_\mathrm{F} \stackrel{\mathrm{def}}{=} \{\emptyset\};$$

2. a set h belongs to the Frege successor of the Frege number n iff for some sets g that belongs to n and for some set x such that is x is not in g, adding x to g results in h:

$$\mathsf{s}_\mathrm{F}(n) \stackrel{\mathrm{def}}{=} \{h \cup \{x\} : h \in n \wedge x \notin h\}.$$

Now the class of Frege numbers should be the class of these equivalence classes. The problem is that such a class does not exist, since the Frege numbers are proper classes except for 0_F. For example, $1_\mathrm{F} = \mathsf{s}_\mathrm{F}(0_\mathrm{F}) = \mathsf{V}$. Therefore it is not convenient in GB to choose Frege numbers as a standard representation of natural numbers in GB.

However, note that it is not a weakness of the concept of Frege numbers that we cannot successfully adopt it in GB. On the contrary, the fact that we cannot successfully adopt the concept of Frege numbers indicates the limits of GB, and more generally, set theories with a principle of limitation of size.

It was Ernst Zermelo's idea to represent natural numbers with sets that in some way reflect the numerosity they stand for. We will see in a moment how Zermelo realized this idea; but first let us see how von Neumann did. His idea was that every natural number is the set of all previous natural numbers. Thus in Neumann's version the number n represents n by the number of its elements:

Neumann numbers 1. Neumann zero is the empty set:

$$0_\mathrm{N} \stackrel{\mathrm{def}}{=} \emptyset;$$

2. the Neumann successor of n is $n \cup \{n\}$:

$$\mathsf{s}_\mathrm{N}(h) \stackrel{\mathrm{def}}{=} n \cup \{n\}.$$

By definition, the set ω introduced in section 4 is the set of Neumann numbers.

Now it is easy to state and prove the well-known Peano axioms for Neumann numbers.

Theorem 5.1. *1. Neumann zero is not a successor of any Neumann number.*

$$\neg(\exists x \in \omega)\, \mathsf{s}_\mathrm{N}(x) = 0_\mathrm{N}$$

2. If the Neumann successors of the Neumann numbers x and y are the same, then x and y are the same.

$$(\forall x \in \omega)\,(\forall y \in \omega)\,(\mathsf{s}_\mathrm{N}(x) = \mathsf{s}_\mathrm{N}(y) \to x = y)$$

3. *If a class X contains Neumann Zero and it is closed under the operation s_N, then ω is a subset of X.*

$$\forall X\,\big(0_\mathrm{N}\in X\wedge(\forall y\in\omega)\,(y\in X\rightarrow \mathsf{s}_\mathrm{N}(y)\in X)\rightarrow\omega\subseteq X\big)$$

Proof. 1. The empty set has no elements.

3. This follows immediately from the definition of ω as the intersection of all inductive classes that contain $\emptyset$.

2. Suppose that for some x and y in ω, $x\cup\{x\}=y\cup\{y\}$, but $x\neq y$. This means that both $x\in y$ and $y\in x$. This violates regularity, but we would like to prove the theorem without using that axiom. We use clause 3.

Let us define H as the class of those elements x of ω for which there is no y in ω such that x and y are elements of each other:

$$H=\{x\in\omega:\neg(\exists y\in\omega)\,x\in y\wedge y\in x\}.$$

First, obviously $\emptyset\in H$. Second, suppose that for some x in ω, $x\in H$ but $x\cup\{x\}\notin H$; that is, for some $y\in\omega$, $x\cup\{x\}\in y$ and $y\in x\cup\{x\}$. We have two cases to consider. (a) $x\in y$ and $y\in x$, which contradicts the hypothesis that $x\in H$. (b) $x=y$, which amounts to $x\in x\cup\{x\}$ and $x\cup\{x\}\in x$, which, again, contradicts the hypothesis. So, H contains Neumann zero and is closed under Neumann successorship. Therefore, by 3, ω is a subset of H; that is, there are no elements of ω that are mutually elements of each other. □

Clause 2 of this proof is an example of what is known as proof by arithmetic induction. Note that the way we represented Peano arithmetic in our system actually has some of the power of second-order Peano arithmetic, since we can quantify over sets of numbers. However, also note that the theory we use is a first order theory.

As an example of arithmetic induction, let us see a theorem that shows that the theory of Neumann numbers in GB has the power of second-order arithmetic. We define the ordering of Neumann numbers as

$$x<_\mathrm{N} y\overset{\text{def}}{\Longleftrightarrow}x\in y.$$

Theorem 5.2. *Every nonempty set of Neumann numbers has a least element.*

$$(\forall h\subseteq\omega)\,\big(h\neq\emptyset\rightarrow(\exists y\in h)\,(\forall z\in h)\,(y=z\vee y<_\mathrm{N} z)\big)$$

Proof. We prove the following by arithmetic induction on x: Every set of Neumann numbers that contains x has a least element.

$$\forall x\,(\forall h\subseteq\omega)\,\big(x\in h\rightarrow(\exists y\in h)\,(\forall z\in h)\,(y=z\vee y<_\mathrm{N} z)\big)$$

The case $x=0$ is obvious. Assume the statement for some x, and consider $x\cup\{x\}$, and let h be a subset of ω such that $x\cup\{x\}\in h$. Then $x\in h$, so by assumption h has a least element. □

The first-order Peano arithmetic version of this theorem is a theorem schema about first-order arithmetic properties, that is, it covers as many subsets as many arithmetic formulas $\varphi(x)$ there are. The set of such formulas is equinumerous with ω. The present version in GB covers every subset of ω. We know from Cantor's theorem that $\mathcal{P}(\omega)$ is not equinumerous with ω. It is not only the case that there are sets of natural numbers that the PA version does not cover but the GB version does; the case is that there are more that are covered in the GB version only than those that are covered in both.

