19th century developments 3: Algebra
The beginnings: Frege

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A substantial change of the views about the objects of mathematics and mathematical truth.

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Researching the foundations of mathematics: supporting or refuting such answers.

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Motivation: there are eternal truths.

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- Syntax should make logical relations fully transparent
- The rules of the calculus must be defined in purely syntactical terms and each step of a deduction must be algorithmically verifiable.

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A result preparing the way for the reduction of arithmetics to logic (chapter 3.):
Definition of the successor-relation from the relation „immediate successor" in a given sequence (in other words: the definition of the transitive closure of a relation).

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The definitions are based on a critical examination of the concepts of number in philosophy and mathematics and a philosophical analysis of the concept of number.

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Neo-Fregeanism: one of the contemporary directions.