Von Neumann's Proof of
Uniqueness of Schrödinger representation of
Heisenberg's commutation relation :

$$QP - PQ = iI$$

Q, P selfadjoint operators
(on a dense subset of) $\mathcal{H}$

A specific example of Q, P satisfying Heisenberg's CCR:
The Schrödinger representation defined by

$$(Qf)(x) = xf(x) \qquad (Pf)(x) = -if'(x) \qquad f \in L^2(\mathbb{R}, \mu)$$

Are there other examples?

Von Neumann-Stone Theorem: NO

Difficulty in investigating uniqueness problem:
Q and P cannot be bounded

$$\Downarrow$$

All the usual domain problems arise

Structure of investigating the uniqueness problem

- Getting rid of unboundedness of the operators involved
- Defining the representation of CCR in terms of bounded operators
- Defining uniqueness of representation of CCR
- Spelling out conditions ensuring uniqueness of representation
- Proving uniqueness

Getting rid of unboundedness

Proposition [Stone's Theroem]: If Q is a selfadjoint operator on $\mathcal{H}$ then

$$\mathbb{R} \ni t \mapsto e^{itQ} \in \mathcal{B}(\mathcal{H}) \tag{1}$$

is a one parameter family of unitary operators e^{itQ} and the map

$$t \mapsto e^{itQ}$$

is continuous in the strong operator topology, i.e.

$$t \mapsto e^{itQ}\xi$$

is continuous for every $\xi \in \mathcal{H}$

Remark: $\mathbb{R} \ni t \mapsto e^{itQ} \in \mathcal{B}(\mathcal{H})$ is a (continuous) representation of $\mathbb{R}$ as an additive group, Q is the generator of the representation

Stone's Theorem

$$\Downarrow$$

Heisenberg's commutation relation

$$QP - PQ = iI$$

can be viewed as the infinitesimal form of a commutation relation and, by Stone's Theorem, it can be reformulated in terms of the one parameter families (groups) U, V of unitary operators determined by Q, P as infinitesimal generators:

$$\begin{aligned} U(a) &= e^{iaQ} \\ V(b) &= e^{ibP} \end{aligned}$$

The commutation relation

$$QP - PQ = iI$$

entails a commutation relation between U and V:

$$U(a)V(b) = e^{iab}V(b)U(a) \qquad a, b \in \mathbb{R} \tag{2}$$

Weyl form of CCR

Von Neumann:

Instead of $U(a)$ and $V(b)$ one can consider the two parameter family

$$S(a,b) \equiv \exp\Big(-\frac{1}{2}iab\Big)U(a)V(b)$$

The Weyl form of CCR entails commutation relation for $S(a,b)$:

$$S(a,b)S(c,d) = \exp\Big(\frac{1}{2}i(ad-bc)\Big)S(a+c,b+d)$$

Defining the representation of CCR in terms of bounded operators

Definition:

$$\mathbb{R} \ni (a, b) \mapsto S(a, b) \in \mathcal{B}(\mathcal{H})$$

is a representation of (the Weyl form) of CCR if

$$\begin{aligned} S(-a, -b) &= S(a, b)^* \\ S(a, b)S(c, d) &= \exp\left(\frac{1}{2} i(ad - bc)\right) S(a + c, b + d) \end{aligned}$$

Defining uniqueness of representation of CCR

Two representations S and S' of CCR on $\mathcal{H}$ are unitarily equivalent if there exists a unitary $U\colon \mathcal{H} \to \mathcal{H}$ such that

$$S(a,b) = US'(a,b)U^* \quad \text{for all} \;\; a, b$$

A representation S of CCR on $\mathcal{H}$ is unique if S is unitarily equivalent to every representation S' of CCR on $\mathcal{H}$

Spelling out conditions ensuring uniqueness of representation

The closed linear subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ is called invariant if

$$S(a,b)\xi \in \mathcal{H}_0 \quad \text{for all } \xi \in \mathcal{H}_0 \text{ and for all } \ a,b$$

The representation $(a,b) \mapsto S(a,b)$ is

- irreducible if there are no non-trivial invariant subspaces
- (strongly) continuous if

$$(a_n,b_n) \to (a,b) \ \text{ entails } \ S(a_n,b_n)\xi \to S(a,b)\xi \ \text{ for all } \xi \in \mathcal{H}$$

Theorem:

