"Unsolved Problems in Mathematics" J. von Neumann's address to the International Congress of Mathematicians Amsterdam, September 2-9, 1954

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The Mathematical Intelligencer **21** (1999) 7-12

1 The invitation

On behalf of the program committee of the International Congress of Mathematicians held between September 2-9, 1954, H.D. Kloosterman, chairman of the committee, wrote a letter [13] in November of 1952 to John von Neumann at the Institute for Advanced Study. In this letter Kloosterman informs von Neumann that a proposal had been made in the committee to consider an address to the congress, an address similar to Hilbert's famous lecture in 1900 in Paris about unsolved problems in mathematics. Kloosterman also points out in his letter the committee's being aware of the increasing tendency of specialization in mathematics, which, in the committe's view, might have gone far enough to prohibit one person from being able to prepare such an address. The committe therefore considered the following three options:

1. A talk prepared and delivered by one mathematician.

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2. A small team of mathematicians prepares and one mathematician deliveres an address.
3. A small team prepares the address and the members of the team report individually.

Writes Kloosterman:

> As I mentioned already the program committee has a preference for the first of these suggestions. On the other hand the committee's opinion is that you are probably the only active mathematician in the world who is master of the whole mathematics to such a degree as to be able to deliver an address of the character expressed above.
>
> For this reason you will oblige me very much to communicate to me if you would kindly accept an invitation to deliver before the International Congress of Mathematicians in Amsterdam an address on unsolved problems in mathematics.
>
> In any case your opinion about these suggestions stated would be most valuable to our committee. [13]

Apparently the November 27 letter of Kloosterman never reached von Neumann. However, Kloosterman contacted von Neumann again in a letter dated March 20, 1953, and he also enclosed a copy of the November 27 letter, thereby renewing the invitation.

Von Neumann had received this second latter on March 25 and replied immediately – but cautiously [24]. Expressing his deep appreciation of the great distinction that the invitation entails and indicating that in view of the exceptional confidence that the invitation expresses he is inclined to accept the invitation, he asked for some more time to consider the matter carefully before making a final committment. Considering von Neumann's famous mental speed he hesitated rather long: it took him two weeks to reach a decision, and the decision was that "If this is the preference of your Committe, and if it is otherwise acceptable to you, I will give an address on the basis of alternative 1. that you mentioned – that is, an individual address 'On Unsolved Problems in Mathematics'." [25].

With his characteristic precision and modesty von Neumann also defines however his task more restrictively by adding

> The total subject of mathematics is clearly too broad for any one of us. I do not think that any mathematician since Gauss has

> covered it uniformly and fully, even Hilbert did not, and all of us are of considerably lesser width (quite apart from the question of depth) than Hilbert. It would, therefore, be quite unrealistic not to admit, that any address I could possibly give would not be biased towards some areas in mathematics in which I have had some experience, to the detriment of others which may be equally or more important. To be specific, I could not avoid a bias towards those parts of analysis, logics, and certain border areas of the applications of mathematics to other sciences, in which I have worked. If your Committee feels that an address which is affected by such imperfections still fits into the program of the Congress, and if the very generous confidence in my ability to deliver continues, I shall be glad to undertake it. The task represents a very interesting and inspiring challenge, and I would certainly try to make the limitations that I have described above as palatable to the audience as I can. [25]

The committe must have felt satisfied with this reply, and in the afternoon session of the first day of the conference von Neumann had delivered the address.

2 Von Neumann's address

2.1 The plan

As far as one can tell on the basis of published and archival material, von Neumann did not prepare a text for his talk: the one-hour invited lectures got published in Volume I of the three volume proceedings of the congress [10]; however, "no manuscript was available" in von Neumann's case.[1] There are two documents in the Von Neumann Archive in the Library of Congress that are related to his talk: one is a handwritten sketch of the topics he planned to discuss [26], the other is a typescript of the talk [27].

As his sketch reveals, von Neumann's plan was to talk about

> ... problems in a particular area of mathematics – operator theory, viewed in its connection with certain other subjects, specifically in its algebraical aspects, and its relationship

[1] Von Neumann was not the only invited speaker without submitting a paper: for instance, A. Tarski did submit one either.

to quantum theory, and through this to logic and to the theory of probability.