Note that there is an amount of naïvity in the argument of the previous paragraph, since we handled first-order formulas as if they were adequately represented by sets in the GB universe, and we have not defined such a representation.

Now let us see Zermelo's original version of representing natural numbers in set theory. In this version, a natural number is the result of successively applying the singleton operation on the empty set n times. Thus, Zermelo's representation of a number n is a set with a single descending chain of length n.

Zermelo numbers 1. Zermelo zero is the empty set:

$$0_Z \stackrel{\text{def}}{=} \emptyset;$$

2. the Zermelo successor of n is the singleton of n:

$$\mathsf{s}_Z(h) \stackrel{\text{def}}{=} \{n\}.$$

Now we define the class of Zermelo numbers just like we defined ω, and prove that it is a set:

$$\mathsf{Z} \stackrel{\text{def}}{=} \big\{x : \forall y \,\big((0_Z \in y \wedge (\forall z \in y)\, \mathsf{s}_Z(z) \in y) \to x \in y)\big)\big\}.$$

Theorem 5.3. *The class of Zermelo numbers is a set.*

Proof. Since the axiom of infinity provides for ω being a set and there is a natural way of projecting Neumann numbers to the corresponting Zermelo numbers, it is clear that we have to use the axiom of replacement in some way to establish the sethood of Z; but it is not clear at first glance how we can do it. This case is similar to that of the definition of ω, since we can define recursively a class function F to be used in replacement as

1. $F(\emptyset) = \emptyset$,
2. $F(x \cup \{x\}) = \{x\}$;

But we also need to find a direct definition in a single formula to be able to apply class comprehension and thus make sure it exists. in other words, we have to find a formula that tells us directly the value $F(x)$ in Z for every x in ω, without referring to other values of the same x. This will not be an easy task, but it is worth the complications.

The idea of finding such a formula is very similar to the way we attempted to find the formula that defines ω. First, although at the moment we don't know whether a total function that maps Neumann numbers to Zermelo numbers exists, we know that for every $x \in \omega$ there are finite approximations g of F in the sense that g is a set function with a domain that includes the Neumann numbers up to x, and with its values satisfying the above clauses up to x. More precisely, we need a function g so that

(a) $\emptyset$ and x is in the domain of g;

(b) for every $z \neq x$ in te domain of g, $z \cup \{z\}$ is in the domain of g;

(c) $g(0_N) = 0_Z$;

(d) for every $z \neq x$ in the domain of g, $g(\mathsf{s}_N(z)) = \mathsf{s}_Z(g(z))$.

Now we can prove as an obvious case of arithmetic induction that for every x in ω and for every pair of functions g and g' that satisfy clauses (a) to (d) above, for every x in ω. So we can choose an arbitrary such g in the definition of F. To avoid long formulas, let us intruduce an ad hoc concept and notation. Let us call a set h x-inductive if u satisfies conditions (a) to(b):

$$\mathsf{Ind}_x(h) \stackrel{\text{def}}{\Longleftrightarrow} \emptyset \in h \wedge x \in h \wedge (\forall z \in h)\,(z \neq x \to z \cup \{z\} \in h).$$

Now we can define F in terms of g:

$$F = \big\{\langle x, y\rangle : x \in \omega \wedge$$
$$\exists g\,\big(\mathsf{Fn}(g) \wedge \mathsf{Ind}_x(\mathsf{dom}(g)) \wedge g(0_N) = 0_Z \wedge (\forall z \in \mathsf{dom}(g))\, g(\mathsf{s}_N(z)) = \mathsf{s}_Z(g(z))\big)\big\}.$$

By definition, $\mathsf{dom}(F) = \omega$ and $\mathsf{rng}(F) = \mathsf{Z}$. So by replacement, Z is a set. □

The construction in the above proof is a special case of a general construction method. in

Note that the function F defined in the above proof is not just an arbitrary function; it is a bijection from ω onto Z. Moreover, note that $F(O_{\mathrm{N}}) = O_{\mathrm{N}}$ and for every x in ω, $F(\mathsf{s}_{\mathrm{N}}(x)) = \mathsf{s}_{\mathrm{Z}}(F(x))$;that is, F is an isomorphism between the two structures.

No we can choose whether to use isomorphism or repeats the proofs with slight modifications to get the same results about Zermelo Numbers that we got about Neumann numbers.

Finally, one paragraph about Frege numbers and Peano arithmetics. Since most Frege numbers are proper classes, the third PA axiom cannot be formulated as a single formula in their context. Instead, we have a formula scheme, just like in ordinary first-order PA. The proofs are more or less straightforward; but the resulting theorems are weaker than in the case of the two other representations of numbers.

We will choose the Neumann version as our standard, and from now on we omit the subscripts. We also introduce smallcase latin letters from the middle of the alphabet as shorthands for variables restricted to Neumann numbers.

$$\forall n\, \varphi(n) \overset{\text{def}}{\Longleftrightarrow} \forall x\, (x \in \omega \to \varphi(x))$$

$$\exists n\, \varphi(n) \overset{\text{def}}{\Longleftrightarrow} \forall x\, (x \in \omega \land \varphi(x))$$

$$\varphi(n) \overset{\text{def}}{\Longleftrightarrow} x \in \omega \land \varphi(x)$$

—where x has no free occurrence in $\varphi(n)$.