Stone-von Neumann's theorem
on the uniqueness of the representation of the CCR relation:

The Schrödinger representation of CCR on $\mathcal{H}$ is the unique irreducible, (strongly) continuous representation of CCR

In detail: The theorem says that if S is any irreducible, continuous representation of CCR on $\mathcal{H}$ and S^{Sch} is the Schrödinger representation on $L^2(\mathbb{R}, \mu)$, then there exists a unitary $U\colon L^2(\mathbb{R}, \mu) \to \mathcal{H}$ such that

$$S(a, b) = U S^{Sch}(a, b) U^* \quad \text{for all} \;\; a, b$$

Von Neumann's proof of uniqueness of the Schrödinger representation of CCR

The uniqueness result was stated by M. Stone first in 1930 with some hints as to the proof but it was von Neumann who had given the full proof in 1931.

Von Neumann's letter to Veblen (September 23, 1930) reports on this result, and his letter to Stone (October 8, 1930) lets us peak into his semi-formal thinking that explains the intuition behind the proof of the uniqueness theorem:

Any proof of this theorem had to construct with the aid of P, Q or

$$U(\alpha) = e^{i\alpha P} \qquad V(\beta) = e^{i\beta Q}$$

some operator, which has easily identifiable properties, determining him in a unique way – and which operator on the other hand can be used to determine some vectors in Hilbert space.

(von Neumann to Stone October 8, 1930)

The general form of the operator determined by U and V is

$$A = \int\int a(\alpha, \beta) U(\alpha) V(\beta) d\alpha d\beta \tag{3}$$

with an integrable function $\mathbb{R}^2 \mapsto a(\alpha, \beta)$. Using the commutation relation between U and V and the definition of S the operator A can also be written in terms of $S(a, b)$, and von Neumann constructs an A which is given by

$$A = \int\int \exp\Big(-\frac{1}{4}(|a|^4 + |b|^2)\Big) S(a, b) da db \tag{4}$$

The crucial observation (has to be proved!) is that the operator

$$P = \frac{1}{2\pi} A$$

is a projection, [surprise !] and if the representation $(a, b) \mapsto S(a, b)$ is irreducible then P is one dimensional, spanned by a unit vector $\xi \in \mathcal{H}$; hence if S and S' are two irreducible representations of CCR and ξ' is the analogously defined vector determined by S', then the map $U\colon \mathcal{H} \to \mathcal{H}$ defined by

$$US(a, b)\xi = S'(a, b)\xi'$$

extends linearly to a unitary operator of $\mathcal{H}$ that intertwines between the two representations S and S'

The explicit and implicit assumptions are essential in von Neumann's uniqueness theorem:

Explicit assumptions:

- continuity property of the map $(a, b) \rightarrow S(a, b)$
- irreducibility of S

Implicit assumption:

- $\mathbb{R}^2$ is a finite dimensional linear space

The importance of these assumptions becomes evident when one tries to generalize the Stone-von Neumann theorem

Generalization of Stone-von Neumann theorem

Replace

$\mathbb{R}^2$	by	an arbitrary linear space H
$\mathbb{R} \ni (a,b) \longrightarrow \frac{1}{2}(ad - bc)$	by	$H \ni (f,g) \mapsto \sigma(f,g)$
		σ a bilinear form
$\mathcal{B}(\mathcal{H})$	by	an arbitrary C^* algebra $\mathcal{A}$
$\mathbb{R} \ni (a,b) \mapsto S(a,b)$	by	$H \ni (f,g) \mapsto W(f,g) \in \mathcal{A}$

such that

$$\begin{aligned} W(-f) &= W(f)^* \\ W(f)W(g) &= \exp(i\sigma(f,g))W(f+g) \end{aligned}$$

Definition : Such a W is a representation of CCR

Definition : The C^* algebra $CCR(H,\sigma)$ generated by $\{W(f) \ : \ f \in H\}$ is called the C^* algebra of the canonical commutation relations determined by H and σ

Theorem [Slawny, 1971]: The $CCR(H,\sigma)$ is unique (up to $*$-isomorphism)

Note

- (strong) continuity of the general representation W of CCR is meaningless
- norm continuity of representation cannot be required because Theorem : if f, g are different then $\| W(f) - W(g) \| \geq \sqrt{2}$)
- Stone-von Neumann uniqueness theorem does not hold for W