The unsolved problems to which attention is called fall into three groups.

1. Problems involving the algebraic structure of rings of operators.
2. The role and meaning of these in view of the present difficulties and uncertainties in quantum theory.
3. Problems of reformulation and unification in logics and probability theory based on this approach. [26]

Von Neumann's handwritten sketch [26] gives a list of 24 more specific issues he intended to discuss in the talk. A comparison of von Neumann's list with the typescript shows that in the address he did not in fact bring up some of the issues and problems he originally had planned to. This is not surprizing in view of the fact that von Neumann's sketch estimates the time needed to discuss each topic on the list to be 89 minutes (he was supposed to give a one hour lecture), and in his sketch he had allocated: only 5 minutes to state and discuss the isomorphism problem of finite von Neumann algebras[2], barely 4 minutes for detailing the characterisation (i.e. classification) problem of infinite von Neumann algebras, and just 3 minutes to deal with the infinite direct product. These are technically involved, deep issues that proved to be extremely challenging for great many talented mathematicians to come. Von Neumann must have realized that he would not be able to do justice to the complexity of these problems in the time frame he had and he skipped them in the lecture. (For a review of the legacy of von Neumann in the theory of operator algebras see [12] and [15].)

What von Neumann decided to do in the lecture was to concentrate on the second and third group of problems: He gave a general motivation for the theory of operator rings and, in particular, he discussed the possible conceptual significance of a particular ring, the finite, continuous ring (the type $\mathbf{II}_1$ von Neumann algebra) both for the mathematical theory of operators and for a better understanding of quantum mechanics, quantum logic and (quantum) probability.

[2]In 1954 von Neumann algebras were called "rings of operators", the name "von Neumann algebra" was proposed by Dieudonne in 1954 – just about the time of von Neumann's talk in 1954.

2.2 Main points in the talk

In the first part of the talk von Neumann makes clear that a satisfactory theory of unbounded operators is absolutely indispensable because unbounded operators are forced upon us by quantum mechanics: the Heisenberg canonical commutation relation cannot be satisfied by bounded operators "... and there could not be two ways about it." [27, p. 5]. This has as a consequence that one has to use infinite dimensional Hilbert spaces to model quantum systems and leads to the trouble that the operators are not everywhere defined in general. "As soon as one observes that non-bounded operators are necessarily not everywhere defined operators, one immediately gets into a host of difficulties and one immediately runs into a number of wide open problems..."[27, p. 7]. The core of these open problems is the problem of finding some reasonable set of requirements that permit forming an algebra of unbounded operators:

> ... it is quite clear that the interesting applications, quite particularly in quantum theory, absolutely call for a settlement of these questions: to tell how to operate on non-bounded operators, how to form sums and products, what to do if their domains have nothing in common, i.e. if there is no point where both are defined, what to do if the points where they are both defined, exist and are dense, but one somehow suspects that the set is not large enough (and this can be given a sharper meaning), generally speaking, how to introduce an algebra. [27, p. 10]

That one is left without any satisfactory calculus of unbounded operators is what von Neumann points out as the chief motivation to look for suitable subsets of bounded operators that determine a well-behaving set of unbounded operators. After an informal discussion of what is now known as von Neumann's "double commutant theorem" (=a selfadjoint set S of bounded Hilbert space operators is strongly dense in its second commutant S'') von Neumann briefly sketches the 1935 Murray-von Neumann classification theory of factors, emphasizing the analogy between the relative dimension and the cardinals:

> ... the whole algorithm of Cantor is such that it goes over on this case. One can prove various theorems on the additivity of equivalence and the transitivity of equivalence, which one would normally expect, so that one can introduce a theory of alephs here, just as in set theory... [27, p. 15]

But the really exciting consequence of the dimension theory is in von Neumann's view that