6 Well-ordered classes

In the first-order language of set theory, there are two different ways of talking about relations. First, one can discuss relations as being represented by open formulas of the language. We call such relations predicate relations. For example, "$x \in y$" and "$X \subseteq Y$" are predicate relations. Second, one can discuss relations as being represented by sets or classes of ordered pairs. We call these set relations and class relations. For example, $\{\langle x, y\rangle : x \in y\}$ is a class relation, and $\{\langle x, y\rangle \in h : x \subseteq y\}$ is a set relation. Not every predicate relation is represented by a class relation (e.g., "$X \subseteq Y$" is not), but if in the formula of a predicate relation (a) the related free variables are set variables, and (b) every bound variable is a set variable, then it is represented by a relation in the second sense. On the other hand, there are many relations in the second sense that cannot be defined by a formula of our language.

In what follows, we talk about relations in the second sense, as represented by classes. Note that the relation R is a subclass of the cartesian product of its domain and range. Domain and range can be defined just like we did in the special case of functions before:

$$\mathsf{dom}(R) = \{x : \exists y\, \langle x, y\rangle \in R\};$$

$$\mathsf{rng}(R) = \{y : \exists x\, \langle x, y\rangle \in R\}.$$

It is convenient to talk about relations over a class H; in this case, we simply require that $R \subseteq H^2$. The formal definition of some well-known relational properties over a class are as follows. As usual, xRy is a shorthand for $\langle x, y\rangle \in R$.

Relation (over H) $\mathsf{rel}_H(R) \Longleftrightarrow^{\text{def}} R \subseteq H^2$

We can introduce some basic properties of relations in the usual way:

Asymmetry (over H) $\mathsf{asym}_H(R) \Longleftrightarrow^{\text{def}} \mathsf{rel}_H(R) \land \neg(\exists x \in H)\,(\exists y \in H)\,(xRy \land yRx)$

Transitivity (over H) $\mathsf{trans}_H(R) \overset{\text{def}}{\Longleftrightarrow}$
$\mathsf{rel}_H(R) \wedge (\forall x \in H)\,(\forall y \in H)\,(\forall z \in H)\,((xRy \wedge yRz) \rightarrow xRz)$

Connectedness (over H) $\mathsf{conn}_H(R) \overset{\text{def}}{\Longleftrightarrow}$
$\mathsf{rel}_H(R) \wedge (\forall x \in H)\,(\forall y \in H)\,(x \neq y \rightarrow (xRy \vee yRx))$

Well-foundedness (over H) $\mathsf{wf}_H(R) \overset{\text{def}}{\Longleftrightarrow}$
$\mathsf{rel}_H(R) \wedge (\forall X \subseteq H)\,(\exists y \in X)\,(\forall z \in X)\,(y \neq z \rightarrow \neg\, zRy)$

Note that well-foundedness is expressed in terms of subclasses (in the case of a set relation, subsets).

Now we are well-equipped to define some basic types of ordering. A partial ordering of a class H is an asymmetric, and transitive relation over H (irreflexivity follows from asymmetry). A total ordering (or simply ordering) of H is a connected partial ordering of H. A well-ordering of H is a well-founded total ordering of H.

Partial ordering of H $\mathsf{po}(R, H) \overset{\text{def}}{\Longleftrightarrow} \mathsf{asym}_H(R) \wedge \mathsf{trans}_H(R)$

Total ordering of H $\mathsf{to}(R, H) \overset{\text{def}}{\Longleftrightarrow} \mathsf{po}(R, H) \wedge \mathsf{conn}_H(R)$

Well-ordering of H $\mathsf{wo}(R, H) \overset{\text{def}}{\Longleftrightarrow} \mathsf{conn}(R, H) \wedge \mathsf{wf}_H(R)$

From now on, we use the symbols $<_H$, $\sqsubset_H$, $\prec_H$ etc. for partial orderings, orderings and well-orderings of a class H, if the emphasis is on the relation. An element of $<_H$ is an ordered pair $\langle x, y \rangle \in H^2$. On the other hand, we use the symbols $H_<$, $H_\sqsubset$, $H_\prec$ etc. for partially ordered, totally ordered or well-ordered classes, that is, if the emphasis is on the class H. An element of $H_<$ is a member of H.

Let us see an example and a counterexample of well-ordering. Take the set of finite strings over a finite alphabet, say, $\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$. The usual alphabetic ordering is a total ordering, but not a well-ordering: a, aa, aaa, aaaa, …, ab, aba, aba, …, b, ba, baa, …. But you get a well-ordering if you collect in the front of the line all the one-letter strings; then the two-letter strings, then the three-letter ones, and so on: a, b, c, aa, ab, …, cc, aaa, …, ccc, aaaa, …. The example shows that the set of strings over a finite alphabet is well-orderable.

The following is just a clarification on the definitions:

Theorem 6.1. *Every well-ordering is a total ordering.*

Proof. Let $<_H$ be a well-ordering. We have to prove that (a) $<_H$ is asymmetric and (b) $<_H$ is transitive. (a) Towards a contradiction, suppose that there are some x and y in H so that $x <_H y$ and $y <_H x$. Then the subset $\{x, y\}$ of H has no least element. (b) Towards a contradiction, suppose that there are some x, y and z in H so that $x <_H y$ and $y <_H z$, but $x \not<_H z$. Then by connectedness either $x = z$, which takes us back to (a), or $z <_H x$, in which case the subset $\{x, y, z\}$ of H has no least element. □

Now a following simple theorem guarantees that well-ordering is preserved from whole to part. But first we have to specify what we mean by part.

Restriction of a relation Let R be a relation over H, and $G \subseteq H$. Then

$$R_H \restriction G \overset{\text{def}}{=} R_H \cap G^2$$

(It is not necessary that G is a subset if H; but it will be in every application of the concept.)

Theorem 6.2. *A restriction of a well-ordering is a well-ordering.*

$$\forall R \forall H\,(\forall G \subseteq H)\,(\mathsf{wo}(R, H) \rightarrow)\mathsf{wo}(R_H \restriction G)$$

Proof. Let $<_H$ be a well-ordering, and $G \subseteq H$. Elements of G are also elements of H; therefore connectedness is inherited from $<_H$ to $<_H \restriction G$. Subclasses of G are also subclasses of H, therefore well-foundedness is inherited as well. □

Remark Not every ordering property is inherited to restrictions. For example, a restriction of a discrete ordering is a discrete ordering; but a restriction of a dense ordering may be discrete. As an example, consider the usual ordering of rational numbers restricted to integers.

We need some method to compare orderings. We adapt the well-known concept of isomorphism, and call it similarity in the present context. Let $H_<$ and $G_\sqsubset$ be partially ordered, ordered or well-ordered classes. $H_<$ and $G_\sqsubset$ are similar if and only if there is a bijection $I : H \longrightarrow G$ such that for every $x \in H$ and $y \in H$,

$$x < y \leftrightarrow I(x) \sqsubset I(y).$$

We use the symbol $\cong$ for similarity. Like isomorphisms in general, similarity is an equivalence relation. (Please note that in this case the term *relation* means predicate relation, which is not represented by a class.)

Theorem 6.3. *Similarity is reflexive, symmetric and transitive.*

1. $\forall H_<\, H_< \cong H_<$;
2. $\forall H_< \forall G_\sqsubset\, (H_< \cong G_\sqsubset \rightarrow G_\sqsubset \cong H_<)$;
3. $\forall H_< \forall G_\sqsubset \forall E_\prec\, ((H_< \cong G_\sqsubset \wedge G_\sqsubset \cong E_\prec) \rightarrow H_< \cong E_\prec)$.

Proof. 1. Take the identical mapping $I(x) = x$ is an isomorphism. 2. Suppose $H_< \cong G_\sqsubset$. Then there is an isomorphism I from $H_<$ to $G_\sqsubset$. Its inverse I^{-1} is an isomorphism from $G_\sqsubset$ to $H_<$. 3. Suppose $H_< \cong G_\sqsubset$ and $G_\sqsubset \cong E_\prec$. Then there are isomorphisms I and J from $H_<$ to $G_\sqsubset$ and from $G_\sqsubset$ to $E_\prec$, respectively. Their composition $I \circ J$ is an isomorphism from $H_<$ to $E_\prec$. □

The concept of an initial segment plays an essential role in the following basic theorems about well-ordered classes.

Initial segment Let $H_<$ be a well-ordered class, and $a \in H$. Then the class

$$\mathsf{seg}_{<,H}(a) \overset{\text{def}}{=} \{\langle x, y\rangle \in H^2 : x < y \wedge y < a\}$$

is a well-ordering, too. It is called the initial segment of $H_<$ given by a.

Theorem 6.4. *No well-ordering is similar to an initial segment of itself.*

Proof. Let $H_<$ be a well-ordered class, and let $a \in H$. Towards a contradiction, suppose that $H_< \cong \mathsf{seg}_{H,<}(a)$. Then there is a bijection $I : H_< \longrightarrow \mathsf{seg}_{H,<}(a)$ such that for all $x \in H$ and $y \in H$, $x <_H y$ iff $I(x) <_{\mathsf{seg}(a)} I(y)$. Since $I(a) \in \mathsf{seg}_{H,<}(a)$, $I(a) < a$. Thus, the class $\{x \in H : I(x) < x\}$ is nonempty, so it has a least element. Let it be b. Now $I(b) < b$, and since I is order-preserving, $I(I(b)) < I(b)$. But this contradicts the claim that b is the least such element in H. □

Remark In the above theorem it is essential that $<$ is a well-ordering. It is not hard to find counterexamples among orderings in general. For example, the open interval $]0,1[_<$ is the initial segment given by 1 of the open interval $]0,2[_<$ of the real numbers. However, $]0,1[_< \cong]0,2[_<$, since $i(x) = \frac{x}{2}$ is an isomorphism.

Theorem 6.5. *The only isomorphism from a well-ordered class to itself is the identical mapping.*

Proof. On the one hand, the identical mapping is clearly an isomorphism. Now let $H_<$ be a well-ordered class and suppose towards a contradiction that there is an isomorphism I on $H_<$ such that for some $a \in H$, $I(a) \neq a$. Since $<_H$ is connected, there are two possibilities. (a) $I(a) < a$. Then the class $\{x \in H : I(x) < x\}$ is nonempty, and hence it has a least element b. Since $I < I(b)$ and I is order-preserving, $I(I(b)) < I(b)$. But then b is not the least such element. (b) $a < I(a)$. Now the class $\{x \in H : x < I(x)\}$ is nonempty, and hence it has a least element b. Since $b < I(b)$ and I is order-preserving, $I^{-1}(b) < b$. But then b is not the least element of the class. □

Remark This is not true of total orderings in general either. Take the set $R^+_<$ of positive reals with the usual ordering, and the mapping $I(x) = \frac{x}{2}$. this is a non-identical automorphism on $R^+_<$.

Finally, let us put forward a very important theorem which says that well-orderings are uniform in the sense that they are either similar, or one is similar to an initial segment of the other.

Theorem 6.6. *Well-orderings are comparable. If $H_<$ and $G_\sqsubset$ are well-ordered classes, exactly one of the following three cases hold:*

(a) for some $b \in G$, $H_< \cong \mathsf{seg}_{G,\sqsubset}(a)$;

(b) for some $a \in H$, $\mathsf{seg}_{H,<}(a) \cong G_\sqsubset$;

(c) $H_< \cong G_\sqsubset$.

Proof. This is going to be a long proof consisting of many small steps.