> One can, however, go a great deal further. Specifically in the case of Hilbert space, one can define finiteness and infiniteness in the same way as Cantor did by equivalence to a proper subset. One can prove most of the Cantoreal properties of finite and infinite, and, finally, one can prove that given a Hilbert space and a ring in it, a simple ring in it, either all linear sets except the null set are infinite (in which case this concept of alephs gives you nothing new), or else the dimensions, the equivalence classes, behave exactly like numbers and there are two qualitatively different cases. The dimensions either behave like integers, or else they behave like all real numbers. There are two subcases, namely there is either a finite top or there is not. So, when they behave like integers, they either behave like all integers from one to a finite n, or like all integers to infinity plus a symbol infinity. When they are continuous, they either behave like all real numbers from null to a finite number a, inclusive, or else like all real numbers up to infinity with a symbolic top at infinity.
>
> In total there are therefore five classes, I mean like the integers which may have a finite top or not, like all real numbers which may have a finite top or not, and, finally, the case where only the infinite dimensions exist, apart from the dimension null.
>
> The case which is entirely finite, where all you have are the dimensions which are integers that have a finite ceiling, is always isomorphic to the matrices of the Euclidean space. The case where you have integers going to infinity, is isomorphic to all matrices of Hilbert space, so there nothing is gained. About the infinite cases very little is known. [27, p. 15-16]

The upshot of this classification theory, in von Neumann's view, is the case "...where the dimensionality is like real numbers with a finite ceiling..", [27, p. 16], which can be chosen to be 1. This case, known as the type $\mathbf{II}_1$ case, fascinated von Neumann from the moment of its discovery in [14], and he made clear more than once that in his view this structure might be more suitable for quantum theory than ordinary Hilbert space theory[3]. It has not escaped the attention of reviewers that von Neumann placed high hopes on

[3] See the Introduction in [14] and footnote 33 in [5].

this structure (see eg. [4], [12]), [1], [8]); however, there is general agreement that physics has not developed in the direction von Neumann seems to have envisaged. Araki even sees in von Neumann's preference of the type $\mathbf{II}_1$ von Neumann algebras a "mathematical Utopia for quantum calculus" [1, p. 119].

But what was the rational behind von Neumann's conviction that the type $\mathbf{II}_1$ structure might be more suitable for quantum mechanics than the other types, in particular the type **I**, which corresponds to the standard Hilbert space quantum mechanics? Surely von Neumann must have had good reasons when he suggested that the Hilbert space formalism, which in its precise mathematical form was largely his own creation, is not entirely appropriate after all. What were then these reasons? Unfortunately, von Neumann never published a paper devoted to a systematic analysis of the conceptual significance of the type $\mathbf{II}_1$ case. His 1954 lecture is thus a major source of information in this regard, since von Neumann consciously addresses this issue in it.

One reason von Neumann gives for the privileged status of the type $\mathbf{II}_1$ algebra is that the unbounded operators affiliated with this algebra are a very well behaving set: "... one can show that any finite number of them, in fact any countable number of them, are simultaneously defined on an everywhere dense set; one can prove that one can indulge in operations like adding and multiplying operators and one never gets into any difficulty whatever. The whole symbolic calculus goes through." [27, p. 16] So, quantum systems modelled by a type $\mathbf{II}_1$ algebra are perfectly well behaving, inluding their unbounded operators. What are the quantum systems that are modeled by this structure?

Von Neumann claims that:

> One can further show that such systems of operators are in many ways very similar to certain operator systems used in quantum theory. I will not attempt to go into detail at this occasion, but it is true that actually the so-called method of second quantization, which introduces the operators of quantum theory depending on certain processes of counting of states, permits a very plausible generalization which leads exactly into this kind of operator ring, and which is therefore immune to the usual pathology of operator rings. [27, p. 16]

It is not entirely clear what it is that von Neumann is referring to in the above passage. He had developed a quantum theory of infinite quantum systems in a manuscript dated 1937 [23], he never published that theory, however. It might be that the procedure in [23] of creating the algebra of observables leads to the type $\mathbf{II}_1$ case – an analysis of this manuscript is yet to be done. Today the typical example of the type $\mathbf{II}_1$ structure is the infinite tensor product of two-by-two (complex) matrices in the representation given by the state whose restriction to any finite tensor product is the product of the two-by-two traces. This structure describes a lattice gas in the infinite temperature state [6].