Let $H_<$ and $G_\sqsubset$ be well-ordered classes, and let us consider the class

$$I = \{\langle x, y\rangle \in H \times G : \mathsf{seg}_{H,<}(x) \cong \mathsf{seg}_{G,\sqsubset}(y)\}.$$

First we show in three steps that I is an isomorphism from a subclass H' of H to a subclass G' of G. 1. If xIy and xIy', then $y = y'$, because otherwise $\mathsf{seg}_{G,\sqsubset}(y) \cong \mathsf{seg}_{G,\sqsubset}(y')$, and since $\sqsubset_G$ is connected, one of these initial segments of G is also an initial segment of the other. Thus I is a function. 2. If xIy and $x'Iy$, then $x = x'$, because otherwise $\mathsf{seg}_{H,<}(x) \cong \mathsf{seg}_{H,<}(x')$, and since $<_H$ is connected, one of these initial segments of H is also an initial segment of the other. Thus I is a bijection between its domain and its range. 3. Now we prove that I is order-preserving. If $x < x'$, then $\mathsf{seg}_{H,<}(x)$ is an initial segment of $\mathsf{seg}_{H,<}(x')$, so by similarity, there is an y in $\mathsf{seg}_{G,\sqsubset}(I(x'))$ such that $\mathsf{seg}_{H,<}(x) \cong \mathsf{seg}_{G,\sqsubset}(y)$. Now either $I(x) < y$ or $y < I(x)$ results in an initial segment being similar with its own initial segment. So, $y = I(x)$. (Since I is a bijection, the other direction of the biconditional follows immediately.)

Now we focus on the domain and the range of I. There are four cases. 1. $\mathsf{dom}(I) = H$, but $\mathsf{rng}(I) \neq G$. $H_<$ is similar to the initial segment of $G_\sqsubset$ given by the least element of $G \setminus \mathsf{rng}(I)$; so, (a) is the case. 2. $\mathsf{dom}(I) \neq H$, but $\mathsf{rng}(I) = G$. $G_\sqsubset$ is similar to the initial segment of $H_<$ given by the least element of $H \setminus \mathsf{dom}(I)$; so, (b) is the case. 3. $\mathsf{dom}(I) = H$ and $\mathsf{rng}(I) = G$. In this case the two well-orderings are similar; so, (c) is the case. 4. $\mathsf{dom}(I) \neq H$ and $\mathsf{rng}(I) \neq G$. This is impossible, because then the initial segment of of $H_<$ given by the least element of $H \setminus \mathsf{dom}(I)$ and the initial segment of $G_\sqsubset$ given by the least element of $G \setminus \mathsf{rng}(I)$ are similar, so these least elements are both inside and outside of $\mathsf{rng}(I)$ and $\mathsf{dom}(I)$, respectively.

Finally, if any two of (a), (b) and (c) were simultaneously the case, it would once again violate the above result that no well-ordered set is similar to an initial segment of itself. □

7 Ordinal numbers

Ordinal numbers—or simply ordinals—are tools for measuring finite and transfinite well-orderings in the same way that natural numbers measure finite orderings. We have already seen in theorem 6.6 that well-orderings are comparable; in simple words, one is either shorter, of the same length, or longer than the other. Ordinal numbers measure the length of a well-ordering in this sense.

Since similarity is an equivalence relation, similar well-ordered sets belong to equivalence classes, called order types. It would be convenient to identify the ordinal number of a well-ordered set with its order type in a Fregean fashion. This method does not work in our framework for the same reason that we could not use a Fregean definition of natural numbers; order types are too large to be considered as sets.

Theorem 7.1. *Let $H_<$ be a nonempty well-ordered class. Then the class $\{g_\sqsubset : h_< \cong g_\sqsubset\}$ is a proper class.*

Proof. Let a be the $<$-minimal element of h. Let $h_<[x/a]$ be the result of replacing a with the set x. For every value of x that is not in h, $h_<[x/a]$ is a different well-ordering similar to $h_<$. So, there is a one-one correspondence between the class $V \setminus h$ of all sets outside of h, and the class $\{h_<[x/a] : x \notin h\}$. $V \setminus h$ is a proper class, so by the axiom of replacement $\{h_<[x] : x \notin h\}$ has to be one, too. But it is a subclass of $h_<$'s order type; so the order type is a proper class aas well. □

Now we develop the concept of Neumann ordinals, as a generalization of the concept of Neumann natural numbers. Neumann ordinals are well-ordered sets that represent their order types. The relation that well-orders an ordinal is the elementhood relation. First we define this for an arbitrary class H:

$$\in_H \stackrel{\text{def}}{=} \{\langle x, y\rangle \in H \times H : x \in y\}.$$

We will also need the concept of a transitive class in the definition of ordinals. We call a class H transitive if every element of H is a subset of H as well:

$$\mathsf{tr}(H) \stackrel{\text{def}}{\iff} \forall x\,(x \in H \to x \subseteq H).$$

Note that the converse cannot be the case because of Cantor's theorem. Also note that the transitivity of the set h and the transitivity of the relation $\in_h$ are independent. A set h can be transitive without $\in_h$ or even $\in_{h\cup\{h\}}$ being transitive; whereas $\in_h$ or $\in_{h\cup\{h\}}$ can be transitive without h being so.

Let us see two obvious corollaries of the definition of a transitive set:

Theorem 7.2. *1. A class H is transitive if and only if $\bigcup H \subseteq H$.*

2. A class H is transitive if and only if $H \subseteq \mathcal{P}(H)$.

Proof. Both clauses follow immediately from the definitions. □

Now we can apply Neumann's definition of ordinals. A class H is an Neumann ordinal (or simply ordinal) if it is transitive, and the relation $\in_H$ well-orders H.

$$\mathsf{on}(H) \stackrel{\text{def}}{\iff} \mathsf{tr}(H) \wedge \mathsf{wo}(\in_H).$$

From now on, we stick to the above definition, and whenever we talk about ordinals, we will mean Neumann ordinals. Please note that this definition is slightly different from the one to be found in the literature. Usually the concept is restricted to sets. We will show soon that there is exactly one proper class that satisfies our ordinal definition. This proper class ordinal will be the class of set ordinals:

$$\mathsf{Ord} \stackrel{\text{def}}{=} \{x : \mathsf{on}(x)\}.$$

[... ordinal variables]
We define the successor of an ordinal the same way as the successor of a natural number:

$$\mathsf{s}(\alpha) \stackrel{\text{def}}{=} \alpha \cup \{\alpha\}.$$

As usual, we omit the brackets whenever it soesn't cause confusion. We want the series of ordinals to be a continuation of the series of integers over ω, making ω the first transfinite ordinal. But ω is not a successor of any of the ordinals below it. In fact, it is the second example of a limit ordinal, an ordinal that is not a successor ordinal—the first one being zero. Anyway, we have to distinguish between successor ordinals and limit ordinals:

$$\mathsf{son}(\alpha) \stackrel{\text{def}}{\iff} \exists\beta\, \alpha = \mathsf{s}\beta;$$

$$\mathsf{lon}(\alpha) \stackrel{\text{def}}{\iff} \neg\,\mathsf{son}(\alpha).$$

Now we start proving a series of simple but important theorems concerning ordinals. This will be a long run, but it is worth; at the end, the infamous Burali-Forti paradox is waiting for us, in the form of a harmless theorem. The proofs could be considerably simpler if we used the axiom of regularity, but we choose not to. The reason is that we want to show that none of the properties of the ordinals depend on regularity.

Theorem 7.3. *1. The empty set is an ordinal.*

$$\mathsf{on}(\emptyset)$$

2. *Every element of an ordinal is an ordinal, and it is the same as the initial segment given by it.*

$$\forall X\,(\mathsf{on}(X) \to (\forall y \in X)\,\big(\mathsf{on}(y) \land y_\in = \mathsf{seg}_{\in,X}(y)\big)$$

3. *The successor of an ordinal class is an ordinal class.*

$$\forall X\,(\mathsf{on}(X) \to \mathsf{OC}(\mathsf{s}X)$$

4. *The intersection class of a class H of ordinals is an ordinal class.*

$$\forall H\,\big((\forall x \in H)\,\mathsf{on}(x) \to \mathsf{on}(\bigcap H)\big)$$

5. *If an ordinal class X is a proper subclass of another ordinal class Y, then X is also an element of Y.*

$$\forall X\,\forall Y\,((\mathsf{OC}(X) \land \mathsf{OC}(Y) \land X \subsetneq Y) \to X \in Y)$$

6. *For all ordinal classes X and Y, eiher $X \subseteq Y$, or $Y \subseteq X$.*

$$\forall X\,\forall Y\,((\mathsf{OC}(X) \land \mathsf{OC}(Y)) \to (X \subseteq Y \lor Y \subseteq X)$$

7. *Ordinal classes are connected by the elementhood relation.*

$$\forall X\,\forall Y\,\big((\mathsf{OC}(X) \land \mathsf{OC}(Y) \land X \neq Y) \to (X \in Y \lor Y \in X)\big)$$

8. *The union class of a class of ordinals is an ordinal class.*

$$\forall X\,\big((\forall x \in H)\,\mathsf{OC}(x) \to \mathsf{OC}(\bigcup H)\big)$$

9. *Every class H of ordinals is well-ordered by $\in_H$.*

$$\forall H\,\big((\forall x \in H)\,\mathsf{OC}(x) \to \mathsf{WO}(\in_H)\big)$$

10. *(Burali-Forti) There is exactly one proper class ordinal: the class* Ord *of ordinal sets.*

$$\forall X\,((\mathsf{OC}(X) \wedge \neg\,\mathsf{M}(X)) \leftrightarrow X = \mathsf{Ord})$$

Proof. We prove the clauses one at a time. We refrain from using the axiom of regularity, because we want to show that the theory of ordinals remains unchanged in a non-wellfounded set-theoretical environment, too.

1. Since $\emptyset$ has no elements, it is obviously both transitive and well-ordered by $\in_\emptyset$.

2. Let X be an ordinal class, and $y \in X$. We prove that y is (a) transitive; (b) well-ordered by $\in_y$; and (c) $y_\in = \mathsf{seg}_{\in,X}(y)$.

   (a) Suppose that for some $z \in y$, $z \not\subseteq y$; that is, there is a $u \in z$ such that $u \notin y$. By transitivity of X, $z \in X$ and $u \in X$. So by connectedness of $\in_X$, either $y \in u$ or $y = u$. So, either $u \in z$, $z \in y$ and $y \in u$, or $z \in y$ and $y \in z$. Both of these violate the fact that X is well-ordered by $\in_X$.

   (b) Since X is transitive, $y \subseteq X$, so $\in_y$ is a restriction of $\in_X$. We saw in the previous section that a restriction of a well-ordering is a well-ordering, too.

   (c) The claim follows immediately from the definition of initial segment.

3. Let X be an ordinal, and consider $\mathsf{s}X$. First of all, it cannot be the case that $\mathsf{s}X \in X$, because then $\in_X$ would be one of its own initial segments. Now we prove with indirect arguments that (a) $\mathsf{s}X$ is transitive, (b) $\in_{\mathsf{s}X}$ is connected, and (c) $\in_{\mathsf{s}X}$ is well-founded.

   (a) Suppose that $\mathsf{s}X$ is not a transitive class. Then there is a $y \in \mathsf{s}X$ such that $y \not\subseteq \mathsf{s}X$. It cannot be the case that $y = X$, since $X \subseteq \mathsf{s}X$. But $y \in X$ cannot be the case either, since $X \subseteq \mathsf{s}X$ and $y \subseteq X$.