Von Neumann's main motivation to prefer the type $\mathbf{II}_1$ is, however, related to the third group of problems he set out to discuss: the relation of logic and probability in quantum mechanics. To appreciate fully von Neumann's position in his address one would have to reconstruct the development of von Neumann's ideas from 1927 on, when he published his three "fundational papers" [18]-[20], through his 1932 book [21] and his joint paper with G. Birkhoff on quantum logic in 1936 [5]. This cannot be done here (see [17] and chapter 7 in [16] for the details of this intellectual history). Disregarding the fine structure of the historical development of von Neumann's ideas, one can formulate the main points needed to understand his view of quantum probability in his address as follows.

The closed linear subspaces of the (infinite dimensional) Hilbert space describing a quantum system form an orthocomplemented lattice ("Hilbert lattice") with respect to the set theoretical intersection as $\wedge$, the orthogonal complement as orthocomplementation $A \mapsto A^{\perp}$ and $\vee$ defined by $A \vee B = \left(A^{\perp} \wedge B^{\perp}\right)^{\perp}$. In quantum logic one interprets the elements of this lattice as quantum propositions and the lattice operations $\wedge, \vee, \perp$ as the logical connectives corresponding to *and*, *or* and *not*, respectively. On this "quantum logic interpretation" the Hilbert lattice is the quantum analog of the Boolean algebra that represents the propositional system of a classical propositional logic. A Boolean algebra also features in another role, however: it represents the algebra of random events in classical probability theory, with probability being an additive measure μ on the algebra. The measure μ has the following property:

$$\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B) \tag{1}$$

where the interpretation of $A \cap B$ is that it represents the joint occurrance of the events A and B.

Property (1) is crucial if the probabilites $\mu(A), \mu(B)$ are to be viewed as relative frequencies: if $N\mu(X)$ ($X = A, B, A \cap B, A \cup B$) are the numbers of occurances of the events in a fixed ensemble of N events then (1) obviously holds.

Von Neumann wanted to interpret also quantum logic as representing random events of a non-commutative probability theory; furthermore, he, too, viewed probabilities as relative frequencies in the years 1927-1936. Trouble is that the non-commutative probability measures, (i.e. the normalized maps ϕ defined on the Hilbert lattice which take values in $[0,1]$ and which are additive on orthogonal elements) violate (1) in general: the equation (2) below cannot hold for all A and B

$$\phi(A) + \phi(B) = \phi(A \wedge B) + \phi(A \vee B) \tag{2}$$

So von Neumann faced the dilemma of the following options:

(i) Give up the frequency interpretation of probability in favor of an interpretation that is capable, in principle, of handling infinite probabilities.

(ii) Give up the interpretaion of quantum logic as random event structure.

(iii) Give up Hilbert lattice (of an ∞-dimensional Hilbert space) as quantum logic.

None of these options is particularly attractive. In his 1936 paper published with G. Birkhoff [5] von Neumann chose option (iii): in that paper quantum logic is postulated to be a lattice that admitts a normalized non-commutative probability measure [4] satisfying (2). What made such a choice possible was that by the time of publishing the paper with Birkhoff in 1936, Murray and von Neumann had already discovered the type $\mathbf{II}_1$ von Neumann algebras [14], which are distinguished by the fact that on their projection lattice there exists a unique (up to constant) finite, non-commutative probability measure τ that satisfies (2). Soon it also became known that τ can be extended from the projection lattice to a trace on the algebra, where "trace" means that for all X, Y we have

$$\tau(XY) = \tau(YX) \tag{3}$$

In short, the projection lattice of a type $\mathbf{II}_1$ von Neumann algebra with the trace τ giving the probabilities is a non-commutative probability structure probabilities of which could be viewed as relative frequencies.

[4] In that paper this measure is called "the apriori theormodynamic weight of states". For an explanation of this terminology see [17].