   (b) Suppose that $\in_{\mathsf{s}X}$ is not connected. Then there are some elements y and z in $\mathsf{s}X$ such that $y \neq z$, $y \notin z$ and $z \notin y$. There are two cases. If both y and z are in X, then $\in_X$ is not connected. If one of them is X itself, then by definition of $\mathsf{s}X$ the other is in X; but we assumed that it is not.

   (c) Suppose that $\in_{\mathsf{s}X}$ is not well-founded. Then there is a nonempty $Y \subseteq \mathsf{s}X$ such that it has no least element. Now either $Y \subseteq X$, which contradicts the wellfoundedness of $\in_X$, or $X \in Y$. In the latter case either $Y = \{X\}$, which contradicts the wellfoundedness of $\in_X$, too; or $Y \setminus \{X\}$ is nonempty, so it has a least element z. By assumption Y has no least element, so $X \in z$. But $z \in X$, which once again contradicts the wellfoundedness of $\in_X$.

4. If $\bigcap H$ is empty, then the claim is obvious. So, let H be a class or ordinals such that $\bigcap H$ is nonempty. We prove that $\bigcap H$ is (a) transitive and (b) well-ordered.

   (a) Let x be in $\bigcap H$. By definition of intersection, for all $y \in H$, $x \in y$, and since y is transitive, $x \subseteq y$. So for all z in x, z is an element of every element of H, so $z \in \bigcap H$. Thus $x \subseteq \bigcap H$. That is, $\bigcap H$ is transitive.

   (b) Let x be an arbitrary element of H. Every subclass of $\bigcap H$ is also a subclass of x. x is well-ordered by $\in_x$, and since $\bigcap H \subseteq x$, $\in_{\bigcap H}$ is a restriction of $\in_x$. But a restriction of a well-ordering is a well-ordering, too.

5. Let X and Y be ordinal classes such that X is a proper subset of Y. Then $Y \setminus X$ is a nonempty subclass of Y. Since Y is well-ordered by $\in_Y$, $Y \setminus X$ has a minimal element; let it be z. Now X is the initial segment of Y given by z, otherwise neither Y nor X would be well-ordered. We seen in 2 that the initial segment of Y given by some $z \in Y$ is z itself. Thus, $X = z$, so $X \in Y$.

6. Towards a contradiction, suppose that for some ordinal classes X and Y, both $X \setminus Y$ and $Y \setminus X$ are nonempty. Since these are subclasses of well-ordered classes, they have least elements; let these be x' and y', respectively. Since they are members of disjoint sets, $x' \neq y'$. But $X \cap Y$ is both the initial segment of X given by x', and the initial segment of Y given by y'. Thus, both $x' = X \cap Y$ and $y' = X \cap Y$, and we have reached a contradiction.

7. The claim follows immediately from clauses 5 and 6.

8. Let H be a class of ordinals, and consider $\bigcup H$. If $\bigcup H$ is empty, the claim is obvious; so let us assume it is not. Since every element of $\bigcup H$ is an element of some ordinal, and we saw in 2 that elements of ordinals are ordinals, $\bigcup H$ itself is a class of ordinals, too. We prove that (a) $\bigcup H$ is transitive, (b) $\in_{\bigcup H}$ is connected, ad $\in_{\bigcup H}$ is well-founded.

   (a) Let x be in $\bigcup H$. By definition of union, there is a $y \in H$ such that $x \in y$, and since y is transitive, $x \subseteq y$. But $y \subseteq \bigcup H$, so $x \subseteq \bigcup H$.

   (b) We have seen in clause 7 that for any ordinal classes X and Y such that $X \neq Y$, either $X \in Y$ or $Y \in X$. This holds of the elements of $\bigcup H$, too.

   (c) Suppose that $\in_{\bigcup H}$ is not well-founded. Then there is a nonempty $G \subseteq \bigcup H$ such that G has no least element. Let x be an element of H such that $G \cap x$ is nonempty. The set $G \cap x$ has a least element y. Since G has no least element, there is a $z \in G$ such that $z \in y$. But then $z \in y$, $y \in x$, and $z \notin x$, which contradicts the fact that x is transitive.

   Note that the sethood of h was necessary only because the union class of a proper class is a proper class, and ordinals are sets.

9. Let H be a class of ordinals. We prove that (a) $\in_H$ is connected; (b) $\in_H$ is well-founded.

   (a) If H has less than two elements, the claim is obvious. Otherwise let x and y be in H such that $x \neq y$. We have just seen in clause 7 that either $x \in y$ or $y \in x$ is the case.

   (b) If H is empty, the claim os obvious. Otherwise let G be a nonempty subclass of H, and let x be in G. The set $G \cap x$ is a subset of x, so either it is empty, or it has a least element y. In the first case x is the least element of G. Consider the second case. Since for every element z of G, either $x \in z$ or $z \in x$, y is also the least element of G.

10. We prove that (a) Ord is an ordinal class, (b) it is a proper class, and (c) any other ordinal class is a set.

   (a) Since Ord is a class of ordinals, it is well-ordered by $\in_{\mathsf{Ord}}$. On the other hand, it is transitive, because by 2 elements of ordinals are ordinals. So, Ord itself is an ordinal class.

   (b) Suppose that Ord is a set. Then $\mathsf{Ord} \in \mathsf{Ord}$; But then $\{\mathsf{Ord}\}$ is a nonempty subclass of Ord that has no least element; but this contradicts the fact that $\in_{\mathsf{Ord}}$ well-orders Ord.

   (c) Let X be an ordinal class such that $X \neq \mathsf{Ord}$. As we saw in clause 7, either $X \in \mathsf{Ord}$, or $\mathsf{Ord} \in X$. $\mathsf{Ord} \in X$ cannot be the case, since elements of classes are sets; so $X \in \mathsf{Ord}$, which implies that X is a set.

□

Since in what follows we will use ordinals quite often, it is convenient to introduce special variables restricted to set ordinals. We use smallcase greek letters for this purpose:

$$\forall\alpha\,\varphi(\alpha) \stackrel{\text{def}}{\Longleftrightarrow} (\forall x \in \mathsf{Ord})\,\varphi(x)$$

$$\exists\alpha\,\varphi(\alpha) \stackrel{\text{def}}{\Longleftrightarrow} (\exists x \in \mathsf{Ord})\,\varphi(x)$$

$$\varphi(\alpha) \stackrel{\text{def}}{\Longleftrightarrow} x \in \mathsf{Ord} \wedge \varphi(x)$$

—where x has no free occurrence in $\varphi(\alpha)$.

Not only is $\in_\alpha$ a well-ordering of any ordinal α, we will also see that $\in_H$ well-orders any class H of ordinals. Hence, it is convenient to use the symbol $<_\alpha$, $<_H$ in this context.

Theorem 7.4. *1. Every element of ω is an ordinal.*

2. Zero is the only limit ordinal in ω.

3. ω is a limit ordinal.

Proof. 1. This is an obvious case of arithmetic induction. We have seen in clause 1 of theorem 7.3 that 0 is in Ord, and clause 3 of that theorem showed that if n is in Ord, then its successor is in Ord, too. Thus, $\omega \subseteq \mathsf{Ord}$.

2. Another obvious case of arithmetic induction. Consider the class

$$H = \{x :\ x = 0 \vee \exists y\, x = y \cup \{y\}\}.$$

Obviously, 0 is in H and if n is in H, then its successor is in H, too. Thus, $\omega \subseteq C$; that is, every $n \in \omega$ is either 0 or a successor.

3. First, observe that $\omega = \bigcup\omega$. We have just seen that ω is a set of ordinals; thus by clause 8 of theorem 7.3, ω is an ordinal. Now suppose that ω is a successor ordinal; that is, for some n in ω, $\mathsf{s}n = \omega$. But then $\omega \in \omega$, which contradicts the well-orderedness of ω.

□

We introduce a method of proving propositions about ordinals, called transfinite induction. Like many concepts in the theory of ordinals, this is a generalization of a concept in the theory of natural numbers. This concept is a version of arithmetic induction known as total induction, an easy consequence of the induction schema in Peano arithmetic. Total induction says that if an arithmetic property P is inherited to every natural number n from all its predecessors, then every natural number has the property P. Transfinite induction expresses a similar idea with ordinals instead of natural numbers, and set-theoretic properties represented by classes.

Theorem 7.5 (Transfinite induction)**.** *Let H be a class so that for every ordinal α, if every ordinal $\beta < \alpha$ is in a class H, then α is in H. Then* Ord *is a subclass of H.*

$$\forall H\,\big(\forall\alpha\,((\forall\beta < \alpha)\,\beta \in H \to \alpha \in H) \to \mathsf{Ord} \subseteq H\big).$$

Proof. Let H be a class and α an ordinal so that $\alpha \notin H$. Then the class $\{\beta : \beta \notin H\}$ is not empty, therefore by clause **??** of theorem 7.3 it has a least element γ. Now every ordinal less then γ is in H, but γ is not in H. thus, a counterexample to the consequent of the conditional in the theorem is also a counterexample to the antecedent. □

Theorem 7.6 (Transfinite recursion)**.** *(. . .)*

Proof. (. . .) □

Now we define the basic operations of ordinal arithmetic via transfinite recursion.

Addition 1. $\alpha + 0 = \alpha$;

2. $\alpha + (\mathsf{s}\beta) = \mathsf{s}(\alpha + \beta)$ for all ordinals β;
3. $\alpha + \beta = \bigcup_{\gamma<\beta} \alpha + \gamma$ for limit ordinals β.

Multiplication 1. $\alpha \cdot 0 = 0$;

2. $\alpha \cdot (\mathsf{s}\beta) = (\alpha \cdot \beta) + \alpha$ for all ordinals β;
3. $\alpha \cdot \beta = \bigcup_{\gamma<\beta} \alpha \cdot \gamma$ for limit ordinals β.

Exponentiation 1. $\alpha^0 = 1$;

2. $\alpha^{\mathsf{s}\beta} = \alpha^\beta \cdot \alpha$ for all ordinals β;
3. $\alpha^\beta = \bigcup_{\gamma<\beta} \alpha^\gamma$ for limit ordinals β.

Theorem 7.7. *Ordinal addition and multiplication are associative, but not commutative.*

Proof. Associativities can be shown by transfinite induction. As for commutativity, take a limit and a successor ordinal to construct a counterexample; e.g., $\omega + 2 \neq 2 + \omega$; $\omega \cdot 2 \neq 2 \cdot \omega$. □

Theorem 7.8 (Cantor)**.** *For every nonzero ordinal α, there is a unique $n \in \omega$, $\langle \beta_1, \ldots, \beta_n \rangle \in Ord^n$, and $\langle k_1, \ldots, k_n \rangle \in \omega^n$ such that*

$$\alpha = \sum_{i \leq n} \omega^{\beta_i} \cdot k_i.$$

.

Proof. We prove the following lemmas by transfinite induction:

1. For every α, there is a greatest β such that $\omega^\beta \leq \alpha$.
2. For every such α and β, there is a $k \in \omega$ and a $\gamma < \omega^\beta$ such that $\alpha = \omega^\beta \cdot k_i + \gamma$.
3. These k and γ are unique.

(...)

Now it is easy to prove the theorem by yet another transfinite induction. (...) □