Thus it would seem that remaining within the mathematical framework of type $\mathbf{II}_1$ von Neumann algebras one can restore the harmonious classical picture: random events can be identified with the propositions stating that the event happens, and probabilities can be viewed as relative frequences of the occurrances of the events. But this restored harmony is deceiving since of all the non-commutative probability measures definable on the type $\mathbf{II}_1$ algebra only the trace τ satisfies the condition (2), which is necessary for a frequency interpretation – and the trace is exactly the functional which is insensitive (in the sense of (3)) for the non-commutativity of the algebra. In other words, there are no "properly non-commutative" probability spaces – as long as one insists on the frequency interpretation of probability; hence, if one wants to entertain the idea of non-commutative probability spaces, the frequency view has to go.

And it did: von Neumann abandoned the frequency interpretation after 1936. In an unfinished manuscript written about 1937 and entitled "Quantum logic (strict- and probability logics)" he writes: "This view, the so-called 'frequency theory of probability' has been very brilliantly upheld and expounded by R. von Mises. This view, however, is not acceptable to us, at least not in the present 'logical' context." [22] ([30, p. 196]. Instead, von Neumann embraces in this unfinished note a "logical theory of probability", which he associates with J. N. Keynes, but which he does not spell out in detail. It is in his 1954 address where he is more explicit and explains the logical interpretation at length without ever mentioning relative frequencies:

> Essentially if a state of a system is given by one vector, the transition probability in another state is the inner product of the two which is the square of the cosine of the angle between them. In other words, probability corresponds precisely to introducing the angles geometrically. Furthermore, there is only one way to introduce it. The more so because in the quantum mechanical machinery the negation of a statement, so the negation of a statement which is represented by a linear set of vectors, corresponds to the orthogonal complement of this linear space. And therefore, as soon as you have introduced into the projective geometry the ordinary machinery of logics, you must have introduced the concept of orthogonality. This actually is rigorously true and any axiomatic elaboration of the subject bears it out. So in order to have logics you need in this set a projective geometry with a concept of orthogonality in it.

> In order to have probability all you need is a concept of all angles, I mean angles other than 90°. Now it is perfectly quite true that in geometry, as soon as you can define the right angle, you can define all angles. Another way to put it is that if you take the case of an orthogonal space, those mappings of this space on itself, which leave orthogonality intact, leave all angles intact, in other words, in those systems which can be used as models of the logical background for quantum theory, it is true that as soon as all the ordinary concepts of logic are fixed under some isomorphic transformation, all of probability theory is already fixed. ... This means, however, that one has a formal mechanism, in which logics and probability theory arise simultaneously and are derived simultaneously. [27, p. 21-22]

It was the simultaneous emergence and mutual determination of probability and logic what von Neumann found intriguing and not at all well understood. He very much wanted to have a detailed axiomatic study of this phenomenon because he hoped that it would shed "... a great deal of new light on logics and probably alter the whole formal structure of logics considerably, if one succeeds in deriving this system from first principles, in other words from a suitable set of axioms." [27, p. 22] He emphasized – and this was his last thought in his address – that it was an entirely open problem whether/how such an axiomatic derivation can be carried out.

Von Neumann's call seems to have remained unanswered so far. And if one takes only a small portion of the enormous philosophical and foundational literature on quantum mechanics seriously, one has to conclude that we are in the peculiar situation where non-commutative measure theory has been developed into a rich mathematical discipline; yet, a satisfactory interpretation of non-commutative measure as probability and of the relation of this non-commutative (quantum) probability to (quantum) logic is still lacking. It shows von Neumann's deep interest in philosophical and conceptual issues that he had chosen this particular topic for a plenary lecture at the world congress of mathematicians in 1954.

Acknowledgement

I wish to thank Marina von Neumann-Whitman for her granting me permission to quote from the unpublished material held in the von Neumann Archive in the Library of Congress. This work was supported by AKP, by

OTKA (contract numbers: T 025841 and F 023447) and by the Dibner Institute MIT, where I was staying during the 1997/98 academic year as a Resident Fellow.

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